Chapter 1 **Rates of Change**

Chapter 1 Prerequisite Skills

Chapter 1 Prerequisite Skills **Question 1 Page 2**

a)

x	y	First Differences
–4	**9**	
		–4
–3	**5**	
		–2
–2	**3**	
		0
–1	**3**	
		2
0	**5**	
		4
1	**9**	
		6
2	**15**	

Answers may vary. For example:
The first differences are not equal, but they progress by an equal amount.

b) Answers may vary. For example:
Yes. Since the first differences are not equal, the function is not linear. Also, since the first differences increase by the same amount each time, the curve is quadratic.

Chapter 1 Prerequisite Skills **Question 2 Page 2**

a) $\text{slope} = \dfrac{3-1}{-2-4}$

$= -\dfrac{2}{6}$ or $-\dfrac{1}{3}$

b) $\text{slope} = \dfrac{-7-(-1)}{3-0}$

$= -\dfrac{6}{3}$ or -2

c) $\text{slope} = \dfrac{1-0}{5-0}$

$= \dfrac{1}{5}$

d) $\text{slope} = \dfrac{4-4}{0-(-9)}$

$= 0$

Chapter 1 Prerequisite Skills **Question 3 Page 2**

a) $y = \frac{1}{2}x - \frac{7}{4}$; slope: $\frac{1}{2}$; y-intercept: $-\frac{7}{4}$

b) $y = -\frac{5}{3}x + \frac{1}{3}$; slope: $-\frac{5}{3}$; y-intercept: $\frac{1}{3}$

c) $y = -2x - \frac{10}{9}$; slope: -2; y-intercept: $-\frac{10}{9}$

d) $y = \frac{7}{5}x + \frac{2}{5}$; slope: $\frac{7}{5}$; y-intercept: $\frac{2}{5}$

Chapter 1 Prerequisite Skills **Question 4 Page 2**

a) $y = 5x + 3$

b) $\text{slope} = \frac{3-1}{-5-1}$

$= -\frac{2}{6}$ or $-\frac{1}{3}$

Substitute the point (1, 1) into $y = -\frac{1}{3}x + b$ to find b.

$1 = -\frac{1}{3}(1) + b$

$b = \frac{4}{3}$

The equation is $y = -\frac{1}{3}x + \frac{4}{3}$.

c) Substitute the point (4, 7) into $y = -2x + b$ to find b.

$7 = -2(4) + b$

$b = 15$

The equation is $y = -2x + 15$.

d) $\text{slope} = \frac{0-(-1)}{3-2}$

$= 1$

Substitute the point (3, 0) into $y = x + b$ to find b.

$0 = (3) + b$

$b = -3$

The equation is $y = x - 3$.

Chapter 1 Prerequisite Skills **Question 5 Page 2**

a) $(a+b)^2 = a^2 + 2ab + b^2$

b) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

c) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

d) $(a+b)^4 = (a^2 + 2ab + b^2)^2$
$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

e) $(a-b)^5 = (a-b)^2(a-b)^3$
$= a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$

f) $(a+b)^5 = (a+b)^2(a+b)^3$
$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Chapter 1 Prerequisite Skills **Question 6 Page 2**

a) $2x^2 - x - 1 = (2x+1)(x-1)$

b) $6x^2 + 17x + 5 = 6x^2 + 15x + 2x + 5$
$= 3x(2x+5) + (2x+5)$
$= (2x+5)(3x+1)$

c) $x^3 - 1 = (x-1)(x^2 + x + 1)$

d) $2x^4 + 7x^3 + 3x^2 = x^2(2x^2 + 7x + 3)$
$= x^2[2x(x+3) + (x+3)]$
$= x^2(2x+1)(x+3)$

e) $x^4 - x^3 - x + 1 = x^3(x-1) - 1(x-1)$
$= (x-1)(x^3 - 1)$
$= (x-1)(x-1)(x^2 + x + 1)$
$= (x-1)^2(x^2 + x + 1)$

f) $t^3 + 2t^2 - 3t = t(t^2 + 2t - 3)$
$= t(t-1)(t+3)$

Chapter 1 Prerequisite Skills **Question 7 Page 2**

b) $(a-b)(a+b)$

c) $a^3 - b^3$

d) $(a-b)(a^3 + a^2b + ab^2 + b^3)$

e) $(a-b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$

f) $(x+h-x)((x+h)^{n-1} + x(x+h)^{n-2} + x^2(x+h)^{n-3} + \cdots + x^{n-3}(x+h)^2 + x^{n-2}(x+h) + x^{n-1})$
$= h((x+h)^{n-1} + x(x+h)^{n-2} + x^2(x+h)^{n-3} + \cdots + x^{n-3}(x+h)^2 + x^{n-2}(x+h) + x^{n-1})$

Chapter 1 Prerequisite Skills **Question 8 Page 2**

a) $\left(\sqrt{x} - \sqrt{2}\right)\left(\sqrt{x} + \sqrt{2}\right) = \left(\sqrt{x}\right)^2 + \sqrt{2x} - \sqrt{2x} - \left(\sqrt{2}\right)^2$
$= x - 2$

b) $\left(\sqrt{x+1} - \sqrt{x}\right)\left(\sqrt{x+1} + \sqrt{x}\right) = \left(\sqrt{x+1}\right)^2 + \sqrt{x(x+1)} - \sqrt{x(x+1)} - \left(\sqrt{x}\right)^2$
$= x + 1 - x$
$= 1$

c) $\left(\sqrt{x+1} - \sqrt{x-1}\right)\left(\sqrt{x+1} + \sqrt{x-1}\right) = \left(\sqrt{x+1}\right)^2 + \sqrt{(x+1)(x-1)} - \sqrt{(x+1)(x-1)} - \left(\sqrt{x-1}\right)^2$
$= x + 1 - (x-1)$
$= 2$

d) $\left(\sqrt{3(x+h)} - \sqrt{3x}\right)\left(\sqrt{3(x+h)} + \sqrt{3x}\right) = \left(\sqrt{3(x+h)}\right)^2 + \sqrt{9x(x+h)} - \sqrt{9x(x+h)} - \left(\sqrt{3x}\right)^2$
$= 3(x+h) - 3x$
$= 3h$

Chapter 1 Prerequisite Skills **Question 9 Page 2**

a) $\frac{1}{2+h} - \frac{1}{2} = \frac{2-(2+h)}{2(2+h)}$
$= \frac{-h}{2(2+h)}$

b) $\frac{1}{x+h} - \frac{1}{x} = \frac{x-(x+h)}{x(x+h)}$
$= \frac{-h}{x(x+h)}$

c) $\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}=\dfrac{x^2-(x+h)^2}{x^2(x+h)^2}$

$=\dfrac{-2xh-h^2}{x^2(x+h)^2}$

$=-\dfrac{h(2x+h)}{x^2(x+h)^2}$

d) $\dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}=\dfrac{\dfrac{x-(x+h)}{x(x+h)}}{h}$

$=\dfrac{-h}{xh(x+h)}$

$=\dfrac{-1}{x(x+h)}$

Chapter 1 Prerequisite Skills **Question 10 Page 3**

a) $f(-2)=3(-2)+12$ $f(3)=3(3)+12$

$=6$ $=21$

The points are (–2, 6) and (3, 21).

b) $f(-2)=-5(-2)^2+2(-2)+1$ $f(3)=-5(3)^2+2(3)+1$

$=-23$ $=-38$

The points are (–2, –23) and (3, –38).

c) $f(-2)=2(-2)^3-7(-2)^2+3$ $f(3)=2(3)^3-7(3)^2+3$

$=-41$ $=-6$

The points are (–2, –41) and (3, –6).

Chapter 1 Prerequisite Skills **Question 11 Page 3**

a) $f(3+h)=6(3+h)-2$

$=16+6h$

b) $f(3+h)=3(3+h)^2+5(3+h)$

$=3(h^2+6h+9)+15+5h$

$=3h^2+23h+42$

c) $f(3+h)=2(3+h)^3-7(3+h)^2$

$=2(h^3+3h^2x+3hx^2+x^3)-7(h^2+6h+9)$

$=2h^3+11h^2+12h-9$

Chapter 1 Prerequisite Skills **Question 12 Page 3**

a) $\dfrac{f(2+h)-f(2)}{h} = \dfrac{6(2+h)-6(2)}{h}$

$= 6$

b) $\dfrac{f(2+h)-f(2)}{h} = \dfrac{2(2+h)^3-2(2)^3}{h}$

$= \dfrac{2(2^3+3(2^2)h+3(2)h^2+h^3)-16}{h}$

$= 2h^2+12h+24$

c) $\dfrac{f(2+h)-f(2)}{h} = \dfrac{\dfrac{1}{(2+h)}-\dfrac{1}{2}}{h}$

$= \dfrac{\dfrac{-h}{2(2+h)}}{h}$

$= -\dfrac{1}{2(2+h)}$

d) $\dfrac{f(2+h)-f(2)}{h} = \dfrac{-\dfrac{4}{(2+h)}-\left(-\dfrac{4}{2}\right)}{h}$

$= \dfrac{\dfrac{-8+4(2+h)}{2(2+h)}}{h}$

$= \dfrac{4}{2(2+h)}$

Chapter 1 Prerequisite Skills **Question 13 Page 3**

a) $\{x \mid x \in \mathbb{R}\}$

b) $\{x \mid x \neq 8,\ x \in \mathbb{R}\}$

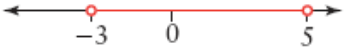
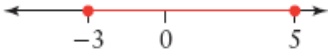
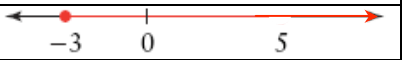
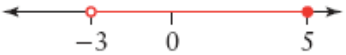
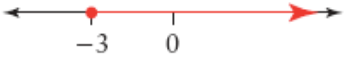
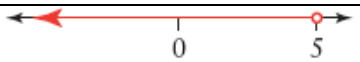
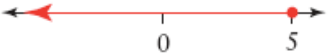
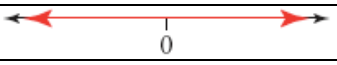
c) $\{x \mid x \in \mathbb{R}\}$

d) $\{x \mid x \geq 0;\ x \in \mathbb{R}\}$

e) $\{x \mid x \neq -3 \text{ and } x \neq 2;\ x \in \mathbb{R}\}$

f) $\{x \mid 0 \leq x \leq 9,\ x \in \mathbb{R}\}$

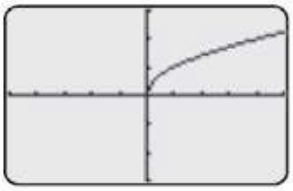
Chapter 1 Prerequisite Skills **Question 14 Page 3**

Interval Notation	Inequality	Number Line
$(-3, 5)$	$-3 < x < 5$	
$[-3, 5]$	$-3 \le x \le 5$	
$[-3, 5)$	$-3 \le x < 5$	
$(-3, 5]$	$-3 < x \le 5$	
$(-3, \infty)$	$x > -3$	
$[-3, \infty)$	$x \ge -3$	
$(-\infty, 5)$	$x < 5$	
$(-\infty, 5]$	$x \le 5$	
$(-\infty, \infty)$	$\mathbb{R}$	

Chapter 1 Prerequisite Skills **Question 15 Page 3**

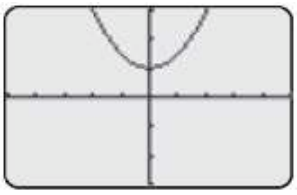
a) domain: $\{x \mid x \in R\}$; range: $\{y \mid y \in \mathbb{R}\}$

b) domain: $\{x \mid x \ge 0,\ x \in R\}$; range: $\{y \mid y \ge 0,\ y \in \mathbb{R}\}$



c) domain: $\{x \mid x \ne -2,\ x \in R\}$; range: $\{y \mid y \ne -4,\ y \in \mathbb{R}\}$

d) domain: $\{x \mid x \in R\}$; range: $\{y \mid y \geq 1,\ y \in \mathbb{R}\}$



Chapter 1 Section 1 **Rates of Change and the Slope of a Curve**

Chapter 1 Section 1 **Question 1 Page 9**

a) average rate of change $= \dfrac{6-(-1)}{2-(-4)}$

$= \dfrac{7}{6}$

b) average rate of change $= \dfrac{17-(-6.7)}{-5-3.2}$

$= \dfrac{23.7}{-8.2}$

$= -\dfrac{237}{82}$

c) average rate of change $= \dfrac{\frac{3}{4}-\left(-\frac{4}{5}\right)}{-1\frac{1}{2}-\left(\frac{2}{3}\right)}$

$= \dfrac{\frac{31}{20}}{-\frac{13}{6}}$

$= -\dfrac{93}{130}$

Chapter 1 Section 1 **Question 2 Page 9**

a) i) average rate of change $= \dfrac{-3-5}{1-(-3)}$

$= -2$

ii) average rate of change $= \dfrac{5-5}{3-(-3)}$

$= 0$

iii) average rate of change $= \dfrac{45-(-3)}{7-1}$

$= 8$

iv) average rate of change $= \frac{6-(-5)}{5-(-1)}$

$= \frac{11}{6}$

b) Answers may vary. For example: Sketch the graph (scatter plot) to estimate the instantaneous rates by choosing small intervals and using the formula for the average rate of change as in a).

i) –1

ii) 5

iii) 0.5

iv) 10

Chapter 1 Section 1 **Question 3 Page 9**

a) Choosing the points (2, 9) and (1, 7): $\frac{7-9}{1-2} = 2$

b) Choosing the points (–2, 2) and (10, 14): $\frac{14-2}{10-(-2)} = 1$

c) Choosing the points (1, 0) and (5, 7): $\frac{7-0}{5-1} = \frac{7}{4}$

Chapter 1 Section 1 **Question 4 Page 9**

Answers may vary. For example:

a) i) They are all zero. B and F are local minima and D is a local maximum.

ii) They have the same magnitude but are opposite in sign. The instantaneous rate of change is negative at A and positive at G.

iii) They are both positive since the function is increasing at both points.

iv) They are both negative since the function is decreasing at both points.

b) i) The instantaneous rate of change is negative at B and positive at C.

ii) They are all negative since the function is decreasing at all three points.

iii) They are both positive since the function is increasing at both points.

iv) The instantaneous rate of change is negative at A and positive at C.

Chapter 1 Section 1 **Question 5 Page 10**

a) The dependent variable is surface area in square centimetres and the independent variable is time in seconds.
The rate of change of surface area over time is expressed in square centimetres per second.

b) i) $\text{average rate of change} = \dfrac{324.0 - 10.0}{10 - 0}$

$= 31.4$

The average rate of change during the first 10 s is 31.4 cm^2/s.

ii) $\text{average rate of change} = \dfrac{2836.0 - 1266.0}{30 - 20}$

$= 157$

The average rate of change between 20 s and 30 s is 157 cm^2/s.

iii) $\text{average rate of change} = \dfrac{2836.0 - 1818.6}{30 - 24}$

$\doteq 169.57$

The average rate of change during the last 6 s is about 169.57 cm^2/s.

c) i) $\text{instantaneous rate of change} = \dfrac{60.24 - 10.0}{4 - 0}$

$\doteq 13$

The instantaneous rate of change at t = 2 s is an estimated 13 cm^2/s.

ii) $\text{instantaneous rate of change} = \dfrac{813.8 - 462.16}{16 - 12}$

$\doteq 88$

The instantaneous rate of change at t = 14 s is an estimated 88 cm^2/s.

iii) $\text{instantaneous rate of change} = \dfrac{2836.0 - 2132.6}{30 - 26}$

$\doteq 176$

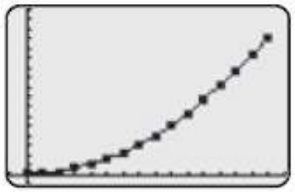
The instantaneous rate of change at t = 28 s is an estimated 176 cm^2/s.

d) Answers may vary. For example:

i) 38 cm^2/s

ii) 100 cm^2/s

iii) 163 cm^2/s



e) Answers may vary. For example:

The instantaneous rate of change is increasing rapidly as the time is increasing. The values I found in part d) agree with this statement.

Chapter 1 Section 1 **Question 6 Page 10**

C, since it is the smallest interval.

Chapter 1 Section 1 **Question 7 Page 10**

a) i)

x	y	First Differences	Average Rate of Change
–3	–50		
		38	
–2	–12		**38**
		14	
–1	2		**14**
		2	
0	4		**2**
		2	
1	6		**2**
		14	
2	20		**14**

ii)

x	y	First Differences	Average Rate of Change
–6	–26		
		52	
–4	26		**26**
		–4	
–2	22		**–2**
		–12	
0	10		**–6**
		28	
2	38		**14**
		116	
4	154		**58**

Parts **b)–d)**: Answers may vary. For example:

b) The average rates of change are equal to the first differences in part i), and they are half the value of the first differences in part ii).

c) The values of the calculated first differences and the average rates of change of y are the not the same in part ii) because the difference between successive x-values is two. In this case, the first differences must be divided by two to calculate the average rate of change.

d) The first differences and average rates of change for a function will be equal if the difference between successive x-values is equal to one.

Chapter 1 Section 1 **Question 8 Page 10**

Explanations may vary. For example:

a) Instantaneous rate of change: The rate of change occurs at the specific instant when the radius is 4 cm.

b) Average rate of change: The rate of change refers to a distance over an interval of 5 h.

c) Instantaneous rate of change: The rate of change occurs at the specific instant when the time is 1 P.M.

d) Average rate of change: The rate of change refers to the stock price over an interval of time of one week.

e) Average rate of change: The rate of change refers to the water level of a lake over an interval from the beginning of March to the end of May.

Chapter 1 Section 1 **Question 9 Page 10**

Answers may vary. For example:

a) The initial temperature of the water was 10° C; After 3 min the water reached its boiling point.

b) The graph shows that the rise in temperature of water is rapid during the first 40 s or so, slowing further until it reaches its boiling point at t = 180 s. After 180 s, the curve is flat, and the instantaneous rate of change is zero after this point.

At t = 60 s, the instantaneous rate of change is about $\frac{87-57}{90-30}$ or 0.5°C/s.

At t = 120 s, the instantaneous rate of change is about $\frac{97-87}{90-30}$ or 0.17°C/s.

At t = 120 s, the instantaneous rate of change is 0°C/s.

Chapter 1 Section 1 **Question 10 Page 11**

a) i) The average rate of change is $\frac{32\ 299\ 496-23\ 143\ 192}{2005-1975}$ or approximately 305 210 people per year.

ii) The average rate of change is $\frac{27\ 697\ 530-24\ 516\ 071}{1990-1980}$ or approximately 318 146 people per year.

iii) From 1975 to 1985: $\frac{25\ 842\ 736-23\ 143\ 192}{1985-1975}$ or approximately 269 954 people per year

From 1985 to 1995: $\frac{29\ 302\ 091-25\ 842\ 736}{1995-1985}$ or approximately 345 936 people per year

From 1995 to 2005: $\frac{32\ 299\ 496-29\ 302\ 091}{2005-1995}$ or approximately 299 741 people per year

Parts **b)–e)**: Answers may vary. For example:

b) While the population has steadily increased, the rate at which it has increased varies.

The estimated instantaneous rate of change in 1983 is:

$\frac{25\ 607\ 651 - 25\ 117\ 442}{1984 - 1982}$ or approximately 245 105 people per year

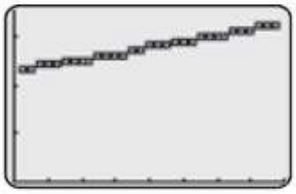
The estimated instantaneous rate of change in 1993 is:

$\frac{28\ 999\ 006 - 28\ 366\ 737}{1994 - 1992}$ or approximately 316 135 people per year

The estimated instantaneous rate of change in 2003 is:

$\frac{31\ 989\ 454 - 31\ 372\ 587}{2004 - 2002}$ or approximately 308 434 people per year

c)



The rate of change of Canada's population has increased steadily since 1975.

d) Canada's population is increasing with respect to time.

e) Q: What do you predict will be Canada's population in the year 2015? Explain.

A: The average rate of change of Canada's population between 1975 and 2005 was 305 210 people per year. Therefore, the estimated population in 2015 is 32 299 496 + (10)(305 210) = 35 351 596.

Q: What do you predict will be the instantaneous rate of change of Canada's population in the year 2015? Explain.

A: The prediction assumes a change equal to the average rate of change over the previous 30 years, which is 305 210 people per year. Therefore, the estimated instantaneous rate of change will be the same figure.

Chapter 1 Section 1 Question 11 Page 11

Solutions to the Achievement Checks are shown in the Teacher's Resource.

Chapter 1 Section 1 **Question 12 Page 11**

Answers may vary. For example:

a) The resistance increases as the voltage increases since the slope of the graph increases.

b) Use the points (0.8, 40) and (1.2, 85).

R = instantaneous rate of change of V

$$= \frac{85 - 40}{1.2 - 0.8}$$

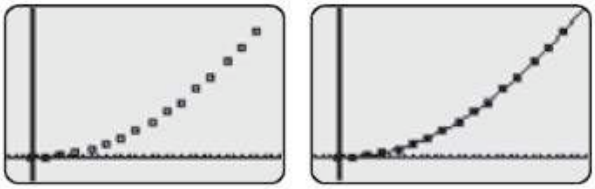
$$\doteq 113$$

The estimated instantaneous rate of change of V when V = 60 V is 113 V/A or Ω.

Chapter 1 Section 1 **Question 13 Page 12**

a)

Time (min)	Radius (m)	Area (m^2)
0	0	0
2	4	50.3
4	8	201.1
6	12	452.4
8	16	804.2
10	20	1256.6
12	24	1809.6
14	28	2463.0
16	32	3217.0
18	36	4071.5
20	40	5026.5
22	44	6082.1
24	48	7238.2
26	52	8494.9
28	56	9852.0
30	60	11 309.7



b) i) The average rate of change during the first 4 min is $\frac{201.1-0}{4-0}$ or approximately 50.3 m^2/min.

ii) The average rate of change during the next 10 min is $\frac{2463.0-201.1}{14-4}$ or approximately 226.2 m^2/min.

iii) The average rate of change during the entire 30 min is $\frac{11\ 309.7-0}{30-0}$ or approximately 377.0 m^2/min.

c) Answers may vary. For example:

The instantaneous rate of change at t = 5 min is $\frac{452.4-201.1}{6-4}$ or approximately 125.7 m^2/min.

The instantaneous rate of change at t = 25 min is $\frac{8494.9-7238.2}{26-24}$ or approximately 628.4 m^2/min.

d) Answers may vary. For example: The information might be useful to determine when the oil spill will reach shore in order to protect the birds, animals and the environment.

Chapter 1 Section 1 **Question 14 Page 12**

a) Since x represents t in this case: at t = 0, y = 2; at t = 5 y = 12; at t = 10, y = 22; at t = 15, y = 12; at t = 20, y = 2

From a hand sketch of these points, it is clear that the function can be expressed using either a sine or cosine function.

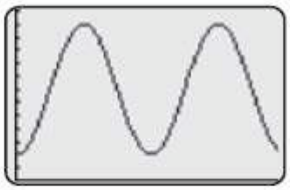
Use a cosine function of the form $y = a\cos(b(x + c)) + d$. Since the function appears upside down compared to cos x, use a negative sign in front of the cosine. The diameter of the windmill is 10 m, so the amplitude of the function is half the diameter, which is 5.
Thus, $a = -5$. It takes 20 s to complete one full revolution, so the period is 20.

Thus, b is equal to $\frac{2\pi}{20} = \frac{\pi}{10}$. There does not need to be a horizontal shift so c = 0. So far, the function is $y = -5\cos\left(\frac{\pi}{10}t\right) + d$. At time t = 0 s, the height of the ladybug is 2 m, so substitute the point (0, 2) into the function to find d.

$2 = -5\cos\left(\frac{\pi}{10}(0)\right) + d$

$d = 7$

Graph $y = -5\cos\left(\frac{\pi}{10}t\right) + 7$ using radians or $y = -5\cos(18t) + 7$ using degrees.



b) Answers may vary. For example:

No. The rate of change of the ladybug's height will not be constant because the rate of change of the height is affected by the position of the blade.

c) Yes. The rate of change of the height of the blade is constantly changing since the slope of the graph is constantly changing.

Chapter 1 Section 1 **Question 15 Page 12**

Answers will vary. For example:

a) If the wind speed increased the blades would turn faster and the period of the function would decrease. The rate of change of the height of the ladybug would increase since the slope of the graph would be steeper.

If the wind speed decreased the blades would turn more slowly and the period of the function would increase. The rate of change of the height of the ladybug would decrease since the slope of the graph would be less steep.

b) The amplitude of the function representing the motion of the ladybug would be reduced from 10 m to 8 m. Therefore the rate of change of the height would decrease since the graph is less steep.

Chapter 1 Section 1 **Question 16 Page 12**

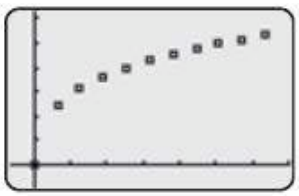
a) i) first 3 s: $\frac{3.58-0}{3-0}$ or approximately 1.19 cm/s; last 3 s: $\frac{5.35-4.75}{10-7}$ or 0.20 cm/s

ii) at 3 s: $\frac{3.94-3.13}{4-2}$ or approximately 0.41 cm/s; at 9 s: $\frac{5.35-4.96}{10-8}$ or approximately 0.2 cm/s

b) i) Use the slopes of the secants from (0,0) to (3, 3.58) and from (7, 4.75) to (10, 5.35).

ii) Use the slopes of the secants from (2, 3.13) to (4, 3.94) and from (8, 4.96) to (10, 5.35).

c)



If the cup was a cylinder, the graph would be a straight line.

d)

t (s)	H (cm)	r (cm)	V (cm^3)
0	0	0	**0**
1	2.48	1.24	**4**
2	3.13	1.565	**8**
3	3.58	1.79	**12**
4	3.94	1.97	**16**
5	4.24	2.12	**20**
6	4.51	2.255	**24**
7	4.75	2.375	**28**
8	4.96	2.48	**32**
9	5.16	2.58	**36**
10	5.35	2.675	**40**

The water is being poured at a constant rate.

Chapter 1 Section 1 **Question 17 Page 12**

$$\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy}$$
$$=\frac{8}{12}$$
$$=\frac{2}{3}$$

Chapter 1 Section 1 **Question 18 Page 12**

$$\log_9 3^{\sqrt{g}}=\log_9 5$$
$$\sqrt{g}\log_9 3=\log_9 5$$
$$\sqrt{g}\left(\frac{1}{2}\right)=\log_9 5$$
$$\sqrt{g}=2\log_9 5$$
$$\sqrt{g}=\log_9 5^2$$
$$\log_9\sqrt{g}=\log_9(\log_9 5^2)$$
$$\log_9 g^{\frac{1}{2}}=\log_9(2\log_9 5)$$
$$\frac{1}{2}\log_9 g=\log_9(2\log_9 5)$$
$$\log_9 g=2\log_9(2\log_9 5)$$

Chapter 1 Section 2 **Rates of Change Using Equations**

Chapter 1 Section 2 **Question 1 Page 20**

a) average rate of change: $\frac{4-1}{4-1}=1$

b) average rate of change: $\frac{4^2-1^2}{4-1}=5$

c) average rate of change: $\frac{4^3-1^3}{4-1}=21$

d) average rate of change: $\frac{7-7}{4-1}=0$

Chapter 1 Section 2 **Question 2 Page 20**

a) $\frac{f(2+h)-f(2)}{h}=\frac{(2+h)-2}{h}$

$=1$

The instantaneous rate of change is 1.

b) $\frac{f(2+h)-f(2)}{h}=\frac{(2+h)^2-2^2}{h}$

$=\frac{h^2+4h+4-4}{h}$

$=h+4$

The instantaneous rate of change is 4.

c) $\frac{f(2+h)-f(2)}{h}=\frac{(2+h)^3-2^3}{h}$

$=\frac{h^3+3(2)h^2+3(2)^2h+(2)^3-8}{h}$

$=h^2+6h+12$

The instantaneous rate of change is 12.

d) $\frac{f(2+h)-f(2)}{h}=\frac{7-7}{h}$

$=0$

The instantaneous rate of change is 0.

Chapter 1 Section 2 **Question 3 Page 20**

$$\frac{f(4+h)-f(4)}{h}$$

Chapter 1 Section 2 **Question 4 Page 20**

$$\frac{(-3+h)^2-(-3)^2}{h}$$

Chapter 1 Section 2 **Question 5 Page 20**

$$\frac{(5+h)^3-(5)^3}{h}$$

Chapter 1 Section 2 **Question 6 Page 20**

$$\frac{(-1+h)^3-(-1)^3}{h}$$

Chapter 1 Section 2 **Question 7 Page 20**

a) True. For $f(x)=4x^3$, $\frac{f(1+h)-f(1)}{h}=\frac{4(1+h)^3-4}{h}$

b) False. The tangent point occurs at $x = 1$.

c) True. For $f(x)=4x^3-4$, $\frac{f(1+h)-f(1)}{h}=\frac{(4(1+h)^3-4)-(4(1)^3-4)}{h}$

$$=\frac{4(1+h)^3-4}{h}$$

d) True. The difference quotient is not defined for $h = 0$.

Chapter 1 Section 2 **Question 8 Page 20**

a) $h = 0.1$: $\frac{(-3+0.1)^2-(-3)^2}{0.1}=-5.9$

$h = 0.01$: $\frac{(-3+0.01)^2-(-3)^2}{0.01}=-5.99$

$h = 0.001$: $\frac{(-3+0.001)^2-(-3)^2}{0.001}=-5.999$

b) $$\frac{(-3+h)^2-(-3)^2}{h} = \frac{9-6h+h^2-9}{h}$$
$$= h-6$$

$h = 0.1$: $0.1 - 6 = -5.9$
$h = 0.01$: $0.01 - 6 = -5.99$
$h = 0.001$: $0.001 - 6 = -5.999$

c) Answers may vary. For example: The answers from part a) and part b) are the same. This makes sense since the expression that is used in part b) is a simplified form of the expression in part a). As the interval h is decreased, the calculated result for the difference quotient is getting closer to -6. The final estimate of the instantaneous rate of change is -6.

Chapter 1 Section 2 **Question 9 Page 20**

$$\frac{(4+h)^4-4^4}{h} = \frac{4^4+4(4)^3h+6(4)^2h^2+4(4)h^3+h^4-4^4}{h}$$
$$= 256+96h+16h^2+h^3$$

$h = 0.1$: $256 + 96(0.1) + 16(0.1)^2 + (0.1)^3$
$= 265.761$
$h = 0.01$: $256 + 96(0.01) + 16(0.01)^2 + (0.01)^3$
$= 256.961\,601$
$h = 0.001$: $256 + 96(0.001) + 16(0.001)^2 + (0.001)^3$
$= 256.096\,016$

The final estimate of the slope at $x = 4$ is 256.

Chapter 1 Section 2 **Question 10 Page 20**

a) average rate of change: $$\frac{(-3)^2+3(-3)-(2^2+3(2))}{-3-2} = \frac{-10}{-5}$$
$$= 2$$

b) average rate of change: $$\frac{2(-3)-1-(2(2)-1)}{-3-2} = \frac{-10}{-5}$$
$$= 2$$

c) average rate of change: $$\frac{7(-3)^2-(-3)^4-(7(2)^2-2^4)}{-3-2} = \frac{-30}{-5}$$
$$= 6$$

d) average rate of change: $$\frac{-3-2(-3)^3-(2-2(2)^3)}{-3-2} = \frac{65}{-5}$$
$$= -13$$

Chapter 1 Section 2 **Question 11 Page 20**

a) $\dfrac{(2+h)^2+3(2+h)-(2^2+3(2))}{h} = \dfrac{4+4h+h^2+6+3h-4-6}{h}$

$= 7+h$

The instantaneous rate of change at $x = 2$ is 7.

b) $\dfrac{2(2+h)-1-(2(2)-1)}{h} = \dfrac{4+2h-1-3}{h}$

$= 2$

The instantaneous rate of change at $x = 2$ is 2.

c) $\dfrac{7(2+h)^2-(2+h)^4-(7(2)^2-(2)^4)}{h} = \dfrac{7(h^2+4h+4)-(h^4+4(2)h^3+6(2)^2h^2+4(2)^3h+2^4)-12}{h}$

$= -h^3-8h^2-17h-4$

The instantaneous rate of change at $x = 2$ is -4.

d) $\dfrac{(2+h)-2(2+h)^3-(2-2(2)^3)}{h} = \dfrac{2+h-2(h^3+3h^2(2)+3h(2)^2+2^3)-(-14)}{h}$

$= -23-12h-2h^2$

The instantaneous rate of change at $x = 2$ is -23.

Chapter 1 Section 2 **Question 12 Page 20**

a) i) $\dfrac{2(a+h)^2-2a^2}{h} = \dfrac{2a^2+4ah+2h^2-2a^2}{h}$

$= 4a+2h$

$4(-3) + 2(0.01) = -11.98$

ii) $\dfrac{(a+h)^3-a^3}{h} = \dfrac{a^3+3a^2h+3ah^2+h^3-a^3}{h}$

$= 3a^2+3ah+h^2$

$3(-3)^2 + 3(-3)(0.01) + (0.01)^2 = 26.9101$

iii) $\dfrac{(a+h)^4-a^4}{h} = \dfrac{a^4+4a^3h+6a^2h^2+4ah^3+h^4-a^4}{h}$

$= 4a^3+6a^2h+4ah^2+h^3$

$4(-3)^3 + 6(-3)^2(0.01) + 4(-3)(0.01)^2 + (0.01)^3 = -107.461\,199$

b) Answers may vary. For example:

Each answer represents the estimate of the slope of the tangent line to the function at the point where $x = -3$.

Chapter 1 Section 2 **Question 13 Page 20**

a) i) $f(x) = x^2$

ii) $a = 4$

iii) $h = 0.01$

iv) $(a, f(a)) = (4, 16)$

b) i) $f(x) = x^3$

ii) $a = 6$

iii) $h = 0.0001$

iv) $(a, f(a)) = (6, 216)$

c) i) $f(x) = 3x^4$

ii) $a = -1$

iii) $h = 0.1$

iv) $(a, f(a)) = (-1, 3)$

d) i) $f(x) = -2x$

ii) $a = 8$

iii) $h = 0.1$

iv) $(a, f(a)) = (8, -16)$

Chapter 1 Section 2 **Question 14 Page 21**

a) $\dfrac{-4.9(1+h)^2 + 15(1+h) + 1 - (-4.9(1)^2 + 15(1) + 1)}{h}$

b) Answers may vary. For example:
The expression is not valid for $h = 0$. Division by zero is not defined in the real number system.

c) i) $\dfrac{-4.9(1+0.1)^2+15(1+0.1)+1-11.1}{0.1} = 4.71$

ii) $\dfrac{-4.9(1+0.01)^2+15(1+0.01)+1-11.1}{0.01} = 5.151$

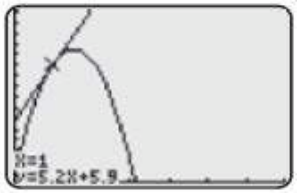
iii) $\dfrac{-4.9(1+0.001)^2+15(1+0.001)+1-11.1}{0.001} = 5.1951$

iv) $\dfrac{-4.9(1+0.0001)^2+15(1+0.0001)+1-11.1}{0.0001} = 5.199\,51$

d) Answers may vary. For example:
The instantaneous rate of change of the height of the soccer ball after 1 s is 5.2 m/s because as h gets smaller, the rate of change of the height gets closer to 5.2 m/s.

e) Answers may vary. For example:
At time t = 1 s, the ball is moving upwards at a speed of 5.2 m/s.

f)



Chapter 1 Section 2 **Question 15 Page 21**

a) first 10 min: $\dfrac{60(25-10)^2-60(25-0)^2}{10-0} = -2400$

The average rate of change of volume during the first 10 min is –2400 L/min.

last 10 min: $\dfrac{60(25-25)^2-60(25-15)^2}{25-15} = -600$

The average rate of change of volume during the last 10 min is –600 L/min.

Both rates of change are negative, but the rate of change of volume during the first 10 min is much more negative since the oil is draining more quickly during this time.

b) i) $$\frac{60(25-(5+h))^2-60(25-5)^2}{h}=\frac{60(h^2-40h+400)-60(400)}{h}$$
$$=60h-2400$$

The instantaneous rate of change of volume at t = 5 min is –2400 L/min.

ii) $$\frac{60(25-(10+h))^2-60(25-10)^2}{h}=\frac{60(h^2-30h+225)-60(225)}{h}$$
$$=60h-1800$$

The instantaneous rate of change of volume at t = 10 min is –1800 L/min.

iii) $$\frac{60(25-(15+h))^2-60(25-15)^2}{h}=\frac{60(h^2-20h+100)-60(100)}{h}$$
$$=60h-1200$$

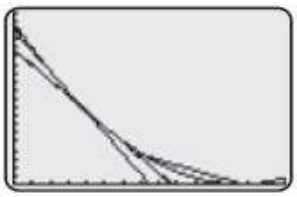
The instantaneous rate of change of volume at t = 15 min is –1200 L/min.

iv) $$\frac{60(25-(20+h))^2-60(25-20)^2}{h}=\frac{60(h^2-10h+25)-60(25)}{h}$$
$$=60h-600$$

The instantaneous rate of change of volume at t = 20 min is –600 L/min.

Answers may vary. For example:
The rate of change in the flow of oil may be slowing because of the shape of the tank and a lessening of pressure.

c)



Chapter 1 Section 2 **Question 16 Page 21**

a) S.A.: $\frac{4\pi(25^2)-4\pi(20^2)}{25-20}$ or approximately 565.5 cm^2/cm

V: $\frac{\frac{4}{3}\pi(25^3)-\frac{4}{3}\pi(20^3)}{25-20}$ or approximatley 6387.9 cm^3/cm

Since the surface area and volume are decreasing, the average rate of change of the surface area is –565.5 cm^2/cm and the average rate of change of the volume is –6387.9 cm^3/cm when r decreases from 25 cm to 20 cm.

b) S.A.: $\dfrac{4\pi(10+h)^2 - 4\pi(10)^2}{h} = \dfrac{40\pi h^2 + 80\pi h + 400\pi - 400\pi}{h}$

$= 40\pi h + 80\pi$

V: $\dfrac{\frac{4}{3}\pi(10+h)^3 - \frac{4}{3}\pi(10)^3}{h} = \dfrac{\frac{4}{3}\pi(h^3 + 3h^2(10) + 3h(10)^2 + 10^3) - \frac{4000}{3}\pi}{h}$

$= \dfrac{4}{3}\pi h^2 + 40\pi h + 400\pi$

Since the surface area and volume are decreasing, the instantaneous rate of change of the surface area is $-80\pi = -251.3$ cm^2/cm and the instantaneous rate of change of the volume is $-400\pi = -1256.6$ cm^3/cm when $r = 10$ cm.

c) Answers may vary. For example:
The rate of change of the volume is greater than the rate of change of the surface area because it is a larger quantity.

Chapter 1 Section 2 **Question 17 Page 21**

a) $\dfrac{80 - 5(3)^2 - (80 - 5(0)^2)}{3 - 0} = -15$

The average rate of change of the height of the branch from $t = 0$ s to $t = 3$ s is –15 m/s.
This value represents the average velocity over the time interval [0, 3].

b) $\dfrac{80 - 5(a+h)^2 - (80 - 5(a)^2)}{h} = \dfrac{80 - 5a^2 - 10ah - 5h^2 - 80 + 5a^2}{h}$

$= -10a - 5h$

i) $-10(0.5) - 5(0.001)$ or -5.005 m/s

ii) $-10(1) - 5(0.001)$ or -10.005 m/s

iii) $-10(1.5) - 5(0.001)$ or -15.005 m/s

iv) $-10(2) - 5(0.001)$ or -20.005 m/s

v) $-10(2.5) - 5(0.001)$ or -25.005 m/s

vi) $-10(3) - 5(0.001)$ or -30.005 m/s

c) Answers may vary. For example:
The values found in part b) represent the rate of change of the height of the branch at different moments in time during the time that it is falling.

Chapter 1 Section 2 **Question 18 Page 22**

a)

Tangent Point $(a, f(a))$	**Side Length Increment, *h***	**Second Point** $(a+h, f(a+h))$	**Slope of Secant** $\frac{f(a+h)-f(a)}{h}$
(4, −4)	1	**(5, −10)**	**−6**
(4, −4)	0.1	**(4.1, −4.51)**	**−5.1**
(4, −4)	0.01	**(4.01, −4.0501)**	**−5.01**
(4, −4)	0.001	**(4.001, −4.005 001)**	**−5.001**
(4, −4)	0.0001	**(4.0001, −4.000 500 01)**	**−5.0001**

b) Answers may vary. For example:
The values in the last column indicate that the slope of the tangent line to the function $f(x) = 3x - x^2$ at the point $x = 4$ is −5.

Chapter 1 Section 2 **Question 19 Page 22**

a) $$\frac{-(12)^2+16(12)+3-(-(4)^2+16(4)+3)}{12-4} = \frac{51-51}{8}$$
$$= 0$$

The average rate of change of the price is \$0 per year.

b) $$\frac{-(a+h)^2+16(a+h)+3-(-a^2+16a+3)}{h} = \frac{-a^2-2ah-h^2+16a+16h+3+a^2-16a-3}{h}$$
$$= -2a-h+16$$

i) $h = 0.1$: $-2(2) - 0.1 + 16$ or \$11.90 per year
$h = 0.01$: $-2(2) - 0.01 + 16$ or \$11.99 per year
$h = 0.001$: $-2(2) - 0.001 + 16$ or \$11.999 per year

The instantaneous rate of change of the price at $t = 2$ years is \$12 per year.

ii) $h = 0.1$: $-2(5) - 0.1 + 16$ or \$5.90 per year
$h = 0.01$: $-2(5) - 0.01 + 16$ or \$5.99 per year
$h = 0.001$: $-2(5) - 0.001 + 16$ or \$5.999 per year

The instantaneous rate of change of the price at $t = 5$ years is \$6 per year.

iii) $h = 0.1$: $-2(10) - 0.1 + 16$ or −\$4.10 per year
$h = 0.01$: $-2(10) - 0.01 + 16$ or −\$4.01 per year
$h = 0.001$: $-2(10) - 0.001 + 16$ or −\$4.011 per year

The instantaneous rate of change of the price at $t = 10$ years is −\$4.00 per year.

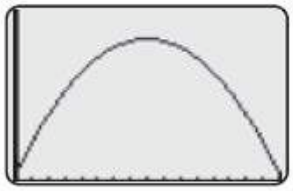
iv) $h = 0.1$: $-2(13) - 0.1 + 16$ or $-\$10.10$ per year
$h = 0.01$: $-2(13) - 0.01 + 16$ or $-\$10.01$ per year
$h = 0.001$: $-2(13) - 0.001 + 16$ or $-\$10.011$ per year

The instantaneous rate of change of the price at $t = 13$ years is $-\$10.00$ per year.

v) $h = 0.1$: $-2(15) - 0.1 + 16$ or $-\$14.10$ per year
$h = 0.01$: $-2(15) - 0.01 + 16$ or $-\$14.01$ per year
$h = 0.001$: $-2(15) - 0.001 + 16$ or $-\$14.011$ per year

The instantaneous rate of change of the price at $t = 15$ years is $-\$14.00$ per year.

c)



Chapter 1 Section 2 **Question 20 Page 22**

a) The slope of the secant PQ is $\frac{(2x - x^2) - 1}{x - 1} = \frac{-(x^2 - 2x + 1)}{x - 1}$

$$= \frac{-(x-1)^2}{x-1}$$

$$= 1 - x$$

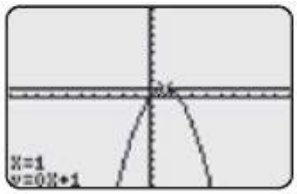
b) $x = 1.1$: $1 - 1.1 = -0.1$
$x = 1.01$: $1 - 1.01 = -0.01$
$x = 1.001$: $1 - 1.001 = -0.001$

$x = 0.9$: $1 - 0.9 = 0.1$
$x = 0.99$: $1 - 0.99 = 0.01$
$x = 0.999$: $1 - 0.999 = 0.001$

c) The slope of the tangent at P is 0.

d) The equation of the tangent is $y = 1$.

e)



Chapter 1 Section **Question 21 Page 22**

a) The slope of the secant lines at a are:

$a = 5.9$: $\dfrac{\sqrt{6}-\sqrt{5.9}}{6-5.9} \doteq 0.204\ 98$

$a = 5.99$: $\dfrac{\sqrt{6}-\sqrt{5.99}}{6-5.99} \doteq 0.204\ 21$

$a = 5.999$: $\dfrac{\sqrt{6}-\sqrt{5.999}}{6-5.999} \doteq 0.204\ 13$

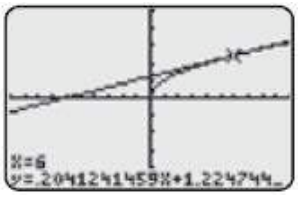
$a = 6.1$: $\dfrac{\sqrt{6}-\sqrt{6.1}}{6-6.1} \doteq 0.203\ 28$

$a = 6.01$: $\dfrac{\sqrt{6}-\sqrt{6.01}}{6-6.01} \doteq 0.204\ 04$

$a = 6.001$: $\dfrac{\sqrt{6}-\sqrt{6.001}}{6-6.001} \doteq 0.204\ 12$

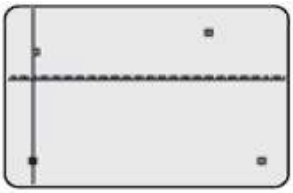
The instantaneous rate of change at $x = 6$ is 0.2.

b)



Chapter 1 Section 2 **Question 22 Page 22**

a)



b) Answers may vary. For example:
From 7:30 A.M. to 7:32 A.M., the rate of change of temperature was 13.50°C/min. From 7:32 A.M. to 9:00 A.M., the rate of change of temperature was 0.06°C/min. From 9:00 A.M. to 9:27 A.M., the rate of change of temperature was −1.19°C/min.

c) average rate of change: $\dfrac{-20-(-20)}{117} = 0$

The average rate of change of temperature over the entire period is 0°C/min.

d) Equation of the line of best fit for the data using quadratic regression:

$y = -0.0127x^2 + 1.383x - 8.098$

e) $\dfrac{f(a+h)-f(a)}{h} = \dfrac{-0.0254ah - 0.0127h^2 + 1.383h}{h}$

$= -0.0254a - 0.0127h + 1.383$

f) To find the instantaneous rates, let $h = 0$ in the above formula.

i) $a = 2$: $-0.0254(2) + 1.383$ or 1.3322°C/min

ii) $a = 30$: $-0.0254(30) + 1.383$ or 0.621°C/min

iii) $a = 75$: $-0.0254(75) + 1.383$ or -0.522°C/min

iv) $a = 105$: $-0.0254(105) + 1.383$ or -1.284°C/min

g) Answers may vary. For example:
The values from part f) best represent the impact of the Chinook wind. The average rate of change over the entire interval is 0°C/minute. The instantaneous rates of change in part f) approximate the instantaneous rates of change and show the rapid increases and decreases in temperature.

Chapter 1 Section 2 Question 23 Page 23

a) first 60 min: $\dfrac{V(60)-V(0)}{60-0} = \dfrac{0.1(150-60)^2 - 0.1(150-0)^2}{60}$

$= -24$

last 30 min: $\dfrac{V(150)-V(120)}{150-120} = \dfrac{0.1(150-150)^2 - 0.1(150-120)^2}{30}$

$= -3$

The average rate of change of the volume of the water during the first 60 min is –24 L/min and during the last 30 min is –3 L/min.

b) Find an expression for the slope of the secant line from $t = 75$ to $t = 75 + h$.

$$\frac{V(75+h)-V(75)}{h} = \frac{0.1(150-(75+h))^2-0.1(150-75)^2}{h}$$

$$= \frac{0.1(h^2-150h+5625)-562.5}{h}$$

$$= -15+0.1h$$

When h is close to zero, the instantaneous rate of change is –15 L/min.

Alternately, find the average rate of change over a small interval containing $t = 75$ min.

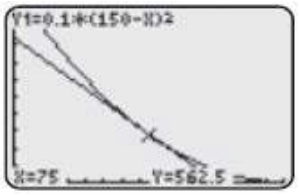
$$\frac{V(75.1)-V(74.9)}{75.1-74.9} = \frac{0.1(150-75.1)^2-0.1(150-74.9)^2}{0.2}$$

$$= \frac{561.001-564.001}{0.2}$$

$$= -15$$

The instantaneous rate of change when $t = 75$ min is –15 L/min.

c)



Chapter 1 Section 2 **Question 24 Page 23**

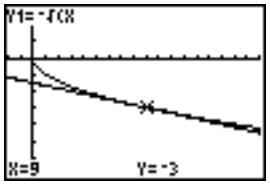
For part ii, use the interval $8.9 \le x \le 9.1$ to estimate the instantaneous rate of change at $x = 9$.

a) i) average rate of change: $\dfrac{-\sqrt{16}-\left(-\sqrt{9}\right)}{16-9} = -\dfrac{1}{7}$

ii) instantaneous rate of change: $\dfrac{-\sqrt{9.1}-(-\sqrt{8.9})}{9.1-8.9} \doteq -0.1667$

$$\doteq -\frac{1}{6}$$

iii)

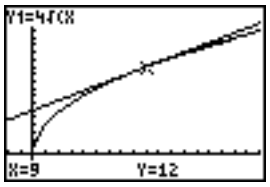


b) i) average rate of change: $\dfrac{4\sqrt{16}-4\sqrt{9}}{16-9}=\dfrac{4}{7}$

ii) instantaneous rate of change: $\dfrac{4\sqrt{9.1}-4\sqrt{8.9}}{9.1-8.9}\doteq 0.6667$

$\doteq\dfrac{2}{3}$

iii)

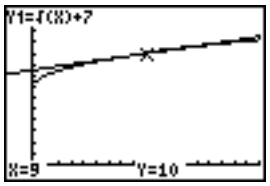


c) i) average rate of change: $\dfrac{\sqrt{16}+7-\left(\sqrt{9}+7\right)}{16-9}=\dfrac{1}{7}$

ii) instantaneous rate of change: $\dfrac{\sqrt{9.1}+7-\left(\sqrt{8.9}+7\right)}{9.1-8.9}\doteq 0.1667$

$\doteq\dfrac{1}{6}$

iii)

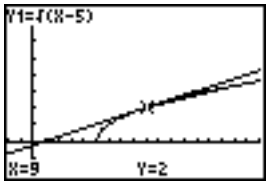


d) i) average rate of change: $\dfrac{\sqrt{16-5}-\sqrt{9-5}}{16-9}=\dfrac{\sqrt{11}-2}{7}$

ii) instantaneous rate of change: $\dfrac{\sqrt{9.1-5}-\sqrt{8.9-5}}{9.1-8.9}\doteq 0.2500$

$\doteq\dfrac{1}{4}$

iii)



Chapter 1 Section 2 **Question 25 Page 23**

For part ii), use the interval $6.9 \le x \le 7.1$ to estimate the instantaneous rate of change at $x = 7$.

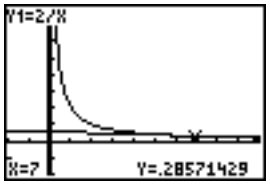
a) i) average rate of change: $\dfrac{\frac{2}{8}-\frac{2}{5}}{8-5} = \dfrac{\frac{10}{40}-\frac{16}{40}}{3}$

$$= -\frac{1}{20}$$

ii) instantaneous rate of change: $\dfrac{\frac{2}{7.1}-\frac{2}{6.9}}{7.1-6.9} = \dfrac{\frac{13.8-14.2}{48.99}}{0.2}$

$$\doteq -\frac{2}{49}$$

iii)



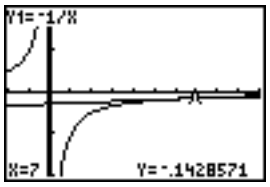
b) i) average rate of change: $\dfrac{-\frac{1}{8}-\left(-\frac{1}{5}\right)}{8-5} = \dfrac{-\frac{5}{40}+\frac{8}{40}}{3}$

$$= \frac{1}{40}$$

ii) instantaneous rate of change: $\dfrac{-\frac{1}{7.1}-\left(-\frac{1}{6.9}\right)}{7.1-6.9} = \dfrac{\frac{-6.9+7.1}{48.99}}{0.2}$

$$\doteq \frac{1}{49}$$

iii)



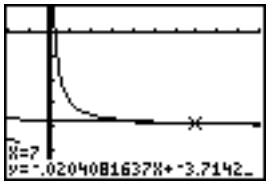
c) i) average rate of change: $\dfrac{\frac{1}{8}-4-\left(\frac{1}{5}-4\right)}{8-5}=\dfrac{\frac{5}{40}-\frac{8}{40}}{3}$

$$=-\frac{1}{40}$$

ii) instantaneous rate of change: $\dfrac{\frac{1}{7.1}-4-\left(\frac{1}{6.9}-4\right)}{7.1-6.9}=\dfrac{\frac{6.9-7.1}{48.99}}{0.2}$

$$\doteq-\frac{1}{49}$$

iii)



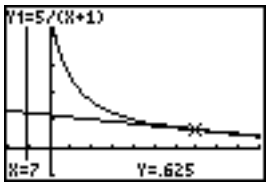
d) i) average rate of change: $\dfrac{\frac{5}{8+1}-\frac{5}{5+1}}{8-5}=\dfrac{\frac{30-45}{54}}{3}$

$$=-\frac{5}{54}$$

ii) instantaneous rate of change: $\dfrac{\frac{5}{7.1+1}-\frac{5}{6.9+1}}{7.1-6.9}=\dfrac{\frac{39.5-40.5}{63.99}}{0.2}$

$$\doteq-\frac{5}{64}$$

iii)



Chapter 1 Section 2 **Question 26 Page 23**

For part ii, use the interval $\frac{\pi}{4}-0.1\le\theta\le\frac{\pi}{4}+0.1$ to estimate the instantaneous rate of change at $\theta=\frac{\pi}{4}$.

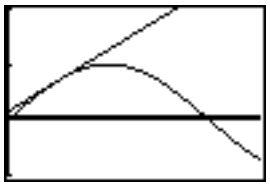
a) i) average rate of change: $\dfrac{\sin\frac{\pi}{3}-\sin\frac{\pi}{6}}{\frac{\pi}{3}-\frac{\pi}{6}}=\dfrac{\frac{\sqrt{3}}{2}-\frac{1}{2}}{\frac{\pi}{6}}$

$$=\frac{3\left(\sqrt{3}-1\right)}{\pi}$$

ii) instantaneous rate of change: $\dfrac{\sin\left(\frac{\pi}{4}+0.1\right)-\sin\left(\frac{\pi}{4}-0.1\right)}{\frac{\pi}{4}+0.1-\left(\frac{\pi}{4}-0.1\right)}\doteq 0.706$

$$\doteq\frac{1}{\sqrt{2}}$$

iii)



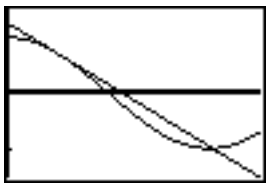
b) i) average rate of change: $\dfrac{\cos\frac{\pi}{3}-\cos\frac{\pi}{6}}{\frac{\pi}{3}-\frac{\pi}{6}}=\dfrac{\frac{1}{2}-\frac{\sqrt{3}}{2}}{\frac{\pi}{6}}$

$$=\frac{3\left(1-\sqrt{3}\right)}{\pi}$$

ii) instantaneous rate of change: $\dfrac{\cos\left(\frac{\pi}{4}+0.1\right)-\cos\left(\frac{\pi}{4}-0.1\right)}{\frac{\pi}{4}+0.1-\left(\frac{\pi}{4}-0.1\right)}\doteq -0.706$

$$\doteq-\frac{1}{\sqrt{2}}$$

iii)



c) i) average rate of change: $\dfrac{\tan\frac{\pi}{3}-\tan\frac{\pi}{6}}{\frac{\pi}{3}-\frac{\pi}{6}}=\dfrac{\sqrt{3}-\frac{1}{\sqrt{3}}}{\frac{\pi}{6}}$

$$=\frac{4\sqrt{3}}{\pi}$$

ii) instantaneous rate of change: $\dfrac{\tan\left(\frac{\pi}{4}+0.1\right)-\tan\left(\frac{\pi}{4}-0.1\right)}{\frac{\pi}{4}+0.1-\left(\frac{\pi}{4}-0.1\right)}\doteq 2.027$

$$\doteq 2$$

iii)

Chapter 1 Section 2 **Question 27 Page 23**

a) f is a constant function, so the average rate of change should be 0.

b) Answers may vary. For example:

Using the function $f(x)=4$, the average rate of change of f over the interval $a\le x\le b$ is $\dfrac{4-4}{b-a}=0$.

c) $\dfrac{f(b)-f(a)}{b-a}=\dfrac{c-c}{b-a}$

$$=\frac{0}{b-a}$$
$$=0$$

d) f is a constant function, so the instantaneous rate of change should be 0.

e) Answers may vary. For example:
A function $f(x)=c$, for any constant c, is a horizontal line. The slope or instantaneous rate of change at any point on the function must be 0.

Chapter 1 Section 2 **Question 28 Page 23**

a) f has slope m, so the average rate of change should be m.

b) Answers may vary. For example:
Using the function $f(x) = 2x + 1$, the average rate of change of f over the interval $1 \le x \le 3$ is

$$\frac{f(3) - f(1)}{3 - 1} = \frac{7 - 3}{2}$$
$$= 2$$

c) For any linear function $f(x) = mx + b$ over the interval $c \le x \le d$, the average rate of change is

$$\frac{f(d) - f(c)}{d - c} = \frac{m(d) + b - \left(m(c) + b\right)}{d - c}$$
$$= \frac{md - mc}{d - c}$$
$$= \frac{m(d - c)}{d - c}$$
$$= m$$

d) f has slope m, so the instantaneous rate of change should be m.

e) Answers may vary. For example:
A function $y = mx + b$, for any interval $a \le x \le b$ is a straight line with slope m. The slope or instantaneous rate of change at any point on the function must be m.

Chapter 1 Section 2 **Question 29 Page 23**

First, estimate the instantaneous rate of change of y at $x = -2$ over the interval $-2.01 \le x \le -1.99$.

$$\frac{f(-1.99) - f(-2.01)}{-1.99 - (-2.01)} \doteq \frac{-31.208 - (-32.808)}{0.02}$$
$$\doteq 80$$

The slope of the tangent is 80, so the slope perpendicular to that is $-\frac{1}{80}$.

When $x = -2$, $y = -32$.

Using the slope $m = -\frac{1}{80}$ and the point $(-2, -32)$, find b in the equation of the perpendicular line, $y = mx + b$.

$$-32 = -\frac{1}{80}(-2) + b$$
$$b = \frac{-1281}{40}$$

The equation of the line is $y = -\frac{1}{80}x - \frac{1281}{40}$.

Chapter 1 Section 2 **Question 30 Page 23**

$$\begin{aligned}
4-|x| &= \sqrt{x^2+4} \\
\left(4-|x|\right)^2 &= x^2+4 \\
16-8|x|+|x||x| &= x^2+4 \\
12-8|x|+x^2 &= x^2 \\
-8|x| &= -12 \\
|x| &= \frac{3}{2} \\
x &= \pm\frac{3}{2}
\end{aligned}$$

Chapter 1 Section 2 **Question 31 Page 23**

$$\begin{aligned}
(a-b)^2 &= 135 \\
a^2-2ab+b^2 &= 135 \\
\log_3(a)+\log_3(b) &= \log_3(ab) \\
\log_3(ab) &= 3 \\
ab &= 3^3 \\
ab &= 27 \\
a^2+b^2 &= 135+2(27) \\
&= 189 \\
(a+b)^2 &= a^2+2ab+b^2 \\
&= 189+54 \\
&= 243 \\
\log_3(a+b)^2 &= \log_3 243 \\
&= 5 \\
2\log_3(a+b) &= 5 \\
\log_3(a+b) &= \frac{5}{2} \\
(a+b) &= 3^{\frac{5}{2}} \\
&= \left(\sqrt{3}\right)^5 \\
\log_{\sqrt{3}}(a+b) &= 5
\end{aligned}$$

Chapter 1 Section 3 **Limits**

Chapter 1 Section 3 **Question 1 Page 29**

Explanations may vary. For example:

a) The limit does not exist.
The sequence continues alternating between 1 and –1, so the limit does not approach any value.

b) The limit is 6.
The sequence approaches 6, so the limit of the sequence is 6.

c) The limit is 0.
Every odd term is half the size of the previous odd term so they eventually will approach 0. Also, all the even terms are 0 so the sequence approaches 0.

d) The limit is 3.
The sequence approaches 3, so the limit of the sequence is 3.

e) The limit is –3.
The even terms get closer to –3 and all odd terms are equal to –3. Thus, the sequence approaches –3.

Chapter 1 Section 3 **Question 2 Page 30**

a) The limit is 4.

b) Explanations may vary. For example:
The limit does not exist.
The odd terms of the sequence begin to grow by the same amount so they will approach infinity. Similarly, the even terms will approach negative infinity, so the sequence does not approach any value.

c) The limit is 3.

Chapter 1 Section 3 **Question 3 Page 30**

$\lim\limits_{x \to 1} f(x)$ does not exist since $\lim\limits_{x \to 1^-} f(x) \neq \lim\limits_{x \to 1^+} f(x)$.

Chapter 1 Section 3 **Question 4 Page 30**

$\lim\limits_{x \to -3} f(x) = 1$

Chapter 1 Section 3 **Question 5 Page 30**

$\lim\limits_{x \to 2} f(x) = 0$

Chapter 1 Section 3 **Question 6 Page 30**

Explanations may vary. For example:

a) False; $\lim\limits_{x \to 3} f(x) \neq f(3)$

b) False; This function is discontinuous, but the right-hand and left-hand limits are equal, so $\lim\limits_{x \to 3} f(x) = 2$

c) True; $\lim\limits_{x \to 3^-} f(x) = 2$

d) False; $\lim\limits_{x \to 3^+} f(x) = 2$

e) False; $f(3) = -1$

Chapter 1 Section 3 **Question 7 Page 30**

Answers may vary. For example:

a) The graph of $y = h(x)$ is continuous at $x = -1$.

b) The graph of $y = h(x)$ is not continuous at $x = -1$ since $\lim\limits_{x \to -1} h(x)$ does not exist.

Chapter 1 Section 3 **Question 8 Page 30**

a) $t_1 = \frac{2}{3}$; $t_2 = \frac{2}{9}$; $t_3 = \frac{2}{27}$; $t_4 = \frac{2}{81}$; $t_5 = \frac{2}{243}$; $t_6 = \frac{2}{729}$

b) Answers may vary. For example:
The sequence is a convergent sequence. The values of the terms in the sequence get smaller and approach 0 as n gets large.

Chapter 1 Section 3 **Question 9 Page 30**

a) $t_1 = 0$; $t_2 = 4$; $t_3 = 18$; $t_4 = 48$; $t_5 = 100$; $t_6 = 180$

b) Answers may vary. For example:
The sequence is a divergent sequence. The values of the terms in the sequence get larger and approach ∞ as n gets large.

Chapter 1 Section 3 **Question 10 Page 30**

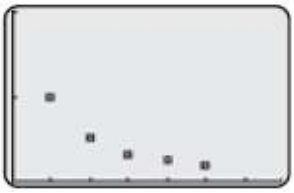
π

Chapter 1 Section 3 **Question 11 Page 30**

$\frac{1}{3}$

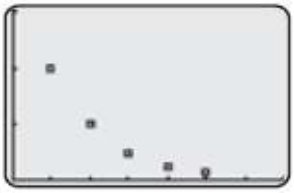
Chapter 1 Section 3 **Question 12 Page 30**

a) i) 0



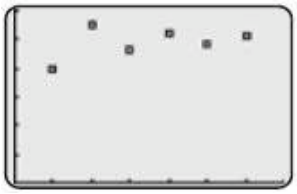
ii) $\lim\limits_{n\to\infty}\left(\frac{1}{n}\right)=0,\ n\in\mathbb{Z}$

b) i) 0



ii) $\lim\limits_{n\to\infty}2^{2-n}=0,\ n\in\mathbb{Z}$

c) i) 5



ii) $\lim\limits_{n\to\infty}\left(5+\frac{(-1)^n}{n}\right)=5,\ n\in\mathbb{Z}$

Chapter 1 Section 3 **Question 13 Page 31**

a) $\{x \mid x \in \mathbb{R}\}$

b) i) $\lim\limits_{x \to -2^-} (-x^3 + 4x) = 0$

ii) $\lim\limits_{x \to -2^+} (-x^3 + 4x) = 0$

iii) $\lim\limits_{x \to -2} (-x^3 + 4x) = 0$

iv) $f(-2) = 0$

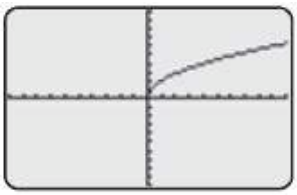
c) Answers may vary. For example:
Since $\lim\limits_{x \to -2} f(x) = f(-2)$, the graph is continuous at $x = -2$.

Chapter 1 Section 3 **Question 14 Page 31**

a) $\lim\limits_{l \to 0^+} 2\sqrt{l} = 0$

b) Answers may vary. For example:
$\lim\limits_{l \to 0^+} 2\sqrt{l}$ exists for $l > 0$; $\lim\limits_{l \to 0^-} 2\sqrt{l}$ does not exist, as the domain of the function $T(l) = 2\sqrt{l}$ is $l \geq 0$.

c)



Answers may vary. For example:
The domain of the function $T(l) = 2\sqrt{l}$ is $l \geq 0$.

Chapter 1 Section 3 **Question 15 Page 31**

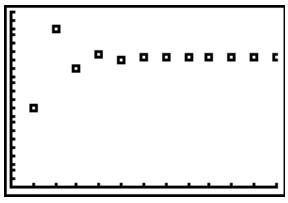
a)

n	f_n	$\frac{f_n}{f_{n-1}}$	**Decimal**
1	1		
2	1	$\frac{\mathbf{1}}{\mathbf{1}}$	1.000 000
3	2	$\frac{\mathbf{2}}{\mathbf{1}}$	2.000 000
4	3	$\frac{\mathbf{3}}{\mathbf{2}}$	1.500 000
5	**5**	$\frac{\mathbf{5}}{\mathbf{3}}$	1.666 667
6	**8**	$\frac{\mathbf{8}}{\mathbf{5}}$	1.600 000
7	**13**	$\frac{\mathbf{13}}{\mathbf{8}}$	1.625 000
8	**21**	$\frac{\mathbf{21}}{\mathbf{13}}$	1.615 385
9	**34**	$\frac{\mathbf{34}}{\mathbf{21}}$	1.619 048
10	**55**	$\frac{\mathbf{55}}{\mathbf{34}}$	1.617 647

b) The ratios approach the Golden Ratio, which is $\frac{1+\sqrt{5}}{2} \doteq 1.618$.

c) $\frac{89}{55} \doteq 1.618\,182$; $\frac{144}{89} \doteq 1.617\,978$; $\frac{233}{144} \doteq 1.618\,056$

d) Answers may vary. For example:
The values of points plotted on the graph approach the value 1.618.



e) $\lim\limits_{n\to\infty}\left(\frac{f_n}{f_{n-1}}\right) = \frac{1+\sqrt{5}}{2}$

Chapter 1 Section 3 **Question 16 Page 32**

a) i)

x	*y*
3.9	**0.316 23**
3.99	**0.1**
3.999	**0.031 623**
3.9999	**0.01**

$\lim\limits_{x \to 4^-} \sqrt{4 - x} = 0$

ii)

x	*y*
4.1	**no value**
4.01	**no value**
4.001	**no value**
4.0001	**no value**

$\lim\limits_{x \to 4^+} \sqrt{4 - x}$ does not exist

iii) Since $\lim\limits_{x \to 4^+} \sqrt{4 - x}$ does not exist, $\lim\limits_{x \to 4} \sqrt{4 - x}$ does not exist.

b) Answers may vary. For example:
The domain of the graph is $x \leq 4$, so this shows that the function has no limit at $x = 4$.

Chapter 1 Section 3 **Question 17 Page 32**

a) i)

x	*y*
–2.1	**no value**
–2.01	**no value**
–2.001	**no value**
–2.0001	**no value**

$\lim\limits_{x \to -2^-} \sqrt{x + 2}$ does not exist

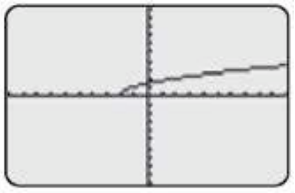
ii)

x	*y*
–1.9	**0.31623**
–1.99	**0.1**
–1.999	**0.031623**
–1.9999	**0.01**

$\lim\limits_{x \to -2^+} \sqrt{x+2} = 0$

iii) Since $\lim\limits_{x \to -2^-} \sqrt{x+2}$ does not exist, $\lim\limits_{x \to -2} \sqrt{x+2}$ does not exist.

b) Answers may vary. For example:
The domain of the graph is $x \geq -2$, so this shows that the function has no limit at $x = -2$.



Chapter 1 Section 3 **Question 18 Page 32**

a) i) \$2.00

ii) $\$1(1+0.5)^2 = \$2.25$

iii) $\$1(1+0.8333)^{12} = \$2.61$

iv) $\$1(1+0.002\,74)^{365} = \$2.71$

v) $\$1(1+0.000\,001\,9)^{525\,600} = \$2.72$

vi) $\$1(1+0.000\,000\,032)^{315\,360\,0} = \$2.72$

b) Answers may vary. For example:
The compounding periods are examples of the sequence for various values of *n*: 1, 12, 365, …

c) Answers may vary. For example:

The limit of the sequence is Euler's Number: $\lim\limits_{n \to \infty} \left(1+\frac{1}{n}\right)^n = 2.72$ for $n \in \mathbb{Z}$.

Chapter 1 Section 3 **Question 19 Page 32**

Answers may vary. For example:

Successive values of the sequence for the continued fraction are: 1, 2, $\frac{3}{2}$, $\frac{5}{3}$, $\frac{8}{5}$, $\frac{13}{8}$, …

From question 15, this is the Fibonacci sequence and $\lim_{n\to\infty}\left(1+\frac{1}{t_n}\right)=\frac{1+\sqrt{5}}{2}$.

The limit represents the golden ratio or the golden mean.

Chapter 1 Section 3 **Question 20 Page 32**

$$\begin{aligned}\sqrt{3\sqrt{3}} &= \left(3\cdot 3^{\frac{1}{2}}\right)^{\frac{1}{2}}\\ &= 3\frac{1}{4}\end{aligned}$$

Successive values of the sequence are:

$\sqrt{3}$, $\sqrt{3^{\frac{3}{2}}}$, $\sqrt{3^{\frac{7}{8}}}$, $\sqrt{3^{\frac{15}{8}}}$, $\sqrt{3^{\frac{31}{16}}}$, ... $\sqrt{3^{\frac{2^{n+1}-1}{2^n}}}$, ...

So, the *n*th term of the sequence is $3^{\frac{2^{n+1}-1}{2^{n+1}}}$.

The limit of the sequence is $\lim_{n\to\infty}\left(3^{\frac{2^{n+1}-1}{2^{n+1}}}\right)=3$.

Chapter 1 Section 3 **Question 21 Page 32**

Let *d* be the common difference between terms, where the terms are $t_1 = cat$, $t_2 = nut$, etc.

$$\begin{aligned}t_6 - t_1 &= bat - cat\\ &= 5d\end{aligned}$$

Since the last two digits are the same *bat – cat* must be a multiple of 100. But the digits "n" and "a" occur between *cat* and *bat*, so $bat - cat \geq 300$.

Therefore, since $bat - cat = 5d$ then $d = 60,\ 80,\ 100,\ 120,\ ...$

But *nut* and *not* are separated by *d*, so $d \leq 100$ means $d = 60$ or 80.

The digits A, U, O, C, R must contain all the odd digits or all the even digits.
For example, A = 3, then, for *d* = 60, (A, U, O, C, R) = (3, 9, 5, 1, 7) and for *d* = 80, (A, U, O, C, R) = (3, 1, 9, 7, 5).
So the digits A, U, O, C, R must be a permutation of all 5 odd digits or all 5 even digits.
Note that C, N, A, B must be consecutive digits, in increasing order. So A – C must be 2.
Also from the above, *d* must be 60. (if *d* = 80, then C- A = 4).

But $d = not - nut$ and $d = art - act$, so O – U = 6 and R – C = 6.
The only solutions are (A, U, O, C, R) = (4, 0, 6, 2, 8) and (A, U, O, C, R) = (5, 1, 7, 3, 9).

In the first case, N = 3 and B = 5
In the second case, N = 4, and B = 6.

Therefore, there are only two sequences that work:

24T, 30T, 36T, 42T, 48T, 54T, 60T
35T, 41T, 47T, 53T, 59T, 65T, 71T

In both cases, the final term of the sequence corresponds to OUT, which is the unique solution.

Chapter 1 Section 3 **Question 22 Page 32**

If (3, k) is a point on the curve $x^2y - y^2x = -30$, then

$$-30 = 9k - 3k^2$$
$$0 = -3k^2 + 9k + 30$$
$$0 = -3(k^2 - 3k - 10)$$
$$0 = -3(k - 5)(k + 2)$$

$k = 5$ or $k = -2$

Chapter 1 Section 4 **Limits and Continuity**

Chapter 1 Section 4 **Question 1 Page 44**

The function has a removable discontinuity at $x = 3$.

Chapter 1 Section 4 **Question 2 Page 44**

The function has an infinite discontinuity at $x = 6$.

Chapter 1 Section 4 **Question 3 Page 44**

a) The function is discontinuous at $x = -2$.

b) The function is discontinuous at $x = -3$ and $x = 1$.

Chapter 1 Section 4 **Question 4 Page 44**

a) Jump discontinuity at $x = -2$: $\lim\limits_{x \to -2^-} f(x) = 1$, $\lim\limits_{x \to -2^+} f(x) = 3$ so $\lim\limits_{x \to -2^-} f(x) \neq \lim\limits_{x \to -2^+} f(x)$.

b) Infinite discontinuity at $x = -6$ and $x = 2$:
$\lim\limits_{x \to -6^-} f(x) = \infty$, $\lim\limits_{x \to -6^+} f(x) = -\infty$; $\lim\limits_{x \to 2^-} f(x) = -\infty$, $\lim\limits_{x \to 2^+} f(x) = \infty$

c) Removable discontinuity at $x = -2$: $f(-2)$ does not exist

Chapter 1 Section 4 **Question 5 Page 44**

a) $\{x \in \mathbb{R} \mid x \neq -1\}$

b) i) $\lim\limits_{x \to -1^+} \dfrac{x^2}{x+1} = \infty$

ii) $\lim\limits_{x \to -1^-} \dfrac{x^2}{x+1} = -\infty$

iii) $\lim\limits_{x \to -1} \dfrac{x^2}{x+1}$ does not exist

iv) $f(-1)$ does not exist

c) Answers may vary. For example:
The graph is discontinuous since it has an infinite discontinuity at $x = -1$. The graph becomes large and negative as x approaches -1 from the left and becomes large and positive as x approaches -1 from the right.

Chapter 1 Section 4 **Question 6 Page 45**

a) $\{x \in \mathbb{R} \mid x \neq 0\}$

b) i) $\lim\limits_{x \to 0^+} \dfrac{x^2 + 4x - 2}{x^2} = -\infty$

ii) $\lim\limits_{x \to 0^-} \dfrac{x^2 + 4x - 2}{x^2} = -\infty$

iii) $\lim\limits_{x \to 0} \dfrac{x^2 + 4x - 2}{x^2} = -\infty$

iv) $f(0)$ does not exist

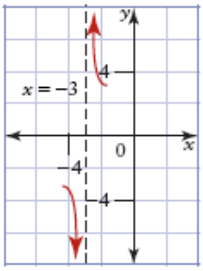
c) Answers may vary. For example:
The graph is discontinuous since it has an infinite discontinuity at $x = 0$. The graph becomes large and negative as x approaches –1 from the left and from the right, but is not continuous since $f(0)$ does not exist.

Chapter 1 Section 4 **Question 7 Page 45**

a) i) There is an infinite discontinuity at $x = -3$.

ii) $\lim\limits_{x \to -3^-} f(x) = -\infty$ and $\lim\limits_{x \to -3^+} f(x) = +\infty$

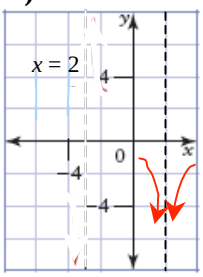
iii)



b) i) There is an infinite discontinuity at $x = 2$.

ii) $\lim\limits_{x \to 2^-} f(x) = +\infty$ and $\lim\limits_{x \to 2^+} f(x) = -\infty$

iii)



Chapter 1 Section 4 **Question 8 Page 45**

a) $\lim\limits_{x \to -2^-} g(x) = 3$

b) $\lim\limits_{x \to -2^+} g(x) = 3$

c) $\lim\limits_{x \to -2} g(x) = 3$

d) $g(-2) = 1$

e) $\lim\limits_{x \to 1^-} g(x) = 0$

f) $\lim\limits_{x \to 1^+} g(x) = 0$

g) $\lim\limits_{x \to 1} g(x) = 0$

h) $g(1) = 0$

Chapter 1 Section 4 **Question 9 Page 45**

a) $\lim\limits_{x \to -1^-} h(x) = 1$

b) $\lim\limits_{x \to -1^+} h(x) = -2$

c) $\lim\limits_{x \to -1} h(x)$ does not exist, since the left- and right-hand limits are not equal.

d) $h(-1) = 1$

e) $\lim\limits_{x \to 3^-} h(x) = 2$

f) $\lim\limits_{x \to 3^+} h(x) = 3$

g) $\lim\limits_{x\to 3} h(x)$ does not exist, since the left- and right-hand limits are not equal.

h) $h(3) = 2$

Chapter 1 Section 4 **Question 10 Page 45**

Refer to page 35 for the list of limit properties.

a) $\lim\limits_{x\to 6} 8 = 8$

Use property 1.

b) $$\begin{aligned}\lim_{x\to -3}\frac{x^2-9}{x+5} &= \frac{\left(\lim\limits_{x\to -3} x\right)^2 - \lim\limits_{x\to -3} 9}{\lim\limits_{x\to -3} x + \lim\limits_{x\to -3} 5}\\ &= \frac{(-3)^2-9}{-3+5}\\ &= 0\end{aligned}$$

Use properties 1, 2, 3, 4, 7 and 8.

c) $$\begin{aligned}\lim_{x\to -5}\frac{6x+2}{x+5} &= \frac{6\left(\lim\limits_{x\to -5} x\right) + \lim\limits_{x\to -5} 2}{\lim\limits_{x\to -5} x + \lim\limits_{x\to -5} 5}\\ &= \frac{6(-5)+2}{-5+5}\\ &= \frac{-28}{0}\end{aligned}$$

The limit does not exist. Use properties 1, 2, 3, 5 and 7.

d) $$\begin{aligned}\lim_{x\to 0}\sqrt[3]{8-x} &= \sqrt[3]{\lim\limits_{x\to 0} 8 - \lim\limits_{x\to 0} x}\\ &= \sqrt[3]{8-0}\\ &= 2\end{aligned}$$

Use properties 1, 2, 4 and 9.

Chapter 1 Section 4 **Question 11 Page 46**

a) $$\begin{aligned}\lim_{x\to 2}\frac{(2+x)^2-16}{x-2} &= \lim_{x\to 2}\frac{x^2+4x-12}{x-2}\\ &= \lim_{x\to 2}\frac{(x-2)(x+6)}{x-2}\\ &= \lim_{x\to 2} x+6\\ &= 8\end{aligned}$$

b) $\lim\limits_{x\to 6}\dfrac{(3-x)^2-9}{x-6}=\lim\limits_{x\to 6}\dfrac{x^2-6x}{x-6}$

$=\lim\limits_{x\to 6}\dfrac{x(x-6)}{x-6}$

$=\lim\limits_{x\to 6}x$

$=6$

c) $\lim\limits_{x\to 2}\dfrac{49-(5+x)^2}{x-2}=\lim\limits_{x\to 2}\dfrac{-(x^2+10x-24)}{x-2}$

$=\lim\limits_{x\to 2}\dfrac{-(x-2)(x+12)}{x-2}$

$=\lim\limits_{x\to 2}-(x+12)$

$=-14$

d) $\lim\limits_{x\to 3}\dfrac{\frac{1}{3}-\frac{1}{x}}{x-3}=\lim\limits_{x\to 3}\dfrac{\frac{x-3}{3x}}{x-3}$

$=\lim\limits_{x\to 3}\dfrac{1}{3x}$

$=\dfrac{1}{9}$

e) $\lim\limits_{x\to -2}\dfrac{x^4-16}{x+2}=\lim\limits_{x\to -2}\dfrac{(x^2+4)(x^2-4)}{x+2}$

$=\lim\limits_{x\to -2}\dfrac{(x^2+4)(x+2)(x-2)}{x+2}$

$=\lim\limits_{x\to -2}(x^2+4)(x-2)$

$=-32$

f) $\lim\limits_{x\to 1}\dfrac{x^2-1}{x^3-x^2-3x+3}=\lim\limits_{x\to 1}\dfrac{(x+1)(x-1)}{(x-1)(x^2-3)}$

$=\lim\limits_{x\to 1}\dfrac{(x+1)}{(x^2-3)}$

$=-1$

Chapter 1 Section 4 **Question 12 Page 46**

a) $\lim\limits_{x\to 0}\dfrac{\sqrt{9+x}-3}{x}=\lim\limits_{x\to 0}\dfrac{\left(\sqrt{9+x}-3\right)\left(\sqrt{9+x}+3\right)}{x\left(\sqrt{9+x}+3\right)}$

$=\lim\limits_{x\to 0}\dfrac{9+x-9}{x\left(\sqrt{9+x}+3\right)}$

$=\lim\limits_{x\to 0}\dfrac{1}{\sqrt{9+x}+3}$

$=\dfrac{1}{6}$

b) $\lim\limits_{x\to 25}\dfrac{5-\sqrt{x}}{x-25}=\lim\limits_{x\to 25}\dfrac{5-\sqrt{x}}{\left(\sqrt{x}-5\right)\left(\sqrt{x}+5\right)}$

$=\lim\limits_{x\to 25}\dfrac{-1}{\sqrt{x}+5}$

$=-\dfrac{1}{10}$

c) $\lim\limits_{x\to 4}\dfrac{x-4}{\sqrt{x}-2}=\lim\limits_{x\to 4}\dfrac{\left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)}{\sqrt{x}-2}$

$=\lim\limits_{x\to 4}\sqrt{x}+2$

$=4$

d) $\lim\limits_{x\to 0}\dfrac{\sqrt{1-x}-1}{3x}=\lim\limits_{x\to 0}\dfrac{\left(\sqrt{1-x}-1\right)\left(\sqrt{1-x}+1\right)}{3x\left(\sqrt{1-x}+1\right)}$

$=\lim\limits_{x\to 0}\dfrac{1-x-1}{3x\left(\sqrt{1-x}+1\right)}$

$=\lim\limits_{x\to 0}\dfrac{-1}{3\left(\sqrt{1-x}+1\right)}$

$=-\dfrac{1}{6}$

e) $\lim\limits_{x\to 0}\dfrac{\sqrt{3-x}-\sqrt{x+3}}{x}=\lim\limits_{x\to 0}\dfrac{\left(\sqrt{3-x}-\sqrt{x+3}\right)\left(\sqrt{3-x}+\sqrt{x+3}\right)}{x\left(\sqrt{3-x}+\sqrt{x+3}\right)}$

$=\lim\limits_{x\to 0}\dfrac{3-x-(x+3)}{x\left(\sqrt{3-x}+\sqrt{x+3}\right)}$

$=\lim\limits_{x\to 0}\dfrac{-2}{\sqrt{3-x}+\sqrt{x+3}}$

$=-\dfrac{1}{\sqrt{3}}$

Chapter 1 Section 4 **Question 13 Page 46**

a) $\lim\limits_{x\to -2}\dfrac{x^2-4}{x+2}=\lim\limits_{x\to -2}\dfrac{(x-2)(x+2)}{x+2}$

$=\lim\limits_{x\to -2}x-2$

$=-4$

b) $\lim\limits_{x\to 0}\dfrac{3x^2-x}{x+5x^2}=\lim\limits_{x\to 0}\dfrac{x(3x-1)}{x(1+5x)}$

$=\lim\limits_{x\to 0}\dfrac{3x-1}{1+5x}$

$=-1$

c) $\lim\limits_{x\to -3}\dfrac{x^2-9}{x+3}=\lim\limits_{x\to -3}\dfrac{(x+3)(x-3)}{x+3}$

$=\lim\limits_{x\to -3}x-3$

$=-6$

d) $\lim\limits_{x\to 0}\dfrac{-2x}{x^2-4x}=\lim\limits_{x\to 0}\dfrac{-2x}{x(x-4)}=\dfrac{1}{9}$

$=\lim\limits_{x\to 0}\dfrac{-2}{x-4}$

$=\dfrac{1}{2}$

e) $\lim\limits_{x\to -5}\dfrac{x^2+4x-5}{25-x^2}=\lim\limits_{x\to -5}\dfrac{(x-1)(x+5)}{(5+x)(5-x)}$

$=\lim\limits_{x\to -5}\dfrac{x-1}{5-x}$

$=\dfrac{-6}{10}$

$=-\dfrac{3}{5}$

f) $\lim\limits_{x\to 3}\dfrac{2x^2-5x-3}{x^2-x-6}=\lim\limits_{x\to 3}\dfrac{(x-3)(2x+1)}{(x-3)(x+2)}=-1$

$=\lim\limits_{x\to 3}\dfrac{2x+1}{x+2}$

$=\dfrac{7}{5}$

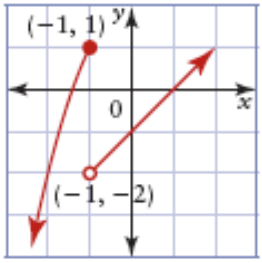
g) $\lim\limits_{x\to -4}\dfrac{3x^2+11x-4}{x^2+3x-4}=\lim\limits_{x\to -4}\dfrac{(x+4)(3x-1)}{(x+4)(x-1)}$

$=\lim\limits_{x\to -4}\dfrac{3x-1}{x-1}$

$=\dfrac{-13}{-5}$

$=\dfrac{13}{5}$

Chapter 1 Section 4 **Question 14 Page 46**



a) $\lim\limits_{x\to -1^+} f(x)=-2$

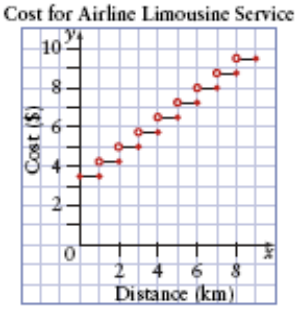
b) $\lim\limits_{x\to -1^-} f(x)=1$

c) $\lim\limits_{x\to -1} f(x)$ does not exist since the left and right limits are not equal.

d) $\lim\limits_{x\to 0} f(x)=-1$

Chapter 1 Section 4 **Question 15 Page 46**

a)

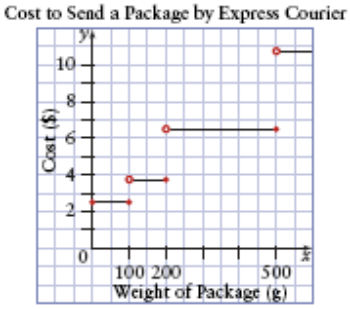


b) Answers may vary. For example: The function is a piecewise linear function.

c) The graph is discontinuous at the following distances: 1 km, 2 km, 3 km, 4 km… In general, the graph is discontinuous for all integer values of the distance. The graph has jump discontinuities.

Chapter 1 Section 4 **Question 16 Page 46**

a)

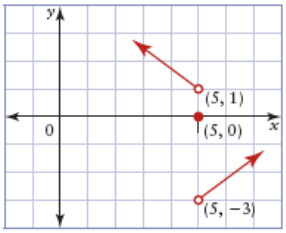


b) Answers may vary. For example:
The function is a piecewise linear function.

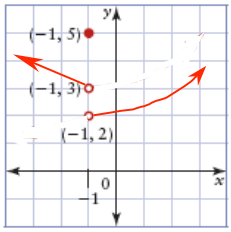
c) The graph is discontinuous for weights of 100 g, 200 g, and 500 g. The graph has jump discontinuities.

Chapter 1 Section 4 **Question 17 Page 46**

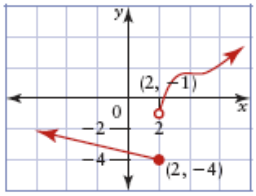
a)



b)

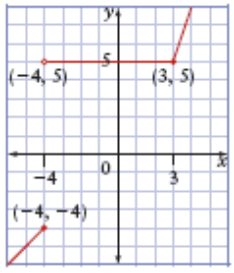


c)



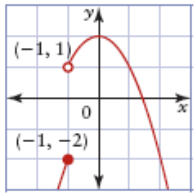
Chapter 1 Section 4 **Question 18 Page 46**

a)



Explanations may vary. For example:
The function is discontinuous at $x = -4$ since $\lim\limits_{x \to -4^-} f(x) \neq \lim\limits_{x \to -4^+} f(x)$.

b)



Explanations may vary. For example:
The function is discontinuous at $x = -1$ since $\lim\limits_{x \to -1^-} f(x) \neq \lim\limits_{x \to -1^+} f(x)$.

Chapter 1 Section 4 **Question 19 Page 47**

a) $\lim\limits_{x \to 0}[4f(x) - 1] = 4\lim\limits_{x \to 0} f(x) - 1$

$= 4(-1) - 1$

$= -5$

b) $\lim\limits_{x\to 0}[f(x)]^3 = [\lim\limits_{x\to 0} f(x)]^3$

$= (-1)^3$

$= -1$

c) $\lim\limits_{x\to 0}\dfrac{[f(x)]^2}{\sqrt{3-f(x)}} = \dfrac{[\lim\limits_{x\to 0} f(x)]^2}{\sqrt{3-\lim\limits_{x\to 0} f(x)}}$

$= \dfrac{(-1)^2}{\sqrt{3-(-1)}}$

$= \dfrac{1}{2}$

Chapter 1 Section 4 **Question 20 Page 47**

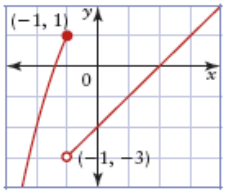
a) Answers may vary. For example:

$\lim\limits_{x\to -1^-} f(x) = a-(-1)^2 = a-1$; $\lim\limits_{x\to -1^+} f(x) = -1-b$

Setting $a - 1 = -1 - b$ gives $a = -b$. However, f is discontinuous when the left and right limits are not equal so when $a \neq -b$.

For instance, the values $a = 2$ and $b = 2$ will make the function discontinuous at $x = -1$.

b) Answers may vary. For example:



c) i) $\lim\limits_{x\to -1^+} f(x) = -1-2$

$= -3$

ii) $\lim\limits_{x\to -1^-} f(x) = 2-(-1)^2$

$= 1$

iii) $\lim\limits_{x\to -1} f(x)$ does not exist since $\lim\limits_{x\to -1^-} f(x) \neq \lim\limits_{x\to -1^+} f(x)$.

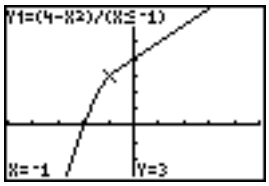
iv) $f(-1) = 2-(-1)^2$

$= 1$

Chapter 1 Section 4 **Question 21 Page 47**

a) Answers may vary. For example:
From the previous question, f was found to be discontinuous when $a \neq -b$, thus f is continuous when $a = -b$.
For instance, the values $a = 4$ and $b = -4$ will make the function continuous at $x = -1$.

b)



c) i) $\lim\limits_{x \to -1^+} f(x) = -1 + 4$

$= 3$

ii) $\lim\limits_{x \to -1^-} f(x) = 4 - (-1)^2$

$= 3$

iii) $\lim\limits_{x \to -1} f(x) = 3$

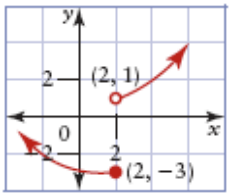
iv) $f(-1) = 4 - (-1)^2$

$= 3$

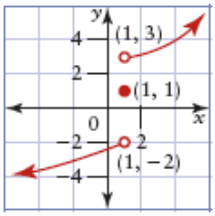
Chapter 1 Section 4 **Question 22 Page 47**

Answers may vary. For example:

a)

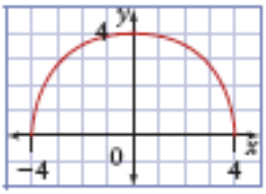


b)



Chapter 1 Section 4 **Question 23 Page 47**

a)



b) $\lim\limits_{x\to 4^-}\sqrt{16-x^2} = \sqrt{16-4^2}$

$= 0$

c) Answers may vary. For example:
The function is not defined in the real number system for x-values that are greater than 4.
Therefore the $\lim\limits_{x\to 4^+}\sqrt{16-x^2}$ does not exist.

d) $\lim\limits_{x\to 4}\sqrt{16-x^2}$ does not exist since $\lim\limits_{x\to 4^+}\sqrt{16-x^2}$ does not exist.

Chapter 1 Section 4 **Question 24 Page 47**

a) $\lim\limits_{x\to 8}\dfrac{2-\sqrt[3]{x}}{8-x} = \lim\limits_{x\to 8}\dfrac{2-\sqrt[3]{x}}{\left(2-\sqrt[3]{x}\right)\left(4+2\sqrt[3]{x}+\sqrt[3]{x^2}\right)}$

$= \lim\limits_{x\to 8}\dfrac{1}{(4+2\sqrt[3]{x}+\sqrt[3]{x^2})}$

$= \dfrac{1}{12}$

b) $\lim\limits_{x\to 2}\dfrac{x^5-32}{x-2} = \lim\limits_{x\to 2}\dfrac{(x-2)(x^4+2x^3+4x^2+8x+16)}{x-2}$

$= \lim\limits_{x\to 2} x^4+2x^3+4x^2+8x+16$

$= 80$

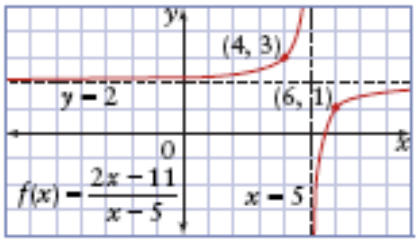
c) $\lim\limits_{x\to 2}\dfrac{6x^3-13x^2+x+2}{x-2} = \lim\limits_{x\to 2}\dfrac{(x-2)(2x-1)(3x+1)}{x-2}$

$= \lim\limits_{x\to 2}(2x-1)(3x+1)$

$= 21$

Chapter 1 Section 4 **Question 25 Page 47**

a)



b) Answers may vary. For example:

A possible equation for this function is $f(x) = \frac{2x-11}{x-5}$. The vertical asymptote is $x = 5$, the horizontal asymptote is $y = 2$, $f(4) = 3$, and $f(6) = 1$.

The function satisfies all of the required conditions when graphed using a graphing calculator.

Chapter 1 Section 4 **Question 26 Page 47**

$$\begin{aligned} 6^x(6-1) &= 3^x(3^4-1) \\ 6^x(5) &= 3^x(80) \\ 6^x &= 3^x(16) \\ 2^x3^x &= 3^x(16) \\ 2^x &= 16 \\ x &= 4 \end{aligned}$$

Chapter 1 Section 4 **Question 27 Page 47**

$$(6x^2-3x)(2x^2-13x-7)(3x+5) = (2x^2-15x+7)(6x^2+13x+5)$$

$$\text{L.S.} = 3x(2x-1)(2x+1)(x-7)(3x+5)$$
$$\text{R.S.} = (2x-1)(2x+1)(x-7)(3x+5)$$

Both sides of the equations are zero when $x = \frac{1}{2}$, $x = -\frac{1}{2}$, $x = 7$, and $x = -\frac{5}{3}$.

When the factors are cancelled, the equation remaining is $3x = 1$, so $x = \frac{1}{3}$ is also a solution.

Chapter 1 Section 4 **Question 28 Page 47**

$$\log_{(2\cos x)} 6 + \log_{(2\cos x)} \sin x = 2$$

$$\log_{(2\cos x)} 6\sin x = 2$$

$$(2\cos x)^2 = 6\sin x$$

$$4\cos^2 x = 6\sin x$$

$$4(1-\sin^2 x) = 6\sin x$$

$$4\sin^2 x + 6\sin x - 4 = 0$$

$$(2\sin x - 1)(2\sin x + 4) = 0$$

$$\sin x = \frac{1}{2} \quad (\sin x \neq -2)$$

$$x = \frac{\pi}{6} \text{ or } 30^\circ$$

Chapter 1 Section 5 **Introduction to Derivatives**

Chapter 1 Section 5 **Question 1 Page 58**

a) The derivative is C, since y is a quadratic function and the slope goes from negative to positive.

b) The derivative is A, since y is a cubic function and the slope is not negative anywhere.

c) The derivative is B, since y is a linear function and the slope of the graph is negative and constant.

Chapter 1 Section 5 **Question 2 Page 58**

a) $f'(x) = 3x^2$

b) i) $f'(-6) = 3(-6)^2$
$= 108$

ii) $f'(-0.5) = 3(-0.5)^2$
$= 0.75$

iii) $f'\left(\frac{2}{3}\right) = 3\left(\frac{4}{9}\right)$
$= \frac{4}{3}$

iv) $f'(2) = 3(2)^2$
$= 12$

c) i) When $x = -6$, $f(-6) = -216$
Use the point $(-6, -216)$ and $m = 108$ in the equation of the tangent $y = mx + b$ to find b.
$-216 = 108(-6) + b$
$b = 432$

The equation of the tangent is $y = 108x + 432$.

ii) When $x = -0.5$, $f(-0.5) = -0.125$
Use the point $(-0.5, -0.125)$ and $m = 0.75$ in the equation of the tangent $y = mx + b$ to find b.
$-0.125 = 0.75(-0.5) + b$
$b = 0.25$

The equation of the tangent is $y = 0.75x + 0.25$.

iii) When $x = \frac{2}{3}$, $f\left(\frac{2}{3}\right) = \frac{8}{27}$

Use the point $\left(\frac{2}{3}, \frac{8}{27}\right)$ and $m = \frac{4}{3}$ in the equation of the tangent $y = mx + b$ to find b.

$$\frac{8}{27} = \frac{4}{3}\left(\frac{2}{3}\right) + b$$

$$b = -\frac{16}{27}$$

The equation of the tangent is $y = \frac{4}{3}x - \frac{16}{27}$.

iv) When $x = 2$, $f(2) = 8$

Use the point (2, 8) and $m = 12$ in the equation of the tangent $y = mx + b$ to find b.

$8 = 12(2) + b$

$b = -16$

The equation of the tangent is $y = 12x - 16$.

Chapter 1 Section 5 **Question 3 Page 58**

Answers may vary. For example:
The function is not differentiable at the given x-value, possibly because it is discontinuous or the function makes an abrupt change.

Chapter 1 Section 5 **Question 4 Page 58**

a) $f'(x) = 1$

b) i) $f'(-6) = 1$

ii) $f'(-0.5) = 1$

iii) $f'\left(\frac{2}{3}\right) = 1$

iv) $f'(2) = 1$

Chapter 1 Section 5 **Question 5 Page 58**

a) $f(x) = 3x$

b) $f(x) = x^2$

c) $f(x) = 4x^3$

d) $f(x) = -6x^2$

e) $f(x) = \frac{5}{x}$

f) $f(x) = \sqrt{x}$

Chapter 1 Section 5 **Question 6 Page 58**

a) $f'(x) = -\frac{1}{x^2}$

b) i) $f'(-6) = -\frac{1}{36}$

ii) $f'(-0.5) = -4$

iii) $f'\left(\frac{2}{3}\right) = -\frac{9}{4}$

iv) $f'(2) = -\frac{1}{4}$

c) i) When $x = -6$, $f(-6) = -\frac{1}{6}$.

Use the point $\left(-6, -\frac{1}{6}\right)$ and $m = -\frac{1}{36}$ in the equation of the tangent $y = mx + b$ to find b.

$-\frac{1}{6} = -\frac{1}{36}(-6) + b$

$b = -\frac{1}{3}$

The equation of the tangent is $y = -\frac{1}{36}x - \frac{1}{3}$.

ii) When $x = -0.5$, $f(-0.5) = -2$.

Use the point $(-0.5, -2)$ and $m = -4$ in the equation of the tangent $y = mx + b$ to find b.

$-2 = -4(-0.5) + b$

$b = -4$

The equation of the tangent is $y = -4x - 4$.

iii) When $x = \frac{2}{3}$, $f\left(\frac{2}{3}\right) = \frac{3}{2}$.

Use the point $\left(\frac{2}{3}, \frac{3}{2}\right)$ and $m = -\frac{9}{4}$ in the equation of the tangent $y = mx + b$ to find b.

$\frac{3}{2} = -\frac{9}{4}\left(\frac{2}{3}\right) + b$

$b = 3$

The equation of the tangent is $y = -\frac{9}{4}x + 3$.

iv) When $x = 2$, $f(2) = \frac{1}{2}$.

Use the point $\left(2, \frac{1}{2}\right)$ and $m = -\frac{1}{4}$ in the equation of the tangent $y = mx + b$ to find b.

$\frac{1}{2} = -\frac{1}{4}(2) + b$

$b = 1$

The equation of the tangent is $y = -\frac{1}{4}x + 1$.

Chapter 1 Section 5 **Question 7 Page 59**

Explanations may vary. For example:

a) $x \in (-\infty, -1)$ or $(-1, \infty)$

The function is differentiable for all values of x except at $x = -1$ since there is an abrupt change in slope at $x = -1$.

b) $x \in (-\infty, \infty)$

The function is differentiable for all values of x since there are no discontinuities or abrupt changes.

c) $x \in (3, \infty)$

The function is differentiable for all values of $x > 3$ since the function is not defined for $x < 3$ and $x = 3$ is an endpoint.

d) $x \in (-\infty, -1)$ or $(-1, \infty)$
The function is differentiable for all values of x except at $x = -1$ since this point is an infinite discontinuity.

Chapter 1 Section 5 **Question 8 Page 59**

Explanations may vary.

a) Linear; The graph of the derivative of a linear function with non-zero slope is a non-zero constant function, in this case 2.

b) Cubic; The graph of the derivative of a cubic function is a quadratic function.

c) Constant; The graph of the derivative of a constant is zero.

d) Quadratic; The graph of the derivative of a quadratic function is a linear function.

Chapter 1 Section 5 **Question 9 Page 59**

a)
$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{(x+h)^2-x^2}{h}\\ &= \lim_{h\to 0}\frac{2xh+h^2}{h}\\ &= \lim_{h\to 0}(2x+h)\\ &= 2x\end{aligned}$$

b) domain for y: $\{x \mid x \in \mathbb{R}\}$; domain for $\frac{dy}{dx}$: $\{x \mid x \in \mathbb{R}\}$

c) Answers may vary. For example: The derivative at a point x represents the slope of the tangent to the original function at x.

Chapter 1 Section 5 **Question 10 Page 59**

a) i)
$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{-3(x+h)^2+3x^2}{h}\\ &= \lim_{h\to 0}\frac{-6xh-3h^2}{h}\\ &= \lim_{h\to 0}(-6x-3h)\\ &= -6x\end{aligned}$$

ii) $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{4(x+h)^2 - 4x^2}{h}$

$= \lim_{h \to 0} \frac{8xh + 4h^2}{h}$

$= \lim_{h \to 0} (8x + 4h)$

$= 8x$

b) Answers may vary. For example:

If $y = ax^2$, where a is a constant, then the derivative of y is $\frac{dy}{dx} = 2ax$.

c) i) $\frac{dy}{dx} = -4x$

ii) $\frac{dy}{dx} = 10x$

d) i) $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{-2(x+h)^2 + 2x^2}{h}$

$= \lim_{h \to 0} \frac{-4xh - 2h^2}{h}$

$= \lim_{h \to 0} (-4x - 2h)$

$= -4x$

ii) $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{5(x+h)^2 - 5x^2}{h}$

$= \lim_{h \to 0} \frac{10xh + 5h^2}{h}$

$= \lim_{h \to 0} (10x + 5h)$

$= 10x$

Chapter 1 Section 5 **Question 11 Page 59**

a) $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{-4 - (-4)}{h}$

$= \lim_{h \to 0} \frac{0}{h}$

$= 0$

b) Answers may vary. For example:
Yes. The slope of a horizontal line is always 0.

c) $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{c - c}{h}$

$= \lim_{h \to 0} \frac{0}{h}$

$= 0$

Chapter 1 Section 5 **Question 12 Page 59**

a) $(x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

b) i) $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{2(x^3 + 3x^2h + 3xh^2 + h^3) - 2x^3}{h}$

$= \lim_{h \to 0} \frac{6x^2h + 6xh^2 + 2h^3}{h}$

$= \lim_{h \to 0} (6x^2 + 6xh + 2h^2)$

$= 6x^2$

ii) $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{-[x^3 + 3x^2h + 3xh^2 + h^3] + x^3}{h}$

$= \lim_{h \to 0} \frac{-3x^2h - 3xh^2 - h^3}{h}$

$= \lim_{h \to 0} (-3x^2 - 3xh - h^2)$

$= -3x^2$

Chapter 1 Section 5 **Question 13 Page 59**

a) Answers may vary. For example:

If $y = ax^3$, where a is a constant, then the derivative of y is $\frac{dy}{dx} = 3ax^2$.

b) i) $\frac{dy}{dx} = -12x^2$

ii) $\frac{dy}{dx} = \frac{3}{2}x^2$

c) i)
$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{-4[x^3+3x^2h+3xh^2+h^3]+4x^3}{h}\\ &= \lim_{h\to 0}\frac{-12x^2h-12xh^2-4h^3}{h}\\ &= \lim_{h\to 0}(-12x^2-12xh-4h^2)\\ &= -12x^2\end{aligned}$$

ii)
$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{\frac{1}{2}[x^3+3x^2h+3xh^2+h^3]-\frac{1}{2}x^3}{h}\\ &= \lim_{h\to 0}\frac{\frac{3}{2}x^2h+\frac{3}{2}xh^2+\frac{1}{2}h^3}{h}\\ &= \lim_{h\to 0}\left(\frac{3}{2}x^2+\frac{3}{2}xh+\frac{1}{2}h^2\right)\\ &= \frac{3}{2}x^2\end{aligned}$$

Chapter 1 Section 5 **Question 14 Page 60**

a) $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{8(x+h) - 8x}{h}$

$= \lim_{h \to 0} \frac{8h}{h}$

$= \lim_{h \to 0} (8)$

$= 8$

b) $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{3(x+h)^2 - 2(x+h) - (3x^2 - 2x)}{h}$

$= \lim_{h \to 0} \frac{6xh + 3h^2 - 2h}{h}$

$= \lim_{h \to 0} (6x + 3h - 2)$

$= 6x - 2$

c) $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{7 - (x+h)^2 - (7 - x^2)}{h}$

$= \lim_{h \to 0} \frac{-2xh - h^2}{h}$

$= \lim_{h \to 0} (-2x - h)$

$= -2x$

d) $y = x(4x + 5)$

$= 4x^2 + 5x$

$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{4(x+h)^2 + 5(x+h) - (4x^2 + 5x)}{h}$

$= \lim_{h \to 0} \frac{8xh + 4h^2 + 5h}{h}$

$= \lim_{h \to 0} (8x + 4h + 5)$

$= 8x + 5$

e) $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{[2(x+h)-1]^2 - (2x-1)^2}{h}$

$= \lim\limits_{h\to 0} \dfrac{4x^2 + 8xh + 4h^2 - 4x - 4h + 1 - 4x^2 + 4x - 1}{h}$

$= \lim\limits_{h\to 0} \dfrac{8xh + 4h^2 - 4h}{h}$

$= \lim\limits_{h\to 0} (8x + 4h - 4)$

$= 8x - 4$

Chapter 1 Section 5 **Question 15 Page 60**

a) $(x + h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$

b) i) $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{(x^4 + 4hx^3 + 6h^2x^2 + 4h^3x + h^4) - x^4}{h}$

$= \lim\limits_{h\to 0} \dfrac{4hx^3 + 6h^2x^2 + 4h^3x + h^4}{h}$

$= \lim\limits_{h\to 0} (4x^3 + 6hx^2 + 4h^2x + h^3)$

$= 4x^3$

ii) $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{2(x^4 + 4hx^3 + 6h^2x^2 + 4h^3x + h^4) - 2x^4}{h}$

$= \lim\limits_{h\to 0} \dfrac{8hx^3 + 12h^2x^2 + 8h^3x + 2h^4}{h}$

$= \lim\limits_{h\to 0} (8x^3 + 12hx^2 + 8h^2x + 2h^3)$

$= 8x^3$

iii) $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{3(x^4 + 4hx^3 + 6h^2x^2 + 4h^3x + h^4) - 3x^4}{h}$

$= \lim\limits_{h\to 0} \dfrac{12hx^3 + 18h^2x^2 + 12h^3x + 3h^4}{h}$

$= \lim\limits_{h\to 0} (12x^3 + 18hx^2 + 12h^2x + 3h^3)$

$= 12x^3$

c) Answers may vary. For example:

If $y = ax^4$, where a is a constant, then the derivative of y is $\frac{dy}{dx} = 4ax^3$.

d) i) $\frac{dy}{dx} = -4x^3$

ii) $\frac{dy}{dx} = 2x^3$

e) i)
$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{-(x^4+4hx^3+6h^2x^2+4h^3x+h^4)+x^4}{h}\\ &= \lim_{h\to 0}\frac{-4hx^3-6h^2x^2-4h^3x-h^4}{h}\\ &= \lim_{h\to 0}(-4x^3-6hx^2-4h^2x-h^3)\\ &= -4x^3\end{aligned}$$

ii)
$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{\frac{1}{2}(x^4+4hx^3+6h^2x^2+4h^3x+h^4)-\frac{1}{2}x^4}{h}\\ &= \lim_{h\to 0}\frac{\frac{1}{2}(4hx^3+6h^2x^2+4h^3x+h^4)}{h}\\ &= \lim_{h\to 0}(2x^3+3hx^2+2h^2x+\frac{1}{2}h^3)\\ &= 2x^3\end{aligned}$$

Chapter 1 Section 5 **Question 16 Page 60**

a)
$$\begin{aligned}H'(t) &= \lim_{h\to 0}\frac{H(t+h)-H(t)}{h}\\ &= \lim_{h\to 0}\frac{-4.9(t+h)^2+3.5(t+h)+1-(-4.9t^2+3.5t+1)}{h}\\ &= \lim_{h\to 0}\frac{-4.9t^2-9.8ht-4.9h^2+3.5t+3.5h+1+4.9t^2-3.5t-1}{h}\\ &= \lim_{h\to 0}\frac{-9.8ht-4.9h^2+3.5h}{h}\\ &= \lim_{h\to 0}(-9.8t-4.9h+3.5)\\ &= -9.8t+3.5\end{aligned}$$

b) $H'(0.5) = -9.8(0.5) + 3.5$
$= -1.4$

The rate of change of the height of the soccer ball at 0.5 s is –1.4 m/s.

c) The ball stops when $H'(t) = 0$.
$0 = -9.8t + 3.5$
$t \doteq 0.357$

The ball momentarily stops when $t \doteq 0.357$ s.

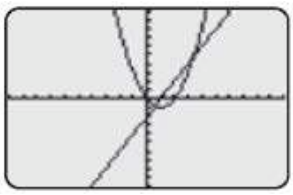
$H(0.357) = -4.9(0.357)^2 + 3.5(0.357) + 1$
$\doteq 1.625$

The height of the ball at this time is 1.625 m.

Chapter 1 Section 5 **Question 17 Page 60**

a) $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
$= \lim_{h \to 0} \frac{(x+h)^2 - 2(x+h) - (x^2 - 2x)}{h}$
$= \lim_{h \to 0} \frac{2xh + h^2 - 2h}{h}$
$= \lim_{h \to 0} (2x + h - 2)$
$= 2x - 2$

b)



c) The slope of the tangent to the function at $x = -3$ is $\left.\frac{dy}{dx}\right|_{x=-3}$

$$\left.\frac{dy}{dx}\right|_{x=-3} = 2(-3) - 2$$
$$= -8$$

When $x = -3$, $y = 15$.
Use the point $(-3, 15)$ and $m = -8$ in the equation of the tangent $y = mx + b$ to find b.
$15 = -8(-3) + b$
$b = -9$

The equation of the tangent is $y = -8x - 9$.

d)

Chapter 1 Section 5 **Question 18 Page 60**

a) i)
$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{2x - 2(x+h)}{x(x+h)}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{-2h}{x(x+h)}}{h}$$
$$= \lim_{h\to 0} \frac{-2}{x^2 + xh}$$
$$= -\frac{2}{x^2}$$

ii) $\frac{dy}{dx} = \lim_{h \to 0} \frac{\frac{-1}{x+h} + \frac{1}{x}}{h}$

$= \lim_{h \to 0} \frac{\frac{-x + (x+h)}{x(x+h)}}{h}$

$= \lim_{h \to 0} \frac{\frac{h}{x(x+h)}}{h}$

$= \lim_{h \to 0} \frac{1}{x^2 + xh}$

$= \frac{1}{x^2}$

iii) $\frac{dy}{dx} = \lim_{h \to 0} \frac{\frac{3}{x+h} - \frac{3}{x}}{h}$

$= \lim_{h \to 0} \frac{\frac{3x - 3(x+h)}{x(x+h)}}{h}$

$= \lim_{h \to 0} \frac{\frac{-3h}{x(x+h)}}{h}$

$= \lim_{h \to 0} \frac{-3}{x^2 + xh}$

$= -\frac{3}{x^2}$

iv) $\frac{dy}{dx} = \lim_{h \to 0} \frac{-\frac{4}{3(x+h)} + \frac{4}{3x}}{h}$

$= \lim_{h \to 0} \frac{\frac{-4x + 4(x+h)}{3x(x+h)}}{h}$

$= \lim_{h \to 0} \frac{\frac{4h}{3x(x+h)}}{h}$

$= \lim_{h \to 0} \frac{4}{3x^2 + 3xh}$

$= \frac{4}{3x^2}$

b) Answers may vary. For example:

If $y = \frac{a}{bx}$, where a and b are constants, then the derivative of y is $\frac{dy}{dx} = -\frac{a}{bx^2}$.

c) i)–iv)

domain of y: $\{x \in \mathbb{R} \mid x \neq 0\}$; domain of $\frac{dy}{dx}$: $\{x \in \mathbb{R} \mid x \neq 0\}$

Chapter 1 Section 5 **Question 19 Page 60**

a) i) $\frac{dy}{dx} = -\frac{5}{x^2}$

ii) $\frac{dy}{dx} = \frac{3}{5x^2}$

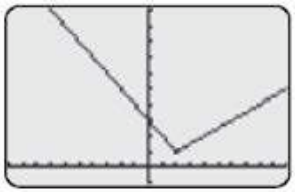
b) i)
$$\begin{aligned}\frac{dy}{dx} &= \lim_{h \to 0} \frac{\frac{5}{x+h} - \frac{5}{x}}{h} \\ &= \lim_{h \to 0} \frac{\frac{5x - 5(x+h)}{x(x+h)}}{h} \\ &= \lim_{h \to 0} \frac{\frac{-5h}{x(x+h)}}{h} \\ &= \lim_{h \to 0} \frac{-5}{x^2 + xh} \\ &= -\frac{5}{x^2}\end{aligned}$$

ii)
$$\begin{aligned}\frac{dy}{dx} &= \lim_{h \to 0} \frac{\frac{-3}{5(x+h)} + \frac{3}{5x}}{h} \\ &= \lim_{h \to 0} \frac{\frac{-3x + 3(x+h)}{5x(x+h)}}{h} \\ &= \lim_{h \to 0} \frac{\frac{3h}{5x(x+h)}}{h} \\ &= \lim_{h \to 0} \frac{3}{5x^2 + 5xh} \\ &= \frac{3}{5x^2}\end{aligned}$$

Chapter 1 Section 5 **Question 20 Page 60**

Answers may vary. For example:

a) A piecewise function defined by $y = -x + 3$ if $x \leq 2$ and $y = 0.5x$ if $x > 2$



b)

c) For $y = -x + 3$ if $x \leq 2$:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{-(x+h) + 3 + x - 3}{h}$$
$$= \lim_{h \to 0} \frac{-h}{h}$$
$$= -1$$

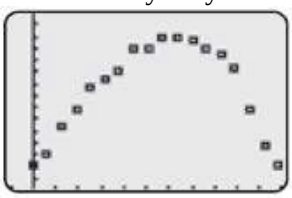
For $y = 0.5x$ if $x > 2$:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{0.5(x+h) - 0.5x}{h}$$
$$= \lim_{h \to 0} \frac{0.5h}{h}$$
$$= 0.5$$

The derivatives of y as x approaches 2 from the left and right sides are not equal, so y is not differentiable at $x = 2$.

Chapter 1 Section 5 **Question 21 Page 60**

a) Answers may vary.



b) Answers may vary. For example:
Use the QuadReg feature on the graphing calculator, which gives:
$y = -1499x^2 + 26\,808x + 356\,532$

c) $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{-1499(x+h)^2 + 26\,808(x+h) + 356\,532 - (-1499x^2 + 26\,808x + 356\,532)}{h}$

$= \lim\limits_{h\to 0} \dfrac{-2998xh + 26\,808h - 1499h^2}{h}$

$= \lim\limits_{h\to 0} (-2998x + 26\,808 - 1499h)$

$= -2998x + 26\,808$

d) i) $\left.\dfrac{dy}{dx}\right|_{x=3} = -2998(3) + 26\,808$

$= 17\,814$

The instantaneous rate of change after 3 years is 17 814 births per year.

ii) $\left.\dfrac{dy}{dx}\right|_{x=7} = -2998(7) + 26\,808$

$= 5822$

The instantaneous rate of change after 7 years is 5822 births per year.

iii) $\left.\dfrac{dy}{dx}\right|_{x=10} = -2998(10) + 26\,808$

$= -3172$

The instantaneous rate of change after 10 years is –3172 births per year.

iv) $\left.\dfrac{dy}{dx}\right|_{x=13} = -2998(13) + 26\,808$

$= -12\,166$

The instantaneous rate of change after 13 years is –12 166 births per year.

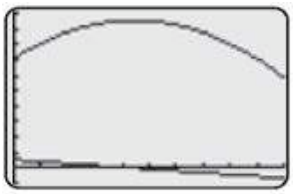
v) $\left.\frac{dy}{dx}\right|_{x=16} = -2998(16) + 26\,808$

$= -21\,160$

The instantaneous rate of change after 16 years is –21 160 births per year.

e) Answers may vary. For example:
Births increased steadily from 1950 until 1959 and started to decline after that as the baby boomers got older.

f)



g) Answers will vary. For example: The equation of the tangent for any given year would give the rate of change of the birth rate (i.e., the slope).

Chapter 1 Section 5 **Question 22 Page 61**

Solutions to the Achievement Checks are shown in the Teacher's Resource.

Chapter 1 Section 5 **Question 23 Page 61**

a) i) $\frac{dy}{dx} = 2x + 3$

ii) $\frac{dy}{dx} = 1 - 6x^2$

iii) $\frac{dy}{dx} = 8x^3 - 1$

b) i) $\frac{dy}{dx} = \lim_{h \to 0} \frac{(x+h)^2 + 3(x+h) - (x^2 + 3x)}{h}$

$= \lim_{h \to 0} \frac{2hx + h^2 + 3h}{h}$

$= \lim_{h \to 0} (2x + h + 3)$

$= 2x + 3$

ii) $\frac{dy}{dx} = \lim_{h\to 0} \frac{(x+h) - 2(x+h)^3 - (x - 2x^3)}{h}$

$= \lim_{h\to 0} \frac{x + h - 2(x^3 + 3hx^2 + 3h^2x + h^3) - x + 2x^3}{h}$

$= \lim_{h\to 0} \frac{h - 6hx^2 - 6h^2x - 2h^3}{h}$

$= \lim_{h\to 0} (1 - 6x^2 - 6hx - 2h^2)$

$= 1 - 6x^2$

iii) $\frac{dy}{dx} = \lim_{h\to 0} \frac{2(x+h)^4 - (x+h) + 5 - (2x^4 - x + 5)}{h}$

$= \lim_{h\to 0} \frac{2(x^4 + 4hx^3 + 6h^2x^2 + 4h^3x + h^4) - x - h + 5 - 2x^4 + x - 5}{h}$

$= \lim_{h\to 0} \frac{8hx^3 + 12h^2x^2 + 8h^3x + 2h^4 - h}{h}$

$= \lim_{h\to 0} (8x^3 + 12hx^2 + 8h^2x + 2h^3 - 1)$

$= 8x^3 - 1$

Chapter 1 Section 5 **Question 24 Page 61**

a) i) $\frac{dy}{dx} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$

$= \lim_{h\to 0} \frac{\frac{x^2 - (x^2 + 2xh + h^2)}{x^2(x+h)^2}}{h}$

$= \lim_{h\to 0} \frac{\frac{-2xh - h^2}{x^2(x+h)^2}}{h}$

$= \lim_{h\to 0} \frac{-2x - h}{x^2(x+h)^2}$

$= \frac{-2x}{x^2 x^2}$

$= -\frac{2}{x^3}$ for $x \neq 0$

ii) $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{\dfrac{1}{(x+h)^3} - \dfrac{1}{x^3}}{h}$

$$= \lim_{h\to 0} \frac{\dfrac{x^3 - (x^3 + 3x^2h + 3xh^2 + h^3)}{x^3(x+h)^3}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{-3x^2h - 3xh^2 - h^3}{x^3(x+h)^3}}{h}$$

$$= \lim_{h\to 0} \frac{-3x^2 - 3xh - h^2}{x^3(x+h)^3}$$

$$= \frac{-3x^2}{x^3x^3}$$

$$= -\frac{3}{x^4} \text{ for } x \neq 0$$

iii) $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{\dfrac{1}{(x+h)^4} - \dfrac{1}{x^4}}{h}$

$$= \lim_{h\to 0} \frac{\dfrac{x^4 - (x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4)}{x^4(x+h)^4}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{-4x^3h - 6x^2h^2 - 4xh^3 - h^4}{x^4(x+h)^4}}{h}$$

$$= \lim_{h\to 0} \frac{-4x^3 - 6x^2h - 4xh^2 - h^3}{x^4(x+h)^4}$$

$$= \frac{-4x^3}{x^4x^4}$$

$$= -\frac{4}{x^5} \text{ for } x \neq 0$$

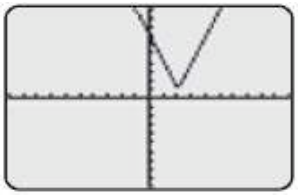
b) i)–iii)

domain of y: $\{x \mid x \neq 0,\ x \in \mathbb{R}\}$; domain of $\dfrac{dy}{dx}$: $\{x \mid x \neq 0,\ x \in \mathbb{R}\}$

c) Answers may vary. For example:
If the degree of the function is $-n$, then the degree of the derivative function is $-n - 1$.

Chapter 1 Section 5 **Question 25 Page 61**

a)



The function is non-differentiable at $x = 2$.

b) Answers may vary. For example:

If $x \le 2$, then $| x - 2 | = -(x - 2)$ so $y = -3(x - 2) + 1$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{-3[(x+h)-2]+1-[-3(x-2)+1]}{h}$$

$$= \lim_{h \to 0} \frac{-3x-3h+6+1+3x-6-1}{h}$$

$$= \lim_{h \to 0} \frac{-3h}{h}$$

$$= -3$$

If $x > 2$, then $| x - 2 | = x - 2$ so $y = 3(x - 2) + 1$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{3[(x+h)-2]+1-[3(x-2)+1]}{h}$$

$$= \lim_{h \to 0} \frac{3x+3h-6+1-3x+6-1}{h}$$

$$= \lim_{h \to 0} \frac{3h}{h}$$

$$= 3$$

Since the derivatives of y as x approaches 2 from the left and right sides are not equal, y is non-differentiable at $x = 2$.

Chapter 1 Section 5 **Question 26 Page 62**

a) i) $\frac{dy}{dx} = 0$

ii) $\frac{dy}{dx} = 1$

iii) $\frac{dy}{dx} = 2x$

iv) $\frac{dy}{dx} = 3x^2$

v) $\frac{dy}{dx} = 4x^3$

b) Answers may vary. For example: The derivatives are equal to the value in the exponent multiplied by the variable, with the variable to the power of one less than the exponent.

c) i) $\frac{dy}{dx} = 5x^4$

ii) $\frac{dy}{dx} = 6x^5$

d) i)
$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{(x+h)^5 - x^5}{h}\\ &= \lim_{h\to 0}\frac{x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5 - x^5}{h}\\ &= \lim_{h\to 0}\left(5x^4 + 10x^3h + 10x^2h^2 + 5xh^3 + h^4\right)\\ &= 5x^4\end{aligned}$$

ii)
$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{(x+h)^6 - x^6}{h}\\ &= \lim_{h\to 0}\frac{x^6 + 6x^5h + 15x^4h^2 + 20x^3h^3 + 15x^2h^4 + 6xh^5 - x^6}{h}\\ &= \lim_{h\to 0}\left(6x^5 + 15x^4h + 20x^3h^2 + 15x^2h^3 + 6xh^4\right)\\ &= 6x^5\end{aligned}$$

e) $\frac{dy}{dx} = nx^{n-1}$ for $n \in \mathbb{N}$

f) Answers may vary. For example:
Consider $y = x^8$.

Then $\frac{dy}{dx}$ should be equal to $8x^{8-1} = 8x^7$.

Use Pascal's triangle in the first principles definition in order to find the derivative.

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{(x+h)^8 - x^8}{h}\\ &= \lim_{h\to 0}\frac{x^8 + 8x^7h + 28x^6h^2 + 56x^5h^3 + 70x^4h^4 + 56x^3h^5 + 28x^2h^6 + 8xh^7 + h^8 - x^8}{h}\\ &= \lim_{h\to 0}(8x^7 + 28x^6h + 56x^5h^2 + 70x^4h^3 + 56x^3h^4 + 28x^2h^5 + 8xh^6 + h^7)\\ &= 8x^7\end{aligned}$$

Chapter 1 Section 5 **Question 27 Page 62**

a) $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

$= \lim\limits_{x \to a} \dfrac{x^2 - a^2}{x - a}$

$= \lim\limits_{x \to a} \dfrac{(x + a)(x - a)}{x - a}$

$= \lim\limits_{x \to a} (x + a)$

$= 2a$

Therefore, $\dfrac{dy}{dx} = 2x$.

It is necessary to factor using the difference of two squares in order to reduce the expression and find the limit.

b) i) $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

$= \lim\limits_{x \to a} \dfrac{x^3 - a^3}{x - a}$

$= \lim\limits_{x \to a} \dfrac{(x^2 + ax + a^2)(x - a)}{x - a}$

$= \lim\limits_{x \to a} (x^2 + ax + a^2)$

$= 3a^2$

Therefore, $\dfrac{dy}{dx} = 3x^2$.

ii) $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

$= \lim\limits_{x \to a} \dfrac{x^4 - a^4}{x - a}$

$= \lim\limits_{x \to a} \dfrac{(x + a)(x - a)(x^2 + a^2)}{x - a}$

$= \lim\limits_{x \to a} (x + a)(x^2 + a^2)$

$= \lim\limits_{x \to a} (x^3 + x^2 a + xa^2 + a^3)$

$= 4a^3$

Therefore, $\dfrac{dy}{dx} = 4x^3$.

iii) $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

$$= \lim_{x \to a} \frac{x^5 - a^5}{x - a}$$

$$= \lim_{x \to a} \frac{(x - a)(x^4 + x^3a + x^2a^2 + xa^3 + a^4)}{x - a}$$

$$= \lim_{x \to a} (x^4 + x^3a + x^2a^2 + xa^3 + a^4)$$

$$= 5a^4$$

Therefore, $\dfrac{dy}{dx} = 5x^4$.

c) Answers may vary. For example:It is easier to factor than to expand.

Chapter 1 Section 5 **Question 28 Page 62**

a) $f'(x) = \lim\limits_{h \to 0} \dfrac{\dfrac{(x+h)+2}{(x+h)-1} - \dfrac{x+2}{x-1}}{h}$

$$= \lim_{h \to 0} \frac{\dfrac{(x+h+2)(x-1) - (x+h-1)(x+2)}{(x+h-1)(x-1)}}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{(x^2 + hx + x - h - 2) - (x^2 + hx + x + 2h - 2)}{(x+h-1)(x-1)}}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{-3h}{(x+h-1)(x-1)}}{h}$$

$$= \lim_{h \to 0} \frac{-3}{(x+h-1)(x-1)}$$

$$= -\frac{3}{(x-1)^2} \quad \text{for } x \neq 1$$

b) $f'(x) = \lim\limits_{h\to 0} \dfrac{\dfrac{3(x+h)-1}{(x+h)+4} - \dfrac{3x-1}{x+4}}{h}$

$$= \lim_{h\to 0} \frac{\dfrac{(3x+3h-1)(x+4)-(x+h+4)(3x-1)}{(x+h+4)(x+4)}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{(3x^2+3hx+11x+12h-4)-(3x^2+3hx+11x-h-4)}{(x+h+4)(x+4)}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{13h}{(x+h+4)(x+4)}}{h}$$

$$= \lim_{h\to 0} \frac{13}{(x+h+4)(x+4)}$$

$$= \frac{13}{(x+4)^2} \quad \text{for } x \neq -4$$

Chapter 1 Section 5 **Question 29 Page 62**

a) $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{\sqrt{(x+h)+1}-\sqrt{x+1}}{h}$

$$= \lim_{h\to 0} \frac{\left(\sqrt{x+h+1}-\sqrt{x+1}\right)\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}$$

$$= \lim_{h\to 0} \frac{x+h+1-(x+1)}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}$$

$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h+1}+\sqrt{x+1}}$$

$$= \frac{1}{2\sqrt{x+1}}$$

domain of $f(x)$ is: $\{x \mid x \geq -1, x \in R\}$; domain of $f'(x)$ is: $\{x \mid x > -1, x \in R\}$

b) $\frac{dy}{dx} = \lim_{h \to 0} \frac{\sqrt{2(x+h)-1} - \sqrt{2x-1}}{h}$

$$= \lim_{h \to 0} \frac{\left(\sqrt{2x+2h-1} - \sqrt{2x-1}\right)\left(\sqrt{2x+2h-1} + \sqrt{2x-1}\right)}{h\left(\sqrt{2x+2h-1} + \sqrt{2x-1}\right)}$$

$$= \lim_{h \to 0} \frac{2x+2h-1-(2x-1)}{h\left(\sqrt{2x+2h-1} + \sqrt{2x-1}\right)}$$

$$= \lim_{h \to 0} \frac{2h}{h\left(\sqrt{2x+2h-1} + \sqrt{2x-1}\right)}$$

$$= \lim_{h \to 0} \frac{2}{\sqrt{2x+2h-1} + \sqrt{2x-1}}$$

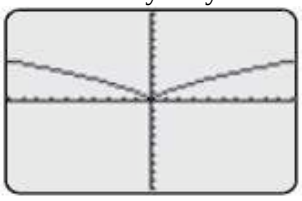
$$= \frac{2}{2\sqrt{2x-1}}$$

$$= \frac{1}{\sqrt{2x-1}}$$

domain of $f(x)$ is: $\{x \mid x \geq 0.5, x \in \mathbb{R}\}$; domain of $f'(x)$ is: $\{x \mid x > 0.5, x \in \mathbb{R}\}$

Chapter 1 Section 5 **Question 30 Page 62**

a) Answers may vary. For example:



The function is non-differentiable at (0, 0) since there is an abrupt change in slope at this point.

b) Answers may vary. For example:

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^{\frac{2}{3}} - x^{\frac{2}{3}}}{h}$$

$$= \lim_{h\to 0} \frac{\left[(x+h)^{\frac{2}{3}} - x^{\frac{2}{3}}\right]\left[(x+h)^{\frac{4}{3}} + x^{\frac{2}{3}}(x+h)^{\frac{2}{3}} + x^{\frac{4}{3}}\right]}{h\left[(x+h)^{\frac{4}{3}} + x^{\frac{2}{3}}(x+h)^{\frac{2}{3}} + x^{\frac{4}{3}}\right]}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h\left[(x+h)^{\frac{4}{3}} + x^{\frac{2}{3}}(x+h)^{\frac{2}{3}} + x^{\frac{4}{3}}\right]}$$

$$= \lim_{h\to 0} \frac{2xh + h^2}{h\left[(x+h)^{\frac{4}{3}} + x^{\frac{2}{3}}(x+h)^{\frac{2}{3}} + x^{\frac{4}{3}}\right]}$$

$$= \lim_{h\to 0} \frac{2x + h}{(x+h)^{\frac{4}{3}} + x^{\frac{2}{3}}(x+h)^{\frac{2}{3}} + x^{\frac{4}{3}}}$$

$$= \frac{2x}{3x^{\frac{4}{3}}}$$

$$= \frac{2}{3x^{\frac{1}{3}}} \quad \text{for } x \neq 0$$

Chapter 1 Section 5 **Question 31 Page 62**

The common differences in an arithmetic sequence are equal.

$3^b - 2^a = 4^c - 3^b$
$3^b - 2^a = (2^2)^c - 3^b$
$3^b - 2^a = 2^{2c} - 3^b$
$2(3^b) = 2^{2c} + 2^a$
$3^b = 2^{2c-1} + 2^{a-1}$

Since 3^b is always odd, the right side of the equation can only be equal when it is odd, so $a = 1$.

$3^b = 2^{2c-1} + 1$

If $c = 1$, $b = 1$ so the ordered triple is (1, 1, 1).
If $c = 2$, $b = 2$ so the ordered triple is (1, 2, 2).

The two solutions are (1, 1, 1) and (1, 2, 2).

Chapter 1 Section 5 **Question 32 Page 62**

Find the area of the first triangle.

$$\text{area} = \frac{1}{2}\left(\sqrt{3}\right)(1)$$
$$= \frac{\sqrt{3}}{2}$$

The area of the first triangle is $\frac{\sqrt{3}}{2}$ cm^2.

The second triangle is equilateral with side length x and all interior angles 60°. The triangle can be split into two right triangles with one leg the unknown height, the other measuring $\frac{x}{2}$ and the hypotenuse x. Let h represent the height of the triangle.

$$\frac{h}{x} = \sin 60°$$
$$h = x \sin 60°$$
$$h = \frac{\sqrt{3}x}{2}$$

Now find x.

$$\text{area} = \frac{1}{2}\left(\frac{\sqrt{3}x}{2}\right)(x)$$
$$\frac{\sqrt{3}}{2} = \frac{\sqrt{3}x^2}{4}$$
$$x^2 = 2$$
$$x = \sqrt{2}$$

Chapter 1 Section 5 **Question 33 Page 62**

$$\frac{x^{12} + 3x^{11} + 2x^{10}}{x^{11} + 2x^{10}} = \frac{x^{10}(x^2 + 3x + 1)}{x^{10}(x + 2)}$$
$$= \frac{(x + 1)(x + 2)}{(x + 2)}$$
$$= x + 1$$

When x = 2008, the value of the expression equals 2009.

Chapter 1 Review

Chapter 1 Review **Question 1 Page 64**

a) Answers may vary. For example:
Since the slope of the graph is negative and becomes less negative, the rate of drainage slows with time.

b) Answers may vary. For example:

i) $\frac{1250-2150}{1-0}=-900$
The average rate of change of the volume of water remaining during the first hour is –900 L/h.

ii) $\frac{400-520}{4-3}=-120$
The average rate of change of the volume of water remaining during the last hour is –120 L/h.

c) Answers may vary. For example:

i) $\frac{1450-1900}{45-15}=-15$
The instantaneous rate of change of the volume of water remaining at 30 min is –15 L/min or –900 L/h.

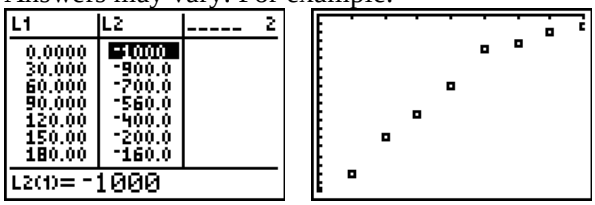
ii) $\frac{800-1100}{105-75}=-10$
The instantaneous rate of change of the volume of water remaining at 1.5 h is –10 L/min or –600 L/h.

iii) $\frac{500-575}{195-165}=-2.5$
The instantaneous rate of change of the volume of water remaining at 3 h is –2.5 L/min or –150 L/h.

d) i) Answers may vary. For example:
The graph would be steeper because the rate of change of the volume of water would become more negative.

ii) Answers may vary. For example:
The graph would be extended further to the right, since the pool would take longer to drain.

e) Answers may vary. For example:



Chapter 1 Review **Question 2 Page 64**

Answers may vary. For example:

a) The volume of gas remaining in a gas tank as a car is driven.

b) The volume of water in a beaker as the beaker is filled with water.

c) The velocity of an airplane the instant it lifts off the runway during takeoff.

d) The velocity of a car the instant that the brakes are applied.

Chapter 1 Review **Question 3 Page 64**

a) $$\text{average rate of change} = \frac{-4.9(4)^2 + 35(4) + 10 - (-4.9(2)^2 + 35(2) + 10)}{4 - 2}$$
$$= \frac{71.6 - 60.4}{2}$$
$$= 5.6$$

The average rate of change is 5.6 m/s.

b) Using the interval $4 \le t \le 6$:

$$\text{instantaneous rate of change} = \frac{-4.9(6)^2 + 35(6) + 10 - (-4.9(4)^2 + 35(4) + 10)}{6 - 4}$$
$$= \frac{43.6 - 71.6}{2}$$
$$= -14$$

The estimated instantaneous rate of change is –14 m/s.

c) Answers may vary. For example:
Use a smaller interval for the secant. For instance, use the interval $4.9 \le t \le 5.1$.

Chapter 1 Review **Question 4 Page 64**

$$\frac{f(a+h)-f(a)}{h} = \frac{3(a+h)^2+2(a+h)-(3a^2+2a)}{h}$$
$$= \frac{6ah+3h^2+2h}{h}$$
$$= 6a+3h+2$$

i) $h = 0.1$: $6(2) + 3(0.1) + 2 = 14.3$
$h = 0.01$: $6(2) + 3(0.01) + 2 = 14.03$
$h = 0.001$: $6(2) + 3(0.001) + 2 = 14.003$

The slope of the tangent is 14.

ii) $h = 0.1$: $6(-3) + 3(0.1) + 2 = -15.7$
$h = 0.01$: $6(-3) + 3(0.01) + 2 = -15.97$
$h = 0.001$: $6(-3) + 3(0.001) + 2 = -15.997$

The slope of the tangent is –16.

b) i) When $x = 2$, $y = 16$. Using the point (2, 16) and the slope $m = 14$, find b in the equation of the tangent $y = mx + b$.

$16 = 14(2) + b$
$b = -12$

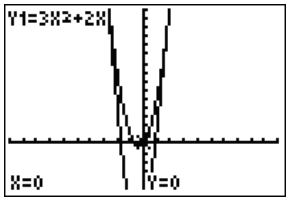
The equation of the tangent at $x = 2$ is $y = 14x - 12$.

ii) When $x = -3$, $y = 21$. Using the point (–3, 21) and the slope $m = -16$, find b in the equation of the tangent $y = mx + b$.

$21 = -16(-3) + b$
$b = -27$

The equation of the tangent at $x = -3$ is $y = -16x - 27$.

c)



Chapter 1 Review **Question 5 Page 64**

a) $t_1 = \frac{4}{3}; t_2 = \frac{1}{6}; t_3 = -\frac{4}{9}; t_4 = -\frac{11}{12}; t_5 = -\frac{4}{3}$

b) Answers may vary. For example:
No. The sequence does not have a limit as $n \to \infty$. The sequence is divergent.

Chapter 1 Review **Question 6 Page 64**

a) $t_1 = \frac{35}{8}; t_2 = \frac{245}{64}; t_3 = \frac{1715}{512}; t_4 = \frac{12\,005}{4096}; t_5 = \frac{84\,035}{32\,768}$

b) The sequence is converging to zero as $n \to \infty$.

c) $t_{12} = 1.007\,09; t_{13} = 0.881\,20$
Therefore, it takes 13 bounces.

Chapter 1 Review **Question 7 Page 65**

Justifications may vary. For example:

a) Yes. The function is continuous at $x = 3$ since the right and left limits approach the value at $x = 3$.

x	2.9	2.99	3	3.01	3.1
$g(x)$	–0.356	–0.335	–0.333	–0.331	–0.311

b) Yes. The function is discontinuous at $x = -3$ since there is a vertical asymptote at this point.

Chapter 1 Review **Question 8 Page 65**

a) domain: $\{x \mid x \neq 0,\ x \in \mathbb{R}\}$; range: $\{y \mid y \leq 2,\ y \in \mathbb{R}\}$

b) i) $\lim\limits_{x \to +\infty} \frac{-2x^2 + 8x - 4}{x^2} = \lim\limits_{x \to +\infty} \left(-2 + \frac{8}{x} - \frac{4}{x^2}\right)$
$= -2$

ii) $\lim\limits_{x \to -\infty} \frac{-2x^2 + 8x - 4}{x^2} = \lim\limits_{x \to -\infty} \left(-2 + \frac{8}{x} - \frac{4}{x^2}\right)$
$= -2$

iii) $\lim\limits_{x \to 0^+} \frac{-2x^2 + 8x - 4}{x^2} = \lim\limits_{x \to 0^+} \left(-2 + \frac{8x - 4}{x^2}\right)$
$= -\infty$

iv) $\lim\limits_{x\to 0^-}\dfrac{-2x^2+8x-4}{x^2}=\lim\limits_{x\to 0^-}\left(-2+\dfrac{8x-4}{x^2}\right)$

$=-\infty$

c) Answers may vary. For example:

The graph has a discontinuity at $x = 0$, since the limits for 0^+ and 0^- are $-\infty$.

Chapter 1 Review **Question 9 Page 65**

a) $\lim\limits_{x\to -1}\dfrac{(-1+x)^2-4}{x+1}=\lim\limits_{x\to -1}\dfrac{x^2-2x-3}{x+1}$

$=\lim\limits_{x\to -1}\dfrac{(x+1)(x-3)}{x+1}$

$=\lim\limits_{x\to -1}(x-3)$

$=-4$

b) $\lim\limits_{x\to 3}\dfrac{-x^2+8x}{2x+1}=\dfrac{15}{7}$

c) $\lim\limits_{x\to 0}\dfrac{\sqrt{x+16}+4}{x}=\lim\limits_{x\to 0}\dfrac{\left(\sqrt{x+16}+4\right)\left(\sqrt{x+16}-4\right)}{x\left(\sqrt{x+16}-4\right)}$

$=\lim\limits_{x\to 0}\dfrac{(x+16)-16}{x\left(\sqrt{x+16}-4\right)}$

$=\lim\limits_{x\to 0}\dfrac{x}{x\left(\sqrt{x+16}-4\right)}$

$=\lim\limits_{x\to 0}\dfrac{1}{\sqrt{x+16}-4}$

$=\dfrac{1}{0}$

The limit does not exist.

d) $\lim\limits_{x\to -2}\dfrac{3x^2+5x-2}{x^2-2x-8}=\lim\limits_{x\to -2}\dfrac{(3x-1)(x+2)}{(x-4)(x+2)}$

$=\lim\limits_{x\to -2}\dfrac{3x-1}{x-4}$

$=\dfrac{7}{6}$

e) $\lim\limits_{x\to 7}\dfrac{x^2-49}{x-7}=\lim\limits_{x\to 7}\dfrac{(x-7)(x+7)}{x-7}$

$=\lim\limits_{x\to 7}(x+7)$

$=14$

Chapter 1 Review **Question 10 Page 65**

Answers may vary. For example:
As x approaches -6 from the left and right, the graph of $y=h(x)$ approaches $y=3$ so the limit as x approaches -6 exists and equals 3. There is a hole in the graph of $y=h(x)$ at $(-6, 3)$. Since $h(-6)\neq 3$, the function is discontinuous at $x=-6$.

Chapter 1 Review **Question 11 Page 65**

a) $\dfrac{dy}{dx}=\lim\limits_{h\to 0}\dfrac{4(x+h)-1-(4x-1)}{h}$

$=\lim\limits_{h\to 0}\dfrac{4h}{h}$

$=4$

b) $h'(x)=\lim\limits_{h\to 0}\dfrac{11(x+h)^2+2(x+h)-(11x^2+2x)}{h}$

$=\lim\limits_{h\to 0}\dfrac{22xh+11h^2+2h}{h}$

$=\lim\limits_{h\to 0}(22x+11h+2)$

$=22x+2$

c) $s'(t)=\lim\limits_{h\to 0}\dfrac{\frac{1}{3}(t+h)^3-5(t+h)^2-\left(\frac{1}{3}t^3-5t^2\right)}{h}$

$=\lim\limits_{h\to 0}\dfrac{\frac{1}{3}(t^3+3t^2h+3th^2+h^3)-5(t^2+2th+h^2)-\left(\frac{1}{3}t^3-5t^2\right)}{h}$

$=\lim\limits_{h\to 0}\dfrac{t^2h+th^2+\frac{1}{3}h^3-10th-5h^2}{h}$

$=\lim\limits_{h\to 0}(t^2+th+\frac{1}{3}h^2-10t-5h)$

$=t^2-10t$

d) $f'(x) = \lim_{h \to 0} \dfrac{(x+h+3)(x+h-1)-(x+3)(x-1)}{h}$

$= \lim_{h \to 0} \dfrac{2xh+2h+h^2}{h}$

$= \lim_{h \to 0} (2x+2+h)$

$= 2x+2$

Chapter 1 Review **Question 12 Page 65**

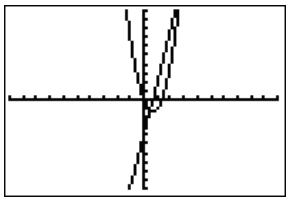
a) $\dfrac{dy}{dx} = \lim_{h \to 0} \dfrac{3(x+h)^2-4(x+h)-(3x^2-4x)}{h}$

$= \lim_{h \to 0} \dfrac{6xh+3h^2-4h}{h}$

$= \lim_{h \to 0} (6x+3h-4)$

$= 6x-4$

b)



c) The slope of the tangent at $x = -2$ is $\left.\dfrac{dy}{dx}\right|_{x=-2}$

$\left.\dfrac{dy}{dx}\right|_{x=-2} = 6(-2)-4$

$= -16$

When $x = -2$, $y = 20$.
Use the point $(-2, 20)$ and $m = -16$ to find b in the equation of the tangent, $y = mx + b$.

$20 = -16(-2) + b$
$b = -12$

The equation of the tangent at $x = -2$ is $y = -16x - 12$.

Chapter 1 Practice Test

Chapter 1 Practice Test **Question 1 Page 66**

C; The function is 0 at $x = 2$, but there is a cusp there.

Chapter 1 Practice Test **Question 2 Page 66**

C; Explanations may vary. For example:
This expression gives the slope of the secant for the interval $x \in (a, h)$.

Chapter 1 Practice Test **Question 3 Page 66**

a) $\lim\limits_{x \to 9}(4x - 1) = 35$

b) $\lim\limits_{x \to -3}(2x^4 - 3x^2 + 6) = 141$

c) $$\begin{aligned}\lim_{x \to 5}\frac{x^2 - 3x - 10}{x - 5} &= \lim_{x \to 5}\frac{(x - 5)(x + 2)}{x - 5}\\ &= \lim_{x \to 5}(x + 2)\\ &= 7\end{aligned}$$

d) $$\begin{aligned}\lim_{x \to 0}\frac{9x}{2x^2 - 5x} &= \lim_{x \to 0}\frac{9x}{x(2x - 5)}\\ &= \lim_{x \to 0}\frac{9}{(2x - 5)}\\ &= -\frac{9}{5}\end{aligned}$$

e) $$\begin{aligned}\lim_{x \to 7}\frac{x^2 - 49}{x - 7} &= \lim_{x \to 7}\frac{(x - 7)(x + 7)}{x - 7}\\ &= \lim_{x \to 7}(x + 7)\\ &= 14\end{aligned}$$

f) $\lim\limits_{x \to \infty}\dfrac{-1}{2 + x^2} = 0$

Chapter 1 Practice Test **Question 4 Page 66**

C; Explanations may vary. For example:
A limit can be used to find the instantaneous rate of change at a point, not the average rate of change.

Chapter 1 Practice Test **Question 5 Page 66**

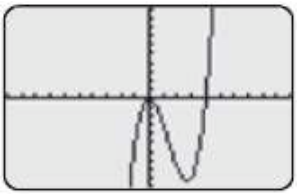
a) $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{(x+h)^3 - 4(x+h)^2 - (x^3 - 4x^2)}{h}$

$= \lim\limits_{h\to 0} \dfrac{3x^2h + 3xh^2 + h^3 - 8xh - 4h^2}{h}$

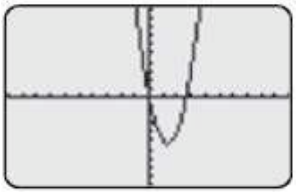
$= \lim\limits_{h\to 0} (3x^2 + 3xh + h^2 - 8x - 4h)$

$= 3x^2 - 8x$

b) $y = x^3 - 4x^2$



$\dfrac{dy}{dx} = 3x^2 - 8x$



c) The slope of the tangent at $x = -1$ is $\left.\dfrac{dy}{dx}\right|_{x=-1}$.

$\left.\dfrac{dy}{dx}\right|_{x=-1} = 3(-1)^2 - 8(-1)$

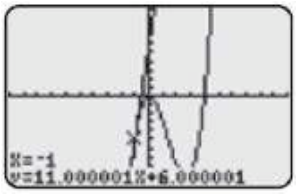
$= 11$

When $x = -1$, $y = -5$.
Use the point $(-1, -5)$ and $m = 11$ to find b in the equation of the tangent, $y = mx + b$.

$-5 = 11(-1) + b$
$b = 6$

The equation of the tangent at $x = -1$ is $y = 11x + 6$.

d)



Chapter 1 Practice Test **Question 6 Page 66**

a) $\lim\limits_{x \to -2} f(x) = -3$

b) $\lim\limits_{x \to 0^-} f(x) = 1$

c) $\lim\limits_{x \to 1^-} f(x) = -2$

d) $\lim\limits_{x \to 1^+} f(x) = -2$

e) $\lim\limits_{x \to -4^-} f(x) = 1$

f) $\lim\limits_{x \to \infty} f(x) = \infty$

Chapter 1 Practice Test **Question 7 Page 67**

a) domain: $\{x \mid x \neq 4,\ x \in \mathbb{R}\}$; range: $\{y \mid y \neq 3,\ y \in \mathbb{R}\}$

b) i) $\lim\limits_{x \to +\infty} \dfrac{3x}{x-4} = 3$ from above

ii) $\lim\limits_{x \to -\infty} \dfrac{3x}{x-4} = 3$ from below

iii) $\lim\limits_{x \to 4^+} \dfrac{3x}{x-4} = +\infty$

iv) $\lim\limits_{x \to 4^-} \dfrac{3x}{x-4} = -\infty$

v) $\lim\limits_{x \to 6} \dfrac{3x}{x-4} = 9$

vi) $\lim\limits_{x \to -2} \dfrac{3x}{x-4} = 1$

c) Explanations may vary. For example:
No. The graph is not continuous. As x approaches 4 from the left, the graph becomes large and negative. As x approaches 4 from the right, the graph becomes large and positive. Therefore, there is a discontinuity at $x = 4$.

Chapter 1 Practice Test **Question 8 Page 67**

a) $V(x) = 4x - \frac{1}{4}x^3$

b) average rate of change: $\frac{V(3) - V(1.5)}{3 - 1.5} = \frac{5.25 - 5.15625}{1.5}$

$= 0.0625$

The average rate of change of the volume of the shed is 0.0625 m^3/m.

c) instantaneous rate of change: $\lim\limits_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim\limits_{x \to a} \frac{4x - \frac{x^3}{4} - \left(4a + \frac{a^3}{4}\right)}{x - a}$

$= \lim\limits_{x \to a} \frac{4(x - a) - \frac{1}{4}(x^3 - a^3)}{x - a}$

$= \lim\limits_{x \to a} \frac{4(x - a) - \frac{1}{4}(x - a)(x^2 + ax + a^2)}{x - a}$

$= \lim\limits_{x \to a} \left(4 - \frac{1}{4}(x^2 + ax + a^2)\right)$

$= 4 - \frac{3a^2}{4}$

When $a = 3$, the instantaneous rate of change of the volume of the shed is –2.75 m^3/m.

Chapter 1 Practice Test **Question 9 Page 67**

a) i) After 1 s, the radius is 0.2 m.

ii) After 3 s, the radius is 0.6 m.

iii) After 5 s, the radius is 1.0 m.

b) The area of a circle is $A = \pi r^2$.

The instantaneous rate of change in the area is $\frac{dA}{dr} = 2\pi r$.

c) i) $\left.\frac{dA}{dr}\right|_{r=0.2} = 2\pi(0.2)$

$\doteq 1.3$

The instantaneous rate of change of the area when $r = 0.2$ m is 1.3 m^2/m.

ii) $\left.\frac{dA}{dr}\right|_{r=0.6} = 2\pi(0.6)$

$\doteq 3.8$

The instantaneous rate of change of the area when $r = 0.6$ m is 3.8 m^2/m.

iii) $\left.\frac{dA}{dr}\right|_{r=1} = 2\pi(1)$

$\doteq 6.3$

The instantaneous rate of change of the area when $r = 1$ m is 6.3 m^2/m.

Chapter 1 Practice Test **Question 10 Page 67**

a) C

b) A

c) D

d) B

Chapter 2 Derivatives

Chapter 2 Prerequisite Skills

Chapter 2 Prerequisite Skills Question 1 Page 70

Justifications may vary. For example:

a) polynomial; The variables and constants in the function are either added or subtracted together.

b) sinusoidal; The sine function is a sinusoidal function.

c) polynomial; The variables and constants in the function are either added or subtracted together.

d) root; The function is the square root of a variable.

e) exponential; The function consists of a constant raised to the power of a variable.

f) rational; The function is expressed as the ratio of two polynomials.

g) logarithmic; The function is logarithmic since it is of the form $\log_b(x)$, where $b = 3$ is the base. The function can be written implicitly as $3^y = x$.

h) polynomial; After expanding the function, all the terms are variables or constants that are either added or subtracted together.

Chapter 2 Prerequisite Skills Question 2 Page 70

a) $m = 2$ so the perpendicular slope to m is $-\frac{1}{2}$.

b) $m = -5$ so the perpendicular slope to m is $-\left(-\frac{1}{5}\right) = \frac{1}{5}$.

c) $y = \frac{2}{3}x - 6$, so $m = \frac{2}{3}$ and the slope perpendicular to m is $-\frac{3}{2}$.

d) $m = 0$ so the perpendicular slope to m is undefined.

e) $m = 1$ so the perpendicular slope to m is -1.

f) m is undefined so the perpendicular slope to m is 0.

Chapter 2 Prerequisite Skills **Question 3 Page 70**

a) $x^{\frac{1}{2}}$

b) $x^{\frac{1}{3}}$

c) $x^{\frac{3}{4}}$

d) $x^{\frac{2}{5}}$

Chapter 2 Prerequisite Skills **Question 4 Page 70**

a) x^{-1}

b) $-2x^{-4}$

c) $x^{-\frac{1}{2}}$

d) $x^{-\frac{2}{3}}$

Chapter 2 Prerequisite Skills **Question 5 Page 70**

a) $(x^3 - 1)(5x + 2)^{-1}$

b) $(3x^4)(5x+6)^{-\frac{1}{2}}$

c) $(9 - x^2)^3(2x + 1)^{-4}$

d) $(x+3)^2(1-7x^2)^{-\frac{1}{3}}$

Chapter 2 Prerequisite Skills **Question 6 Page 70**

a) $x^{-6} = \frac{1}{x^6}$

b) $\frac{2x^3}{x^3} - \frac{x^2}{x^3} + \frac{3x}{x^3} = 2 - \frac{1}{x} + \frac{3}{x^2}$

c) $x^{-3} = \frac{1}{x^3}$

d) $x^{\frac{1}{2}} - x^{-\frac{1}{2}} = x^{\frac{1}{2}} - \frac{1}{x^{\frac{1}{2}}}$

e) $c^6c^3 = c^9$

f) $\frac{(4x-3)^2}{(x^2+3)^{\frac{3}{2}}}$

Chapter 2 Prerequisite Skills **Question 7 Page 70**

a) i) increasing: $(-\infty, -1), (2.5, \infty)$; decreasing: $(-1, 2.5)$

ii) positive: $(-3, 2), (3, \infty)$; negative: $(-\infty, -3), (2, 3)$

iii) zero slope: $x = -1$, $x = 2.5$; positive slope: $(-\infty, -1), (2.5, \infty)$; negative slope: $(-1, -2.5)$

b) i) increasing: $(-2.5, 1.5), (4.5, \infty)$; decreasing: $(-\infty, -2.5), (1.5, 4.5)$

ii) positive: $(-\infty, -4), (-0.5, 4), (5, \infty)$; negative: $(-4, -0.5), (4, 5)$

iii) zero slope: $x = -2.5$, $x = 1.5$, $x = 4.5$; positive slope: $(-2.5, 1.5), (4.5, \infty)$;
negative slope: $(-\infty, -2.5), (1.5, 4.5)$

Chapter 2 Prerequisite Skills **Question 8 Page 71**

a) $x^2 - 8x + 12 = 0$
$(x-2)(x-6) = 0$
$x = 2, x = 6$

b) $4x^2 - 16x - 84 = 0$
$x^2 - 4x - 21 = 0$
$(x+3)(x-7) = 0$
$x = -3, x = 7$

c) $5x^2 - 14x + 8 = 0$
$(5x-4)(x-2) = 0$
$x = \frac{4}{5}, x = 2$

d) $6x^2 - 5x - 6 = 0$
$(3x+2)(2x-3) = 0$
$x = -\frac{2}{3}, x = \frac{3}{2}$

e) $x = \dfrac{-5 \pm \sqrt{5^2 - 4(1)(-4)}}{2(1)}$

$x = \dfrac{-5 \pm \sqrt{41}}{2}$

f) $x = \dfrac{-13 \pm \sqrt{13^2 - 4(2)(-6)}}{2(2)}$

$x = \dfrac{-13 \pm \sqrt{217}}{4}$

g) $x = \dfrac{-(-9) \pm \sqrt{(-9)^2 - 4(4)(3)}}{2(4)}$

$x = \dfrac{9 \pm \sqrt{33}}{8}$

h) $x = \dfrac{-7 \pm \sqrt{7^2 - 4(-1)(-1)}}{2(-1)}$

$x = \dfrac{7 \pm \sqrt{45}}{2}$

$x = \dfrac{7 \pm 3\sqrt{5}}{2}$

Chapter 2 Prerequisite Skills **Question 9 Page 71**

The factor theorem requires using trial and error to find all values of x that satisfy the given equation. Use long or synthetic division to help find the factors of each polynomial.

a) $(x + 1)$ is a factor of $x^3 + 3x^2 - 6x - 8 = 0$

$(x + 1)(x^2 + 2x - 8) = 0$

$(x + 1)(x + 4)(x - 2) = 0$

$x = -4, x = -1, x = 2$

b) $(x + 1)$ is a factor of $2x^3 - x^2 - 5x - 2 = 0$

$(x + 1)(2x^2 - 3x - 2) = 0$

$(x + 1)(2x + 1)(x - 2) = 0$

$x = -1, x = -\dfrac{1}{2}, x = 2$

c) $(x - 3)$ is a factor of $3x^3 + 4x^2 - 35x - 12 = 0$

$$(x - 3)(3x^2 + 13x + 4) = 0$$
$$(x - 3)(3x + 1)(x + 4) = 0$$
$$x = -4,\ x = -\frac{1}{3},\ x = 3$$

d) $(x - 1)$ is a factor of $5x^3 + 11x^2 - 13x - 3 = 0$

$$(x - 1)(5x^2 + 16x + 3) = 0$$
$$(x - 1)(5x + 1)(x + 3) = 0$$
$$x = -3,\ x = -\frac{1}{5},\ x = 1$$

e) $(x - 1)$ is a factor of $3x^3 + 2x^2 - 7x + 2 = 0$

$$(x - 1)(3x^2 + 5x - 2) = 0$$
$$(x - 1)(3x - 1)(x + 2) = 0$$
$$x = -2,\ x = \frac{1}{3},\ x = 1$$

f) $(x + 1)$ is a factor of
$x^4 - 2x^3 - 13x^2 + 14x + 24 = 0$
$(x + 1)(x^3 - 3x^2 - 10x + 24) = 0$

$(x - 2)$ is a factor of $x^3 - 3x^2 - 10x + 24 = 0$

$$(x + 1)(x - 2)(x^2 - x - 12) = 0$$
$$(x + 1)(x - 2)(x + 3)(x - 4) = 0$$
$$x = -3,\ x = -1,\ x = 2,\ x = 4$$

Chapter 2 Prerequisite Skills **Question 10 Page 71**

a) $(x^2 + 4)(5) + 2x(5x - 7) = 5x^2 + 20 + 10x^2 - 14x$
$= 15x^2 - 14x + 20$

b) $(9 - 5x^3)(14x) + (-15x^2)(7x^2 + 2) = 126x - 70x^4 - 105x^4 - 30x^2$
$= -175x^4 - 30x^2 + 126x$

c) $(3x^4 - 6x)(6x^2 + 5) + (12x^3 - 6)(2x^3 + 5x) = 18x^6 + 15x^4 - 36x^3 - 30x + 24x^6 + 60x^4 - 12x^3 - 30x$
$= 42x^6 + 75x^4 - 48x^3 - 60x$

Chapter 2 Prerequisite Skills **Question 11 Page 71**

a) $8(x^3 - 1)^5(2x + 7)^3 + 15x^2(x^3 - 1)^4(2x + 7)^4 = (x^3 - 1)^4(2x + 7)^3[8(x^3 - 1) + 15x^2(2x + 7)]$
$= (x^3 - 1)^4(2x + 7)^3(8x^3 - 8 + 30x^3 + 105x^2)$
$= (x^3 - 1)^4(2x + 7)^3(38x^3 + 105x^2 - 8)$

b) $6(x^3+4)^{-1} - 3x^2(6x-5)(x^3+4)^{-2} = 3(x^3+4)^{-2}[2(x^3+4) - x^2(6x-5)]$
$= 3(x^3+4)^{-2}(2x^3+8-6x^3+5x^2)$
$= (x^3+4)^{-2}(-12x^3+15x^2+24)$

c) $2x^{\frac{7}{2}} - 2x^{\frac{1}{2}} = 2x^{\frac{1}{2}}(x^3-1)$
$= 2\sqrt{x}(x-1)(x^2+x+1)$

d) $1+2x^{-1}+x^{-2} = x^{-2}(x^2+2x+1)$
$= x^{-2}(x+1)(x+1)$
$= \dfrac{(x+1)^2}{x^2}$

Chapter 2 Prerequisite Skills **Question 12 Page 71**

a) $u = \sqrt{4}$ which is 2 so $y = 6(2)^2 - 1$
$= 23$

b) $u = 9 - 2(4)$ which is 1 so $y = -\dfrac{5}{(1)^3}$
$= -5$

c) $u = 5(4) - 18$ which is 2 so $y = -(2)^2 + 3(2) + 1$
$= 3$

Chapter 2 Prerequisite Skills **Question 13 Page 71**

a) $f \circ g(x) = f\left[\dfrac{1}{x-2}\right]$
$= \left(\dfrac{1}{x-2}\right)^3 + 1$

b) $g \circ h(x) = g\left(\sqrt{1-x^2}\right)$
$= \dfrac{1}{\sqrt{1-x^2}-2}$

c) $h[f(x)] = h[x^3+1]$
$= \sqrt{1-(x^3+1)^2}$
$= \sqrt{-x^6-2x^3}$

d) $g[f(x)] = g[x^3 + 1]$

$$= \frac{1}{(x^3 + 1) - 2}$$

$$= \frac{1}{x^3 - 1}$$

Chapter 2 Prerequisite Skills **Question 14 Page 71**

Answers may vary. For example:

a) Set $f(x) = x^2$ and $g(x) = 2x - 3$.
Then $f[g(x)] = (2x - 3)^2$ so $h(x) = f[g(x)]$.

b) Set $f(x) = \sqrt{x}$ and $g(x) = 2 + 4x$.
Then $f[g(x)] = \sqrt{2 + 4x}$ so $h(x) = f[g(x)]$.

c) Set $f(x) = \frac{1}{x}$ and $g(x) = 3x^2 - 7x$.
Then $f[g(x)] = \frac{1}{3x^2 - 7x}$ so $h(x) = f[g(x)]$.

d) Set $f(x) = \frac{1}{x^2}$ and $g(x) = x^3 - 4$.
Then $f[g(x)] = \frac{1}{(x^3 - 4)^2}$ so $h(x) = f[g(x)]$.

Chapter 2 Section 1 **Derivative of a Polynomial Function**

Chapter 2 Section 1 **Question 1 Page 83**

A, B, E, G, and H all have a derivative of zero since they are all constants.

Chapter 2 Section 1 **Question 2 Page 83**

a) $\frac{dy}{dx} = 1(x^{1-1})$

$= 1$

b) $\frac{dy}{dx} = \frac{1}{4}(2x^{2-1})$

$= \frac{1}{2}x$

c) $\frac{dy}{dx} = 5x^{5-1}$

$= 5x^4$

d) $\frac{dy}{dx} = -3(4x^{4-1})$

$= -12x^3$

e) $\frac{dy}{dx} = 1.5(3x^{3-1})$

$= 4.5x^2$

f) $y = x^{\frac{3}{5}}$

$\frac{dy}{dx} = \frac{3}{5}x^{\frac{3}{5}-1}$

$= \frac{3}{5}x^{-\frac{2}{5}}$

$= \frac{3}{5\sqrt[5]{x^2}}$

g) $y = 5x^{-1}$

$\frac{dy}{dx} = 5(-x^{-1-1})$

$= -\frac{5}{x^2}$

h) $y = 4x^{-\frac{1}{2}}$

$$\frac{dy}{dx} = 4\left(-\frac{1}{2}x^{-\frac{1}{2}-1}\right)$$

$$= -2x^{-\frac{3}{2}}$$

$$= -\frac{2}{\sqrt{x^3}}$$

Chapter 2 Section 1 **Question 3 Page 83**

a) $y = 6$ is a constant so $\frac{dy}{dx} = 0$.

The slope of the tangent line is 0 at all values of x so the slope is 0 at $x = 12$.

b) $f'(x) = 2(5x^{5-1})$

$= 10x^4$

The slope of the tangent to f at $x = \sqrt{3}$ is the derivative of f at $x = \sqrt{3}$, so find $f'\left(\sqrt{3}\right)$.

$$f'(\sqrt{3}) = 10\left(\sqrt{3}\right)^4$$

$$= 90$$

c) $g(x) = -3x^{-\frac{1}{2}}$

$$g'(x) = -3\left(-\frac{1}{2}x^{-\frac{1}{2}-1}\right)$$

$$= \frac{3}{2\sqrt{x^3}}$$

The slope of the tangent to g at $x = 4$ is the derivative of g at $x = 4$, so find $g'(4)$.

$$g'(4) = \frac{3}{2\sqrt{4}^3}$$

$$= \frac{3}{16} \text{ or } 0.1875$$

d) $h'(t) = -4.9\left(2t^{2-1}\right)$

$= -9.8t$

The slope of the tangent to h at $t = 3.5$ is the derivative of h at $t = 3.5$, so find $h'(3.5)$.

$$h'(3.5) = -9.8(3.5)$$

$$= -34.3$$

e) $A'(r) = \pi(2r^{2-1})$

$= 2\pi r$

The slope of the tangent to A at $r = \frac{3}{4}$ is the derivative of A at $r = \frac{3}{4}$, so find $A'\left(\frac{3}{4}\right)$.

$A'\left(\frac{3}{4}\right) = 2\pi\left(\frac{3}{4}\right)$

$= \frac{3\pi}{2}$ or $4.712\,389$

f) $y = \frac{1}{3}x^{-1}$

$\frac{dy}{dx} = \frac{1}{3}(-x^{-1-1})$

$= -\frac{1}{3x^2}$

The slope of the tangent to y at $x = -2$ is the derivative of y at $x = -2$, so find $\left.\frac{dy}{dx}\right|_{x=-2}$

$\left.\frac{dy}{dx}\right|_{x=-2} = -\frac{1}{3(-2)^2}$

$= -\frac{1}{12}$

Chapter 2 Section 1 **Question 4 Page 83**

a) $f'(x) = 2(2x^{2-1}) + 3x^{3-1}$

$= 4x + 3x^2$

The sum rule, power rule, and constant multiple rule were used.

b) $\frac{dy}{dx} = \frac{4}{5}(5x^{5-1}) - 3x^{1-1}$

$= 4x^4 - 3$

The difference rule, power rule, and constant multiple rule were used.

c) $h'(t) = -1.1(4t^{4-1}) + 0$

$= -4.4t^3$

The sum rule, power rule, constant rule, and constant multiple rule were used.

d) $V'(r) = \frac{4}{3}\pi(3r^{3-1})$

$= 4\pi r^2$

The power rule and constant multiple rule were used.

e) $p'(a) = \frac{1}{15}(5a^{5-1}) - 2(\frac{1}{2}a^{\frac{1}{2}-1})$

$= \frac{1}{3}a^4 - \frac{1}{\sqrt{a}}$

The difference rule, power rule and constant multiple rule were used.

f) $k'(s) = -1(-2s^{-2-1}) + 7(4s^{4-1})$

$= \frac{2}{s^3} + 28s^3$

The sum rule, power rule, and constant multiple rule were used.

Chapter 2 Section 1 **Question 5 Page 84**

a) i) The slope is given by y'.

$y' = 6(2x^{2-1}) - 3x^{1-1} + 0$

$= 12x - 3$

Set $y' = 0$.

$0 = 12x - 3$

$x = \frac{1}{4}$ or 0.25

At $x = 0.25$, $y = 6(0.25)^2 - 3(0.25) + 4 = 3.625$, so the point is (0.25, 3.625).

ii) The slope is given by y'.

$y' = -(2x^{2-1}) + 5x^{1-1} + 0$

$= -2x + 5$

Set $y' = 0$.

$0 = -2x + 5$

$x = \frac{5}{2}$ or 2.5

At $x = 2.5$, $y = -(2.5)^2 + 5(2.5) - 1 = 5.25$, so the point is (2.5, 5.25).

iii) The slope is given by y'.

$y = \frac{15}{4}x^2 + 4x + 3$

$y' = \frac{15}{4}(2x^{2-1}) + 4x^{1-1} + 0$

$= \frac{15}{2}x + 4$

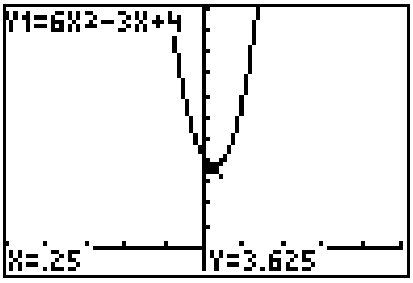
Set $y' = 0$.

$$0 = \frac{15}{2}x + 4$$

$$x = -\frac{8}{15} \text{ or } -0.5\overline{3}$$

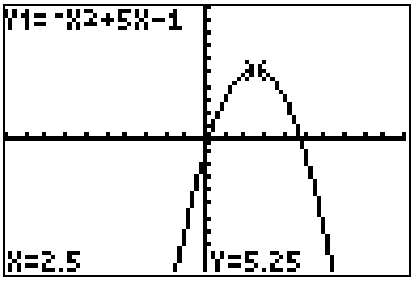
At $x = -0.5\overline{3}$, $y = \frac{15}{4}(-0.5\overline{3})^2 + 4(-0.5\overline{3}) + 3$ or $1.9\overline{3}$, so the point is $(-0.5\overline{3}, -1.9\overline{3})$.

b) i)



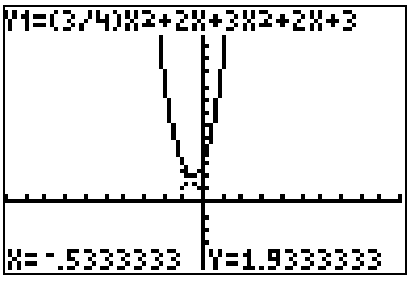
The point (0.25, 3.625) is a local minimum.

ii)



The point (2.5, 5.25) is a local maximum.

iii)



The point $(-0.5\overline{3}, 1.9\overline{3})$ is a local minimum.

Chapter 2 Section 1 **Question 6 Page 84**

a) $f(x) = 5x^2 - 3x$

$f'(x) = 5(2x^{2-1}) - 3x^{1-1}$

$= 10x - 3$

b) $g(x) = 6x^2 + 5x - 4$

$g'(x) = 6(2x^{2-1}) + 5x^{1-1} + 0$

$= 12x + 5$

c) $p(x) = \frac{1}{4}x^5 - x^3 + \frac{1}{2}$

$p'(x) = \frac{1}{4}(5x^{5-1}) - 3x^{3-1} + 0$

$= \frac{5}{4}x^4 - 3x^2$

d) $f(x) = 25x^2 + 20x + 4$

$f'(x) = 25(2x^{2-1}) + 20x^{1-1} + 0$

$= 50x + 20$

Chapter 2 Section 1 **Question 7 Page 84**

Answers may vary. For example:

a) Use a graphing calculator to graph the curve, and then draw the tangent to the curve and determine the equation of the tangent to the curve at the given x-value.

b) The derivative is equal to the slope of the tangent at the given point. The equation of the derivative can then be solved to find the x-value of the tangent point. The x-value is then substituted into the equation of the function to find the y-value of the tangent point.

Chapter 2 Section 1 **Question 8 Page 84**

a) Answers may vary. For example:

The rules in this chapter require that the given function be expanded and then differentiated using the sum, power, and constant multiple rules. This question needs to be in a more expanded form to solve via these methods.

b) $f(x)$ would need to be expanded and simplified in order to differentiate using the rules of this section.

c) $f(x) = (4x^2 - 4x + 1)(x + 3)$

$= 4x^3 + 12x^2 - 4x^2 - 12x + x + 3$

$= 4x^3 + 8x^2 - 11x + 3$

$f'(x) = 4(3x^{3-1}) + 8(2x^{2-1}) - 11x^{1-1} + 0$

$= 12x^2 + 16x - 11$

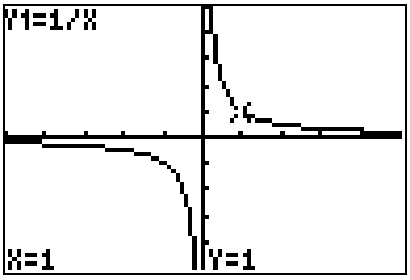
Chapter 2 Section 1 **Question 9 Page 84**

Answers may vary. The question is done **algebraically** in Section 1, Example 2 on page 79.

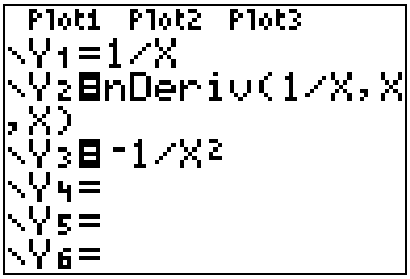
Graphically: Since $y = x^{-1}$, need to show that $\frac{dy}{dx} = -x^{-2}$.

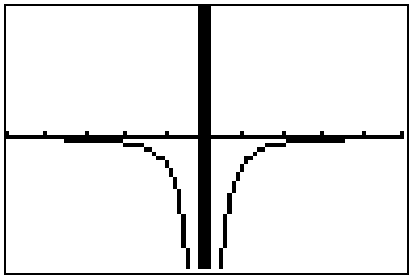
Following the procedure in Section 2.1, Example 1 on page 78:

Graph $y = x^{-1}$:



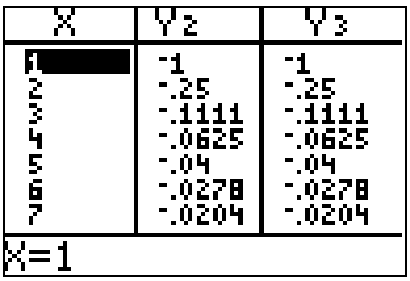
Graph $y_2 = \frac{dy}{dx}$ and $y_3 = x^2$ together:





The two graphs are the same, so the derivative of y is $-\frac{1}{x^2}$.

Numerically: the table of values for $y_2 = \frac{dy}{dx}$ and $y_3 = -\frac{1}{x^2}$ are the same:



Chapter 2 Section 1 **Question 10 Page 84**

$f'(t)$ represents the rate of change of the amount of water flowing into the first barrel, and $g'(t)$ represents the rate of change of the amount of water flowing into the second barrel. So, $f'(t) + g'(t)$ represents the sum of the rates of change of water flowing into the two barrels and by the sum rule, $(f + g)'(t)$ also represents the rate of change of water flowing into the two barrels.

Chapter 2 Section 1 **Question 11 Page 84**

a) Find $h'(5)$.

$$h'(t) = 0 - 4.9(2t^{2-1})$$
$$= -9.8t$$

The rate of change of the skydiver at 5 s is –9.8(5) = –49 m/s.

b) Find t for $h'(t)$ = 1000 m.

$$1000 = 2500 - 4.9t^2$$
$$t = \sqrt{\frac{1500}{4.9}}$$
$$t \doteq 17.5$$

The skydiver's parachute opens at time t = 17.5 s.

c) Find $h'(17.5)$.

$$h'(17.5) = -9.8(17.5)$$
$$= 171.5$$

The rate of change of the height of the skydiver is 171.5 m/s.

Chapter 2 Section 1 **Question 12 Page 84**

$$h(t) = -0.5gt^2 + 250$$
$$h'(t) = -0.5g(2t^{2-1}) + 0$$
$$= -gt$$

Earth:

$$h'(t) = -9.8t$$
$$h'(4) = -9.8(4)$$
$$= -39.2 \text{ m/s}$$

Venus:

$$h'(t) = -8.9t$$
$$h'(4) = -8.9(4)$$
$$= -35.6 \text{ m/s}$$

Mars:

$$h'(t) = -3.7t$$
$$h'(4) = -3.7(4)$$
$$= -14.8 \text{ m/s}$$

Saturn:

$$h'(4) = -10.5t$$
$$h'(4) = -10.5(4)$$
$$= -42.0 \text{ m/s}$$

Neptune:

$$h'(4) = -11.2t$$
$$h'(4) = -11.2(4)$$
$$= -44.8 \text{ m/s}$$

Chapter 2 Section 1 **Question 13 Page 84**

a) $\frac{dy}{dx} = -6(4x^{4-1}) + 2(3x^{3-1}) + 0$

$= -24x^3 + 6x^2$

$m = -24(-1)^3 + 6(-1)^2$

$= 30$

b) Substitute the point (–1, –3) into the equation of the tangent line $y = 30x + b$ to solve for b.

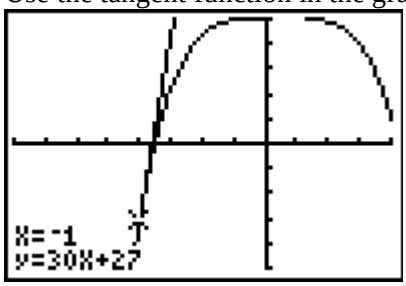
$-3 = 30(-1) + b$

$b = 27$

The equation of the tangent line is $y = 30x + 27$.

c) Answers may vary. For example:

Use the tangent function in the graphing calculator to find the equation of the tangent:



Chapter 2 Section 1 **Question 14 Page 85**

a) $\frac{dy}{dx} = -1.5(3x^{3-1}) + 3x^{1-1} - 2$

$= -4.5x^2 + 3$

At $x = 2$, the slope of the tangent is $4.5(2)^2 + 3 = -15$.

b) Substitute the point (2, –8) into the equation of the tangent line $y = -15x + b$ to solve for b.

$y = -15x + b$

$-8 = -15(2) + b$

$b = 22$

The equation of the tangent at (2, –8) is $y = -15x + 22$.

c) Answers may vary. For example:

Use the tangent function in the graphing calculator to find the equation of the tangent:

Chapter 2 Section 1 **Question 15 Page 85**

a) Find $h(2)$.

$h(2) = -4.9(2)^2 + 24.5(2) + 2$

$= 31.4$

The height is 31.4 m.

b) $h'(t) = -4.9(2t^{2-1}) + 24.5t^{1-1}$

$= -9.8t + 24.5$

$h'(1) = 14.7$ m/s; $h'(2) = 4.9$ m/s; $h'(4) = -14.7$ m/s; $h'(5) = -24.5$ m/s

c) After 5 s, the arrow is at the initial launch height of 2 m off the ground.

d) Solve $0 = -4.9t^2 + 24.5t + 2$ using the quadratic formula.

$$t = \frac{-24.5 \pm \sqrt{(24.5)^2 - 4(-4.9)(2)}}{2(-4.9)}$$

$t = -0.08$ s or $t = 5.08$ s

Since the time cannot be negative, the arrow hits the ground after 5.08 s.

e) The speed of the arrow at any time t is given by $h'(t)$. Solve for $h'(5.08)$.

$h'(5.08) = -9.8(5.08) + 24.5$

$= -25.28$

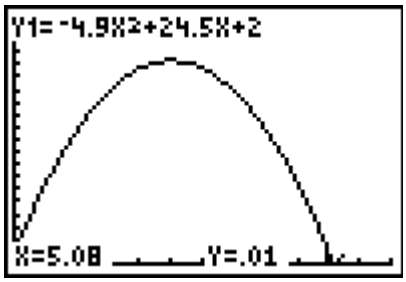
The speed of the arrow is –25.28 m/s.

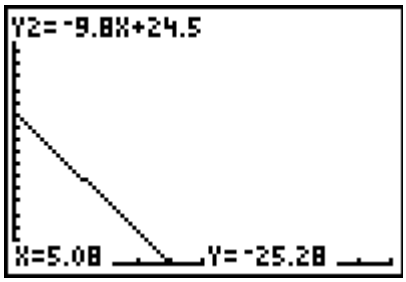
Explanations may vary. For example:

I plugged the time when the arrow hits the ground into the derivative function $h'(t)$. Also, the arrow is travelling in a negative (downwards) direction at this time so the speed is 25.28 m/s.

f) Answers may vary. For example:

Draw the graphs for $h(t)$ and $h'(t)$ from 0 to about 5.08 s and read values off of the plots.





Chapter 2 Section 1 **Question 16 Page 85**

a $f'(x) = 3x^{3-1} - 7$

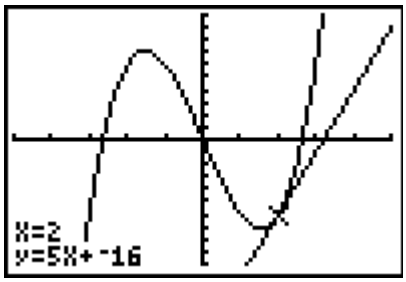
$5 = 3x^2 - 7$

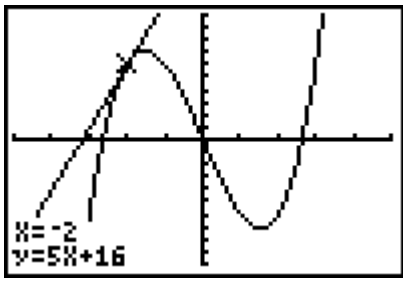
$x = -2$ or $x = 2$

The corresponding values of y are $f(2) = (2)^3 - 7(2)$, which is –6, and $f(-2) = (-2)^3 - 7(-2)$, which is 6. Therefore, the points are (2, –6) and (–2, 6).

b) If the tangent equation is $y = mx + b$ where $m = 5$, $x = 2$ and $y = -6$, then $b = -6 - 5(2) = 16$.
Similarly, when $m = 5$, $x = -2$ and $y = 6$, then $b = 6 - 5(-2) = 16$.
So, the equations of the tangents are $y = 5x - 16$ at (2, –6) and $y = 5x + 16$ at (–2, 6).

c) A CAS solution is shown in the **Technology Extension** on page 87.





Chapter 2 Section 1 **Question 17 Page 85**

a) The slope is given by $f'(x) = 4x$ and $g'(x) = 3x^2$.

If $4x = 3x^2$, then $x = 0$ or $x = \frac{4}{3}$.

b) When $x = 0$, $f(x) = 0$ and $g(x) = 0$ so the equation of the tangent line to f and g at this point is $y = 0$.

For the tangent of f at $x = \frac{4}{3}$, the y-coordinate is $f\left(\frac{4}{3}\right) = \frac{32}{9}$ and the slope, m, is $f'\left(\frac{4}{3}\right) = \frac{16}{3}$.

Substitute these values into $y = mx + b$ to find b.

$$\frac{32}{9} = \left(\frac{16}{3}\right)\left(\frac{4}{3}\right) + b$$

$$b = -\frac{32}{9}$$

The tangent equation for $f(x) = 2x^2$ at $x = \frac{4}{3}$ is $y = \frac{16}{3}x - \frac{32}{9}$.

For the tangent of g at $x = \frac{4}{3}$, the y-coordinate is $g\left(\frac{4}{3}\right) = \frac{64}{27}$ and the slope, m, is also $g'\left(\frac{4}{3}\right) = \frac{16}{3}$.

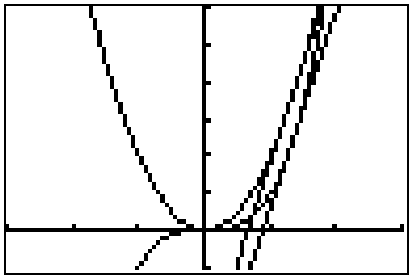
Substitute these values into $y = mx + b$ to find b.

$$\frac{64}{27} = \left(\frac{16}{3}\right)\left(\frac{4}{3}\right) + b$$

$$b = -\frac{128}{27}$$

The tangent equation for $g(x) = x^3$ at $x = \frac{4}{3}$ is $y = \frac{16}{3}x - \frac{128}{27}$.

c)



Chapter 2 Section 1 **Question 18 Page 85**

Use the definition of the derivative to prove that if $h(t) = f(t) + g(t)$, then $h'(t) = f'(t) + g'(t)$.

$$\begin{aligned} h'(t) &= \lim_{x \to 0} \frac{h(t+x) - h(t)}{x} \\ &= \lim_{x \to 0} \frac{\big(f(t+x) + g(t+x)\big) - \big(f(t) + g(t)\big)}{x} \\ &= \lim_{x \to 0} \frac{\big(f(t+x) - f(t)\big) + \big(g(t+x) - g(t)\big)}{x} \\ &= \lim_{x \to 0} \frac{f(t+x) - f(t)}{x} + \lim_{x \to 0} \frac{g(t+x) - g(t)}{x} \\ &= f'(t) + g'(t) \end{aligned}$$

Chapter 2 Section 1 **Question 19 Page 85**

a)

$$\begin{aligned} C(1000) &= 3450 + 1.5(1000) - 0.0001(1000)^2 \\ &= \$4850 \\ R(1000) &= 3.25(1000) \\ &= \$3250 \end{aligned}$$

b)

$$\begin{aligned} C'(x) &= 1.5 - 0.0002x \\ C'(1000) &= 1.5 - 0.0002(1000) \\ &= 1.3 \\ C'(3000) &= 1.5 - 0.0002(3000) \\ &= 0.9 \end{aligned}$$

The values give the rate of change of the cost when 1000 and 3000 yogurt bars are produced, which is the marginal cost of production per unit.

c) If $C'(x) = 0$, then $1.5 - 0.0002x = 0$ with solution $x = 7500$.
This is outside the given domain, $0 \le x \le 5000$, so $x \neq 7500$.

d) $R'(x) = 3.25$
This is the rate of change of revenue per yogurt bar produced.

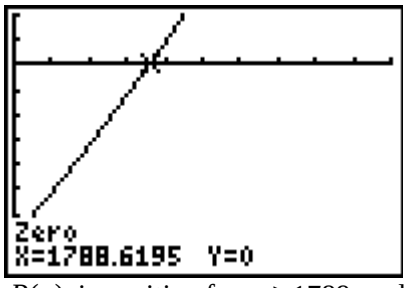
e) The profit function is $P(x) = R(x) - C(x)$.

$$P(x) = (3.25x) - (3450 + 1.5x - 0.0001x^2)$$

$$P(x) = 0.0001x^2 + 1.75x - 3450$$

f) Answers may vary. For example:

Use a graphing calculator to plot the function $P(x)$ and find points on the graph that are positive.



$P(x)$ is positive for $x \geq 1789$ and negative for $0 \leq x \leq 1788$, since x cannot be negative; This information tells the owners whether they are going to make or lose money based on how many yogurt bars they are producing and it shows the break even point.

Chapter 2 Section 1 **Question 20 Page 85**

a) The slope of the tangent to $f(x)$ is $f'(x)$.

$f'(x) = -2(3x^{3-1}) + 5(2x^{2-1}) - x^{1-1} + 0$

$= -6x^2 + 10x - 1$

The slope of the tangent to the curve at $x = 2$ is $f'(2) = -6(2)^2 + 10(2) - 1$, which is –5.

b) The slope of the normal is perpendicular to the slope of the tangent. Use $y = mx + b$ for the normal.

$m = \dfrac{-1}{f'(2)}$, which evaluates to $\dfrac{-1}{-5} = \dfrac{1}{5}$ or 0.2.

The y-coordinate at $x = 2$ is $f(2) = 5$ so (2, 5) lies on the normal. Use (2, 5) and $m = 0.2$ to find b.

$5 = 0.2(2) + b$ so $b = 4.6$

The equation of the normal to $f(x)$ at $x = 2$ is given by $y = 0.2x + 4.6$.

Chapter 2 Section 1 **Question 21 Page 85**

a) $f(x) = -4x^3 + 3x^{-1} + x^{\frac{1}{2}} - 2$

The slope of the tangent to $f(x)$ is $f'(x)$.

$f'(x) = -4(3x^{3-1}) + 3(-x^{-1-1}) + \dfrac{1}{2}x^{\frac{1}{2}-1} + 0$

$= -12x^2 - \dfrac{3}{x^2} + \dfrac{1}{2\sqrt{x}}$

The slope of the tangent to the curve at $x = 1$ is $f'(1) = -12(1)^2 - \dfrac{3}{(1)^2} + \dfrac{1}{2\sqrt{1}}$, which is $-\dfrac{29}{2}$.

b) The slope of the normal is perpendicular to the slope of the tangent. Use $y = mx + b$ for the normal.

$m = \dfrac{-1}{f'(1)}$, which evaluates to $\dfrac{-1}{\left(-\dfrac{29}{2}\right)} = \dfrac{2}{29}$.

The y-coordinate at $x = 1$ is $f(1) = -2$ so $(1, -2)$ lies on the normal. Use $(1, -2)$ and $m = \dfrac{2}{29}$ to find b.

$-2 = \dfrac{2}{29}(1) + b$ so $b = -\dfrac{60}{29}$

The equation of the normal to $f(x)$ at $x = 1$ is given by $y = \dfrac{2}{29}x - \dfrac{60}{29}$.

Chapter 2 Section 1 **Question 22 Page 86**

a) $f(x) = 4 - 4x^{\frac{1}{2}} + x$

The slope of the tangent to $f(x)$ is $f'(x)$.

$f'(x) = 0 - 4\left(\dfrac{1}{2}x^{\frac{1}{2}-1}\right) + x^{1-1}$

$= -\dfrac{2}{\sqrt{x}} + 1$

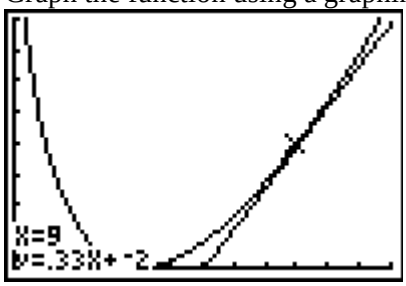
The slope of the tangent to the curve at $x = 9$ is $f'(9) = -\dfrac{2}{\sqrt{9}} + 1$, which is $\dfrac{1}{3}$.

The y-coordinate at $x = 9$ is $f(9) = 1$ so $(9, 1)$ lies on the tangent. Use $(9, 1)$ and $m = \dfrac{1}{3}$ to find b.

$1 = \dfrac{1}{3}(9) + b$ so $b = -2$.

The equation of the tangent to $f(x)$ at $x = 9$ is given by $y = \dfrac{1}{3}x - 2$.

b) Graph the function using a graphing calculator and use the tangent function.



Chapter 2 Section 1 **Question 23 Page 86**

a) $g(x) = 4x^{-2} - 12x^{-3} + x - 3$

The slope of the tangent to $g(x)$ is $g'(x)$.

$g'(x) = 4(-2x^{-2-1}) - 12(-3x^{-3-1}) + x^{1-1} + 0$

$= -\frac{8}{x^3} + \frac{36}{x^4} + 1$

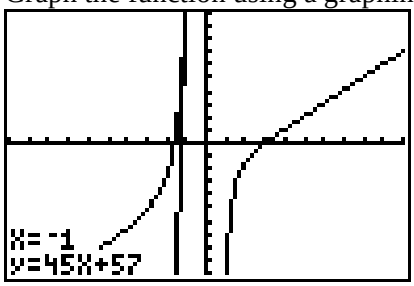
The slope of the tangent to the curve at $x = -1$ is $g'(-1) == -\frac{8}{(-1)^3} + \frac{36}{(-1)^4} + 1$, which is 45.

The y-coordinate at $x = -1$ is $g(-1) = 12$ so $(-1, 12)$ lies on the tangent. Use $(-1, 12)$ and $m = 45$ in $y = mx + b$ to find b.

$12 = 45(-1) + b$ so $b = 57$.

The equation of the tangent to $g(x)$ at $x = -1$ is given by $y = 45x + 57$.

b) Graph the function using a graphing calculator and use the tangent function.



Chapter 2 Section 1 **Question 24 Page 86**

a) The slope of the tangents will be 0 since the slope of the line $x = 1$ is undefined or infinite.

$\frac{dy}{dx} = -4x^{4-1} + 8(2x^{2-1})$

$= -4x^3 + 16x$

Now, find the values of x such that $\frac{dy}{dx} = 0$.

$-4x^3 + 16x = 0$

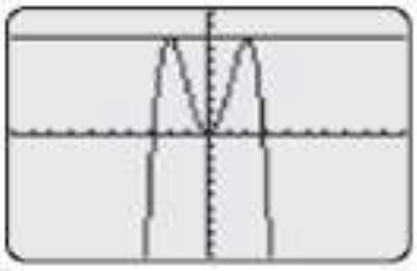
$-4x(x^2 - 4) = 0$

$-4x(x + 2)(x - 2) = 0$

The curve y will have tangents with slope zero at $x = 0$, $x = -2$, $x = 2$.
When $x = 0$, $y = 0$.
When $x = -2$ or $x = 2$, $y = 16$.

The equations of the tangent lines to y that are perpendicular to $x = -1$ are $y = 0$ at the point $(0, 0)$ and $y = 16$ at the point $(2, 16)$.

b)



Chapter 2 Section 1 **Question 25 Page 86**

a) $\frac{dy}{dx} = 6(3x^{3-1}) + 2(2x^{2-1})$

$= 18x^2 + 4x$

Show that there are no points on the curve where the tangents have slope –5, so set $\frac{dy}{dx} = -5$.

$-5 = 18x^2 + 4x$

$18x^2 + 4x + 5 = 0$

Use the quadratic formula to solve for x.

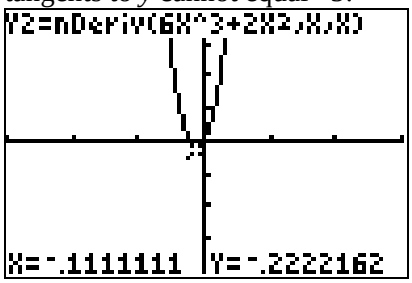
$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(18)(5)}}{2(18)}$$

But the discriminant $4^2 - 4(18)(5) = -344$, which is less than 0.

Therefore, since the equation cannot be solved for real values of x, there are no tangents to the curve with a slope of –5.

b) Answers may vary. For example:

Plot the first derivative. Notice that the value of $\frac{dy}{dx}$ cannot be less than $-\frac{2}{9}$, so the slopes of the tangents to y cannot equal –5.



Chapter 2 Section 1 **Question 26 Page 86**

Solutions to the Achievement Checks are shown in the Teacher's Resource.

Chapter 2 Section 1 **Question 27 Page 86**

$$f'(x) = a(3x^{3-1}) + b(2x^{2-1}) + 3x^{1-1} + 0$$
$$= 3ax^2 + 2bx + 3$$

Substitute $f'(-1) = 14$ and $x = -1$ to get $11 = 3a - 2b$ and call this equation (1).

Substitute $f(2) = 10$ and $x = 2$ into $f(x) = ax^3 + bx^2 + 3x - 2$ to get $6 = 8a + 4b$ and call this equation (2).

Either use substitution or elimination to solve for a and b.

For example, solving (2) + 2 × (1) gives $28 = 14a$ so $a = 2$.

Substitution of $a = 2$ into either of the two equations gives $b = -\frac{5}{2}$.

Chapter 2 Section 1 **Question 28 Page 86**

Let (x_1, y_1) represent the point where the tangent meets the curve y.

Find the slope of the tangent to the curve at any value of x, which is $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 2x^{2-1} + 0$$
$$m = 2x$$

Substitute the values $m = 2x_1$, $y_1 = x_1^2 - 2$ into the rise over run formula for the slope to find the values of x_1.

$$m = \frac{y_1 - (-5)}{x_1 - 1}$$
$$2x_1 = \frac{(x_1^2 - 2) + 5}{x_1 - 1}$$
$$2x_1(x_1 - 1) = x_1^2 + 3$$
$$0 = x_1^2 - 2x_1 - 3$$
$$0 = (x_1 + 1)(x_1 - 3)$$

The tangent lines meet y at the points $x_1 = -1$ and $x_1 = 3$.

When $x_1 = -1$, $m = 2(-1)$ or -2. Substituting $(1, -5)$ into $y = -2x + b$ gives $-5 = -2(1) + b$ so $b = -3$.

When $x_1 = 3$, $m = 2(3)$ or 6. Substituting $(1, -5)$ into $y = 6x + b$ gives $-5 = 6(1) + b$ so $b = -11$.

Therefore, the equations of the two tangents are $y = -2x - 3$ and $y = 6x - 11$.

Chapter 2 Section 1 **Question 29 Page 86**

a) Find the slope of the tangent to the curve at any value of x, which is $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 2(3x^{3-1}) - 3(2x^{2-1}) - 11x^{1-1} + 0$$
$$m = 6x^2 - 6x - 11$$

Either use the factor theorem or graph the function to find the solutions to the function when $y = 2$.

$$2 = 2x^3 - 3x^2 - 11x + 8$$
$$0 = 2x^3 - 3x^2 - 11x + 6$$
$$0 = (x + 2)(2x^2 - 7x + 3)$$
$$0 = (x + 2)(2x - 1)(x - 3)$$

When $y = 2$, the x-coordinates of the points on the graph are $x = -2$, $x = 0.5$, and $x = 3$.

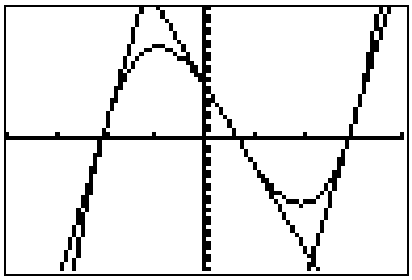
When $x = -2$, $m = 6(-2)^2 - 6(-2) - 11$ or 25.
Substituting $(-2, 2)$ into $y = 25x + b$ gives $2 = 25(-2) + b$ so $b = 52$.

When $x = 0.5$, $m = 6(0.5)^2 - 6(0.5) - 11$ or -12.5.
Substituting $(0.5, 2)$ into $y = -12.5x + b$ gives $2 = -12.5(0.5) + b$ so $b = 8.25$.

When $x = 3$, $m = 6(3)^2 - 6(3) - 11$ or 25.
Substituting $(3, 2)$ into $y = 25x + b$ gives $2 = 25(3) + b$ so $b = -73$.

Therefore, the equations of the three tangents are $y = 25x + 52$, $y = 12.5x + 8.25$, and $y = 25x - 73$.

b)



Chapter 2 Section 1 **Question 30 Page 86**

If the derivative of a polynomial function has an exponent of –1, then the original exponent of the polynomial must be 0. But $x^0 = 1$, and the derivative of 1 is 0, not x^{-1}. Therefore, there does not exist a polynomial function with derivative x^{-1}.

Chapter 2 Section 1 **Question 31 Page 86**

Consider polynomials $p(x)$ and $q(x)$ of degree $m > 0$ and $n > 0$.
If the equation $p(x) - q(x)$ has $m + n$ real roots, which of the following choices must be true? C
p is of degree m and q is of degree n.
So the equation $p(x) - q(x) = 0$ has degree $\max\{m, n\} < m + n$ since $m \neq 0 \neq n$.
That the equation has more than $\max\{m, n\}$ real roots implies that $p(x) - q(x)$ must be identically zero.
Hence the equation has infinitely many real roots.

Chapter 2 Section 1 **Question 32 Page 86**

E

Chapter 2 Section 2 **The Product Rule**

Chapter 2 Section 2 **Question 1 Page 93**

a) i) $f(x) = 2x^2 + 7x - 4$

$f'(x) = 4x + 7$

ii) $f'(x) = \left[\frac{d}{dx}(x+4)\right](2x-1) + (x+4)\left[\frac{d}{dx}(2x-1)\right]$

$= (1)(2x-1) + (x+4)(2)$

$= 4x + 7$

b) i) $h(x) = -10x^2 + 11x - 3$

$h'(x) = -20x + 11$

ii) $h'(x) = \left[\frac{d}{dx}(5x-3)\right](1-2x) + (5x-3)\left[\frac{d}{dx}(1-2x)\right]$

$= (5)(1-2x) + (5x-3)(-2)$

$= -20x + 11$

c) i) $h(x) = -3x^2 - 5x + 8$

$h'(x) = -6x - 5$

ii) $h'(x) = \left[\frac{d}{dx}(-x+1)\right](3x+8) + (-x+1)\left[\frac{d}{dx}(3x+8)\right]$

$= (-1)(3x+8) + (-x+1)(3)$

$= -6x - 5$

d) i) $g(x) = -6x^2 + 11x - 4$

$g'(x) = -12x + 11$

ii) $g'(x) = \left[\frac{d}{dx}(2x-1)\right](4-3x) + (2x-1)\left[\frac{d}{dx}(4-3x)\right]$

$= (2)(4-3x) + (2x-1)(-3)$

$= -12x + 11$

Chapter 2 Section 2 **Question 2 Page 93**

a) $f'(x) = \left[\frac{d}{dx}(5x+2)\right](8x-6) + (5x+2)\left[\frac{d}{dx}(8x-6)\right]$

$= (5)(8x-6) + (5x+2)(8)$

$= 80x - 14$

b) $h'(t) = \left[\frac{d}{dt}(-t+4)\right](2t+1) + (-t+4)\left[\frac{d}{dt}(2t+1)\right]$

$= (-1)(2t+1) + (-t+4)(2)$

$= -4t + 7$

c) $p'(x) = \left[\frac{d}{dx}(-2x+3)\right](x-9) + (-2x+3)\left[\frac{d}{dx}(x-9)\right]$

$= (-2)(x-9) + (-2x+3)(1)$

$= -4x + 21$

d) $g'(x) = \left[\frac{d}{dx}(x^2+2)\right](4x-5) + (x^2+2)\left[\frac{d}{dx}(4x-5)\right]$

$= (2x)(4x-5) + (x^2+2)(4)$

$= 12x^2 - 10x + 8$

e) $f'(x) = \left[\frac{d}{dx}(1-x)\right](x^2-5) + (1-x)\left[\frac{d}{dx}(x^2-5)\right]$

$= (-1)(x^2-5) + (1-x)(2x)$

$= -3x^2 + 2x + 5$

f) $h'(t) = \left[\frac{d}{dt}(t^2+3)\right](3t^2-7) + (t^2+3)\left[\frac{d}{dt}(3t^2-7)\right]$

$= (2t)(3t^2-7) + (t^2+3)(6t)$

$= 4t(3t^2+1)$

Chapter 2 Section 2 **Question 3 Page 93**

a) $M'(u) = \left[\frac{d}{du}(1-4u^2)\right](u+2) + (1-4u^2)\left[\frac{d}{du}(u+2)\right]$

$= (-8u)(u+2) + (1-4u^2)(1)$

$= -12u^2 - 16u + 1$

b) $g'(x) = \left[\frac{d}{dx}(-x+3)\right](x-10) + (-x+3)\left[\frac{d}{dx}(x-10)\right]$

$= (-1)(x-10) + (-x+3)(1)$

$= -2x + 13$

c) $p'(n) = \left[\frac{d}{dn}(5n+1)\right](-n^2+3) + (5n+1)\left[\frac{d}{dn}(-n^2+3)\right]$

$= (5)(-n^2+3) + (5n+1)(-2n)$

$= -15n^2 - 2n + 15$

d) $A'(r) = \left[\frac{d}{dr}(1+2r)\right](2r^2-6) + (1+2r)\left[\frac{d}{dr}(2r^2-6)\right]$

$= (2)(2r^2-6) + (1+2r)(4r)$

$= 12r^2 + 4r - 12$

e) $b'(k) = \left[\frac{d}{dk}(-0.2k+4)\right](2-k) + (-0.2k+4)\left[\frac{d}{dk}(2-k)\right]$

$= (-0.2)(2-k) + (-0.2k+4)(-1)$

$= 0.4k - 4.4$

Chapter 2 Section 2 **Question 4 Page 93**

a) $f(x) = 5x^2 + 7;\ g(x) = 21 - 3x$

b) $f(x) = -4x^3 + 8x;\ g(x) = 2x^2 - 4x$

c) $f(x) = 2x^3 - x;\ g(x) = 0.5x^2 + x$

d) $f(x) = -\frac{3}{4}x^4 + 6x;\ g(x) = 7x - \frac{2}{3}x^2$

Chapter 2 Section 2 **Question 5 Page 94**

a) $f'(x) = \left[\frac{d}{dx}(x^2-2x)\right](3x+1) + (x^2-2x)\left[\frac{d}{dx}(3x+1)\right]$

$= (2x-2)(3x+1) + (x^2-2x)(3)$

$= 9x^2 - 10x - 2$

$f'(-2) = 9(-2)^2 - 10(-2) - 2$

$= 54$

b) $f'(x) = \left[\frac{d}{dx}\left(1-x^3\right)\right](-x^2+2) + (1-x^3)\left[\frac{d}{dx}\left(-x^2+2\right)\right]$

$= (-3x^2)(-x^2+2) + (1-x^3)(-2x)$

$= 5x^4 - 6x^2 - 2x$

$f'(-2) = 5(-2)^4 - 6(-2)^2 - 2(-2)$

$= 60$

c) $f'(x) = \left[\frac{d}{dx}\left(3x-1\right)\right](2x+5) + (3x-1)\left[\frac{d}{dx}\left(2x+5\right)\right]$

$= (3)(2x+5) + (3x-1)(2)$

$= 12x + 13$

$f'(-2) = 12(-2) + 13$

$= -11$

d) $f'(x) = \left[\frac{d}{dx}\left(-x^2+x\right)\right](5x^2-1) + (-x^2+x)\left[\frac{d}{dx}\left(5x^2-1\right)\right]$

$= (-2x+1)(5x^2-1) + (-x^2+x)(10x)$

$= -20x^3 + 15x^2 + 2x - 1$

$f'(-2) = -20(-2)^3 + 15(-2)^2 + 2(-2) - 1$

$= 215$

e) $f'(x) = \left[\frac{d}{dx}\left(2x-x^2\right)\right](7x+4) + (2x-x^2)\left[\frac{d}{dx}\left(7x+4\right)\right]$

$= (2-2x)(7x+4) + (2x-x^2)(7)$

$= -21x^2 + 20x + 8$

$f'(-2) = -21(-2)^2 + 20(-2) + 8$

$= -116$

f) $f'(x) = \left[\frac{d}{dx}\left(-5x^3+x\right)\right](-x+2) + (-5x^3+x)\left[\frac{d}{dx}\left(-x+2\right)\right]$

$= (-15x^2+1)(-x+2) + (-5x^3+x)(-1)$

$= 20x^3 - 30x^2 - 2x + 2$

$f'(-2) = 20(-2)^3 - 30(-2)^2 - 2(-2) + 2$

$= -274$

Chapter 2 Section 2 **Question 6 Page 94**

a) $f'(x) = \left[\frac{d}{dx}\left(x^2 - 3\right)\right](x^2 + 1) + (x^2 - 3)\left[\frac{d}{dx}\left(x^2 + 1\right)\right]$

$= (2x)(x^2 + 1) + (x^2 - 3)(2x)$

$= 4x^3 - 4x$

$f'(-4) = 4(-4)^3 - 4(-4)$

$m = -240$

The y-coordinate at $x = -4$ is $f(-4) = 221$.

Use the point $(-4, 221)$ and $m = -240$ in the equation $y = mx + b$ to find b.

$221 = -240(-4) + b$ so $b = -739$.

The equation of the tangent is $y = -240x - 739$.

b) $g'(x) = \left[\frac{d}{dx}\left(2x^2 - 1\right)\right](-x^2 + 3) + (2x^2 - 1)\left[\frac{d}{dx}\left(-x^2 + 3\right)\right]$

$= (4x)(-x^2 + 3) + (2x^2 - 1)(-2x)$

$= -8x^3 + 14x$

$g'(2) = -8(2)^3 + 14(2)$

$m = -36$

The y-coordinate at $x = 2$ is $g(2) = -7$.

Use the point $(2, -7)$ and $m = -36$ in the equation $y = mx + b$ to find b.

$-7 = -36(2) + b$ so $b = 65$.

The equation of the tangent is $y = -36x + 65$.

c)

$h'(x) = \left[\frac{d}{dx}\left(x^4 + 4\right)\right](2x^2 - 6) + (x^4 + 4)\left[\frac{d}{dx}\left(2x^2 - 6\right)\right]$

$= (4x^3)(2x^2 - 6) + (x^4 + 4)(4x)$

$= 12x^5 - 24x^3 + 16x$

$h'(-1) = 12(-1)^5 - 24(-1)^3 + 16(-1)$

$m = -4$

The y-coordinate at $x = -1$ is $h(-1) = -20$.

Use the point $(-1, -20)$ and $m = -4$ in the equation $y = mx + b$ to find b.

$-20 = -4(-1) + b$ so $b = -24$

The equation of the tangent is $y = -4x - 24$.

d) $p'(x) = \left[\frac{d}{dx}\left(-x^3+2\right)\right](4x^2-3)+(-x^3+2)\left[\frac{d}{dx}\left(4x^2-3\right)\right]$

$= (-3x^2)(4x^2-3)+(-x^3+2)(8x)$

$= -20x^4+9x^2+16x$

$p'(3) = -20(3)^4+9(3)^2+16(3)$

$m = -1491$

The y-coordinate at $x = 3$ is $p(3) = -825$.

Use the point $(3, -825)$ and $m = -1491$ in the equation $y = mx + b$ to find b.

$-825 = -1491(3) + b$ so $b = 3648$

The equation of the tangent is $y = -1491x + 3648$.

Chapter 2 Section 2 **Question 7 Page 94**

a) $\frac{dy}{dx} = \left[\frac{d}{dx}\left(-4x+3\right)\right](x+3)+(-4x+3)\left[\frac{d}{dx}\left(x+3\right)\right]$

$= (-4)(x+3)+(-4x+3)(1)$

$= -8x-9$

Find the value of x that satisfies $\frac{dy}{dx} = 0$.

$0 = -8x-9$

$x = -\frac{9}{8}$

When $x = -\frac{9}{8}$, $y = \frac{225}{16}$ so the point $\left(-\frac{9}{8}, \frac{225}{16}\right)$ on the curve corresponds to the slope $m = 0$.

b) $\frac{dy}{dx} = \left[\frac{d}{dx}\left(5x+7\right)\right](2x-9)+(5x+7)\left[\frac{d}{dx}\left(2x-9\right)\right]$

$= (5)(2x-9)+(5x+7)(2)$

$= 20x-31$

Find the value of x that satisfies $\frac{dy}{dx} = \frac{2}{5}$.

$\frac{2}{5} = 20x-31$

$x = 1.57$

When $x = 1.57$, $y = -87.021$ so the point $(1.57, -87.021)$ on the curve corresponds to the slope $m = \frac{2}{5}$.

c) $\frac{dy}{dx} = \left[\frac{d}{dx}\left(2x-1\right)\right](-4+x^2) + (2x-1)\left[\frac{d}{dx}\left(-4+x^2\right)\right]$

$= (2)(-4+x^2) + (2x-1)(2x)$

$= 6x^2 - 2x - 8$

Find the value of x that satisfies $\frac{dy}{dx} = 3$.

$3 = 6x^2 - 2x - 8$

$0 = 6x^2 - 2x - 11$

$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(6)(-11)}}{2(6)}$

$x = 1.53,\ x = -1.20$

When $x = 1.53$, $y = 3.42$ and when $x = -1.20$, $y = 8.70$ so the points (1.53, 3.42) and (–1.20, 8.70) on the curve correspond to the slope $m = 3$.

d) $\frac{dy}{dx} = \left[\frac{d}{dx}\left(x^2-2\right)\right](2x+1) + (x^2-2)\left[\frac{d}{dx}\left(2x+1\right)\right]$

$= (2x)(2x+1) + (x^2-2)(2)$

$= 6x^2 + 2x - 4$

Find the value of x that satisfies $\frac{dy}{dx} = -2$.

$-2 = 6x^2 + 2x - 4$

$0 = 6x^2 + 2x - 2$

$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(6)(-2)}}{2(6)}$

$x = 0.43,\ x = -0.77$

When $x = 0.43$, $y = -3.38$ and when $x = -0.77$, $y = 0.76$ so the points (0.43, –3.38) and (–0.77, 0.76) on the curve correspond to the slope $m = -2$.

Chapter 2 Section 2 **Question 8 Page 94**

a) In t years from now, there will be $(120 + 10t)$ trees and the yield will be $(280 + 15t)$ apples per tree.

$Y(t) = (120+10t)(280+15t)$

b) $Y'(t) = \left[\frac{d}{dt}(120+10t)\right](280+15t) + (120+10t)\left[\frac{d}{dt}(280+15t)\right]$

$= (10)(280+15t) + (120+10t)(15)$

$= 4600 + 300t$

$Y'(2) = 4600 + 300(2)$

$= 5200$

The yield is 5200 apples/year. This represents the annual rate of change of apple production after 2 years.

c) $Y'(6) = 4600 + 300(6)$

$= 6400$

The yield is 6400 apples/years. This represents the rate of change of apple production at $t = 6$ years.

Chapter 2 Section 2 **Question 9 Page 94**

a) $\frac{dy}{dx} = \left[\frac{d}{dx}(5x^2 - x + 1)\right](x+2) + (5x^2 - x + 1)\left[\frac{d}{dx}(x+2)\right]$

$= (10x - 1)(x+2) + (5x^2 - x + 1)(1)$

$= 15x^2 + 18x - 1$

b) $\frac{dy}{dx} = \left[\frac{d}{dx}(1 - 2x^3 + x^2)\right]\left(\frac{1}{x^3} + 1\right) + (1 - 2x^3 + x^2)\left[\frac{d}{dx}\left(\frac{1}{x^3} + 1\right)\right]$

$= (-6x^2 + 2x)\left(\frac{1}{x^3} + 1\right) + (1 - 2x^3 + x^2)\left(-\frac{3}{x^4}\right)$

$= -6x^2 + 2x - \frac{1}{x^2} - \frac{3}{x^4}$

c) $\frac{dy}{dx} = \left[\frac{d}{dx}(-x^2)\right](4x-1)(x^3+2x+3) + (-x^2)\left[\frac{d}{dx}(4x-1)\right](x^3+2x+3) + (-x^2)(4x-1)\left[\frac{d}{dx}(x^3+2x+3)\right]$

$= (-2x)(4x-1)(x^3+2x+3) + (-x^2)(4)(x^3+2x+3) + (-x^2)(4x-1)(3x^2+2)$

$= (-8x^5 + 2x^4 - 16x^3 - 20x^2 + 6x) + (-4x^5 - 8x^3 - 12x^2) + (-12x^5 + 3x^4 - 8x^3 + 2x^2)$

$= -24x^5 + 5x^4 - 32x^3 - 30x^2 + 6x$

d) $\frac{dy}{dx} = 2\left[\frac{d}{dx}\left(2x^2 - x^{\frac{1}{2}}\right)\right]\left(2x^2 - x^{\frac{1}{2}}\right)$

$= 2\left(4x - \frac{1}{2\sqrt{x}}\right)\left(2x^2 - \sqrt{x}\right)$

$= 16x^3 - 10x^{\frac{3}{2}} + 1$

e) $\frac{dy}{dx} = 2\left[\frac{d}{dx}\left(-3x^2 + x + 1\right)\right]\left(-3x^2 + x + 1\right)$

$= 2(-6x + 1)(-3x^2 + x + 1)$

$= 36x^3 - 18x^2 - 10x + 2$

Chapter 2 Section 2 **Question 10 Page 94**

a) One method is expanding the function and then differentiating.

$R(n) = (81 - 4n)(6.50 + 0.50n)$

$= 526.50 + 40.50n - 26n - 2n^2$

$= 526.50 + 14.50n - 2n^2$

$R'(n) = 14.50 - 4n$

Another method is differentiating the function using the product rule.

$R'(n) = \left[\frac{d}{dn}\left(81 - 4n\right)\right](6.50 + 0.50n) + (81 - 4n)\left[\frac{d}{dn}\left(6.50 + 0.50n\right)\right]$

$= (-4)(6.50 + 0.50n) + (81 - 4n)(0.50)$

$= 14.50 - 4n$

b) $R'(4) = 14.5 - 4(4)$

$= -1.5$

This tells the manager that for 4 increases of \$0.50, the miniature golf club is losing revenue at the ticket price of \$8.50.

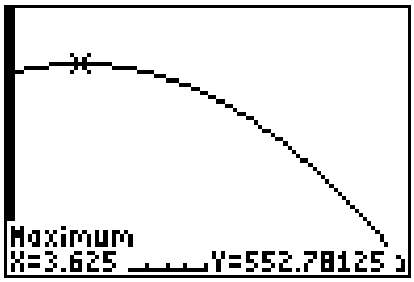
c) $R'(n) = 0$

$14.5 - 4n = 0$

$n = 3.625$

This tells the manager that the maximum price for a ticket is 6.50 + 0.50(3.625) = \$8.31.

d)



The maximum revenue is \$552.78. This occurs for n = 3.625, which is the value of n found in part c). As n grows larger than 3.625, the revenue decreases.

e) The maximum value of the function is found at $R'(n) = 0$. $R(n)$ is maximized at n = 3.625.

Chapter 2 Section 2 **Question 11 Page 95**

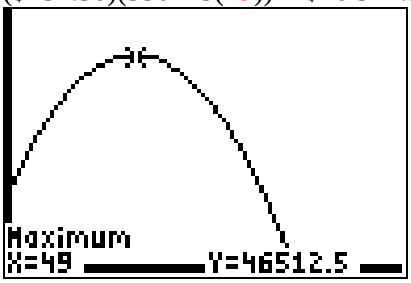
a) For x increases in \$2.50 increments the cost of a visit will be $(30 + 2.50x)$ dollars.
The number of clients for x increases will be $(550 - 5x)$.
$R(x) = (30 + 2.50x)(550 - 5x)$

b) $R'(x) = \left[\frac{d}{dx}(30 + 2.5x)\right](550 - 5x) + (30 + 2.5x)\left[\frac{d}{dn}(550 - 5x)\right]$
$= (2.5)(550 - 5x) + (30 + 2.5x)(-5)$
$= 1225 - 25x$

c) $R'(3) = 1225 - 25(3)$
$= 1150$
This represents the rate of change of revenue at a \$7.50 increase.

d) $R'(x) = 0$
$1225 - 25x = 0$
$x = 49$

e) The owner could maximize revenue by making 49 increases of \$2.50 increments. A visit to the hair salon would then cost 30 + 49(\$2.50) = \$152.50 resulting in a maximum revenue of (\$152.50)(550 – 5(49)) = \$46 512.50.



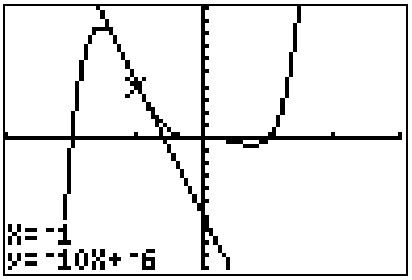
Chapter 2 Section 2 **Question 12 Page 95**

a)

$f'(x) = \left[\frac{d}{dx}(2x^2)\right](x^2 + 2x)(x - 1) + 2x^2\left[\frac{d}{dx}(x^2 + 2x)\right](x - 1) + 2x^2(x^2 + 2x)\left[\frac{d}{dx}(x - 1)\right]$
$= 4x(x^2 + 2x)(x - 1) + 2x^2(2x + 2)(x - 1) + 2x^2(x^2 + 2x)(1)$
$= (4x^4 + 4x^3 - 8x^2) + (4x^4 - 4x^2) + (2x^4 + 4x^3)$
$= 10x^4 + 8x^3 - 12x^2$

$f'(-1) = 10(-1)^4 + 8(-1)^3 - 12(-1)^2$ so the slope at $x = -1$ is $m = -10$.
The y-coordinate of the point on the graph at $x = -1$ is $f(-1) = 4$.
If the tangent to the graph is $y = mx + b$ then use the point $(-1, 4)$ and $m = -10$ to find b.
$4 = -10(-1) + b$
$b = -6$
The equation of the tangent line is $y = -10x - 6$.

b)



Chapter 2 Section 2 **Question 13 Page 95**

a)

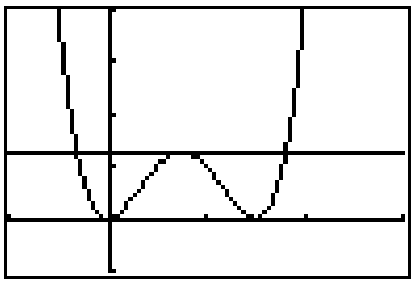
$$f'(x) = 2\left[\frac{d}{dx}\left(3x - 2x^2\right)\right](3x - 2x^2)$$
$$= 2(3 - 4x)(3x - 2x^2)$$
$$= 16x^3 - 36x^2 + 18x$$

If the tangent line is parallel to the x-axis, then its slope is $m = 0$.
To find the points on the graph where the slope of its tangent is 0, set $f'(x) = 0$ and solve for x.

$$16x^3 - 36x^2 + 18x = 0$$
$$2x(8x^2 - 18x + 9) = 0$$
$$2x(4x - 3)(2x - 3) = 0$$

The solutions are $x = 0$, $x = \frac{3}{2}$, $x = \frac{3}{4}$.
Therefore, the points on the graph corresponding to tangents with slope parallel to the x-axis are (0, 0), $\left(\frac{3}{2}, 0\right)$ and $\left(\frac{3}{4}, \frac{81}{64}\right)$.

b)



Chapter 2 Section 2 **Question 14 Page 95**

a) At the start, $t = 0$ so the amount of gas is $V(0) = 90$ L.

b) $V'(t) = 90\left(2\left[\frac{d}{dt}\left(1-\frac{t}{18}\right)\right]\left(1-\frac{t}{18}\right)\right)$

$= 180\left(-\frac{1}{18}\right)\left(1-\frac{t}{18}\right)$

$= -10+\frac{5}{9}t$

After 12 h, the rate of gas leakage is $V'(12) = -10+\frac{5}{9}(12)$, which is –3.33.

So the tank is leaking 3.33 L/h after 12 h.

c) Set $V(t) = 40$ and find t.

$40 = 90\left(1-\frac{t}{18}\right)^2$

$\frac{4}{9} = \left(1-\frac{t}{18}\right)^2$

$\frac{2}{3} = 1-\frac{t}{18}$

$t = \frac{20}{3}$

Substitute $t = \frac{20}{3}$ into the derivative function to find $V'(t)$.

$V'(\frac{20}{3}) = -10+\frac{5}{9}\left(\frac{20}{3}\right)$

$= -6.30$

The gas is leaking at 6.30 L/h when there is 40 L in the tank.

Chapter 2 Section 2 **Question 15 Page 95**

a) For $t = 0$, the population of fish is $p(0) = 15(30)(8)$, which is 3600 fish.

b) $p'(t) = 15(2t)(t+8) + 15(t^2+30)(1)$

$p'(t) = 45t^2 + 240t + 450$

$p'(3) = 45(9) + 720 + 450$

$= 1575$

In 3 years, the rate of change of the fish population is 1575 fish per year.

c) Set $p(t) = 5000$ and solve for t.

$5000 = 15(t^2+30)(t+8)$

$5000 = 15(t^3 + 30t + 8t^2 + 240)$

$5000 = 15t^3 + 120t^2 + 450t + 3600$

$0 = 3t^3 + 24t^2 + 90t - 280$

Use CAS or a graphing calculator to find the values of t.

$t = 1.908$ years

$p'(1.908) = 45(1.908)^2 + 240(1.908) + 450$

$= 1071.74$

The rate of change of the fish is 1071.7 fish per year.

d) Doubling the number of fish gives $2 \times 3600 = 7200$ fish. Set $p(t) = 7200$ and solve for t.

$7200 = 15t^3 + 120t^2 + 450t + 3600$

$0 = 15t^3 + 120t^2 + 450t - 3600$

$0 = 3t^3 + 24t^2 + 90t - 720$

Use CAS or a graphing calculator to find the values of t.

$t = 3.45$ years

$p'(3.45) = 45(3.45)^2 + 240(3.45) + 450$

$= 1813.61$

The population will double 3.45 years from now and the rate of change at this time is 1813.6 fish/year.

Chapter 2 Section 2 **Question 16 Page 95**

a) After n increases the cost will be $(1.75 + 0.25n)$ dollars and the number of sales will be $(150 - 10n)$.
$R(n) = (1.75 + 0.25n)(150 - 10n)$

b) Compare $R'(n)$ for $n = 1, 3, 4, 5$, and 6 increases.
$R'(n) = (0.25)(150 - 10n) + (1.75 + 0,25n)(-10)$
$= 20 - 5n.$
$R'(1) = 15,\ R'(3) = 5,\ R'(4) = 0,\ R'(5) = -5,\ R'(6) = -10$

c) $R'(n) = 0$
$20 - 5n = 0$
$n = 4$
This means that the profit is maximized after 4 increases of \$0.25.

d) The profit function is the revenue function minus the cost function.
The cost function is the cost to make one smoothie multiplied by the number of smoothies.
$P(n) = R(n) - C(n)$
$= R(n) - 0.75(150 - 10n)$
$P'(n) = R'(n) - 0.75(-10)$
$= R'(n) + 7.5$
$P'(1) = 22.50,\ P'(3) = 12.50,\ P'(4) = 7.50,\ P'(5) = 2.50,\ P'(6) = -2.50$

e) The maximum profit occurs when $P'(n) = 0$.
$0 = R'(n) + 7.5$
$0 = (20 - 5n) + 7.5$
$n = 5.5$
When $n = 3.85$, the cost of a smoothie will be \$1.75 + \$0.25(5.5) = \$3.13. This is the price for maximum profit.

f) Answers may vary. For example:

The profit numbers have a rate of change that is 7.5 greater than that of the revenue numbers.

Chapter 2 Section 2 **Question 17 Page 96**

a) i) $\frac{dy}{dx} = (2x-3)(x^2-3x) + (x^2-3x)(2x-3)$

ii) $\frac{dy}{dx} = (6x^2+1)(2x^3+x) + (2x^3+x)(6x^2+1)$

iii) $\frac{dy}{dx} = (-4x^3+10x)(-x^4+5x^2) + (-x^4+5x^2)(-4x^3+10x)$

Both terms in the answers are the same.

b) $\frac{d}{dx}[f(x)]^2 = 2f(x)\frac{d}{dx}[f(x)]$

c) $\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$

Set $f(x) = g(x)$.

$\frac{d}{dx}[f(x)f(x)] = f(x)\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}[f(x)]$

$\frac{d}{dx}[f(x)]^2 = 2f(x)\frac{d}{dx}[f(x)]$

d) i) $\frac{dy}{dx} = 2(x^2-3x)(2x-3)$

ii) $\frac{dy}{dx} = 2(2x^3+x)(6x^2+1)$

iii) $\frac{dy}{dx} = 2(-x^4+5x^2)(-4x^3+10x)$

Since $\frac{d}{dx}[f(x)]^2 = 2f(x)\frac{d}{dx}[f(x)]$, the answers are the same as in part a).

Chapter 2 Section 2 **Question 18 Page 96**

a) $[fgh]' = f'gh + f(gh)'$
$= f'gh + f(g'h + fgh')$
$= f'gh + fg'h + fgh'$

b) $f'(x) = (2x)(3x^4 - 2)(5x + 1) + (x^2 + 4)(12x^3)(5x + 1) + (x^2 + 4)(3x^4 - 2)(5)$

c) Expand and differentiate using the sum rule and power rule.

$f(x) = 15x^7 + 3x^6 + 60x^5 + 12x^4 - 10x^3 - 2x^2 - 40x - 8$
$f'(x) = 105x^6 + 18x^5 + 300x^4 + 48x^3 - 30x^2 - 4x - 40$

d) Parts b) and c) simplify to the same polynomial.

$f'(x) = 2x(3x^4 - 2)(5x + 1) + (x^2 + 4)(12x^3)(5x + 1) + (x^2 + 4)(3x^4 - 2)(5)$
$= 2x(15x^5 + 3x^4 - 10x - 2) + 12x^3(5x^3 + x^2 + 20x + 4) + 5(3x^6 - 2x^2 + 12x^4 - 8)$
$= (30x^6 + 6x^5 - 20x^2 - 4x) + (60x^6 + 12x^5 + 240x^4 + 48x^3) + (15x^6 + 60x^4 - 10x^2 - 40)$
$= 105x^6 + 18x^5 + 300x^4 + 48x^3 - 30x^2 - 4x - 40$

Chapter 2 Section 2 **Question 19 Page 96**

a) $\frac{d}{dx}\left[f(x)\right]^3 = 3f(x)^2 \frac{d}{dx}[f(x)]$

b) Using $[fgh]' = f'gh + fg'h + fgh'$ let $g = h = f$.
$[fff]' = f'ff + ff'f + fff'$
$[f^3]' = f'f^2 + f^2f' + f^2f'$
$\frac{d}{dx}\left[f(x)\right]^3 = 3f(x)^2 \frac{d}{dx}[f(x)]$

c) i)
$\frac{dy}{dx} = 3(4x^2 - x)^2(8x - 1)$
$= 384x^5 - 240x^4 + 48x^3 - 3x^2$

ii) $\frac{dy}{dx} = 3(x^3 + x)^2(3x^2 + 1)$
$= 9x^8 + 21x^6 + 15x^4 + 3x^2$

iii)
$\frac{dy}{dx} = 3(-2x^4 + x^2)^2(-8x^3 + 2x)$
$= -96x^{11} + 120x^9 - 48x^7 + 6x^5$

Chapter 2 Section 2 **Question 20 Page 96**

a) $h'(x) = 3x^2 f(x) + x^3 f'(x)$

b) $p'(x) = g(x)(4x^3 - 6x) + (x^4 - 3x^2)g'(x)$

c) $q'(x) = (-12x^3 - 16x + 5) f(x) + (-3x^4 - 8x^2 + 5x + 6) f'(x)$

d) $r'(x) = 4 f(x)(2x^3 + 5x^2)(3x^2 + 5x) + (2x^3 + 5x^2)^2 f'(x)$

Chapter 2 Section 2 **Question 21 Page 96**

a) $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}\left(\frac{d}{dx}[f(x)]\right)$

b) $n = 4: \ \frac{dy}{dx} = 4(2x^3 + x^2)^3(6x^2 + 2x)$

$n = 5: \ \frac{dy}{dx} = 5(2x^3 + x^2)^4(6x^2 + 2x)$

$n = 6: \ \frac{dy}{dx} = 6(2x^3 + x^2)^5(6x^2 + 2x)$

Chapter 2 Section 2 **Question 22 Page 96**

C

Chapter 2 Section 2 **Question 23 Page 96**

E

Chapter 2 Section 3 **Velocity, Acceleration, and Second Derivatives**

Chapter 2 Section 3 **Question 1 Page 106**

a) $\frac{dy}{dx} = 6x^2,\ \frac{d^2y}{dx^2} = 12x$

b) $s'(t) = -4t^3 + 15t^2 - 4t + 1,\ s''(t) = -12t^2 + 30t - 4$

c) $h'(x) = x^5 - x^4,\ h''(x) = 5x^4 - 4x^3$

d) $f'(x) = \frac{3}{4}x^2 - 4x,\ f''(x) = \frac{3}{2}x - 4$

e) $g'(x) = 5x^4 + 12x^3 - 6x^2,\ g''(x) = 20x^3 + 36x^2 - 12x$

f) $h'(t) = -9.8t + 25,\ h''(t) = -9.8$

Chapter 2 Section 3 **Question 2 Page 106**

a) $f'(x) = 8x^3 - 9x^2 + 12x,\ f''(x) = 24x^2 - 18x + 12$

$f''(3) = 24(3)^2 - 18(3) + 12$

$= 174$

b) $f'(x) = 12x^2 - 5,\ f''(x) = 24x$

$f''(3) = 24(3)$

$= 72$

c) $f'(x) = -2x^4 - 3x^2,\ f''(x) = -8x^3 - 6x$

$f''(3) = -8(3)^3 - 6(3)$

$= -234$

d) $f'(x) = 6x(1 - x) + (3x^2 + 2)(-1)$

$= 6x - 6x^2 - 3x^2 - 2$

$= -9x^2 + 6x - 2$

$f''(x) = -18x + 6$

$f''(3) = -18(3) + 6$

$= -48$

e) $f'(x) = (6x - 5)(2x) + 6(x^2 + 4)$

$f'(x) = 12x^2 - 10x + 6x^2 + 24$

$= 18x^2 - 10x + 24$

$f''(x) = 36x - 10$

$f''(3) = 36(3) - 10$

$= 98$

f) $f'(x) = 20x^4 - 2x^3 - 6x,\ f''(x) = 80x^3 - 6x^2 - 6$

$f''(3) = 80(3)^3 - 6(3)^2 - 6$

$= 2100$

Chapter 2 Section 3 **Question 3 Page 107**

a) $s(t) = 5 + 7t - 8t^3$

$v(t) = s'(t)$ $\quad$ $a(t) = s''(t)$

$= 7 - 24t^2$ $\quad$ $= -48t$

b) $s(t) = (2t + 3)(4 - 5t)$

$= 8t + 12 - 10t^2 - 15t$

$= -10t^2 - 7t + 12$

$v(t) = s'(t)$ $\quad$ $a(t) = s''(t)$

$= -20t - 7$ $\quad$ $= -20$

c) $s(t) = -(t + 2)(3t^2 - t + 5)$

$= -(3t^3 - t^2 + 5t + 6t^2 - 2t + 10)$

$= -3t^3 - 5t^2 - 3t - 10$

$v(t) = s'(t)$ $\quad$ $a(t) = s''(t)$

$= -9t^2 - 10t - 3$ $\quad$ $= -18t - 10$

d) $s(t) = \dfrac{-2t^4 - t^3 + 8t^2}{4t^2}$

$= -\dfrac{1}{2}t^2 - \dfrac{1}{4}t + 2$

$v(t) = s'(t)$ $\quad$ $a(t) = s''(t)$

$= -t - \dfrac{1}{4}$ $\quad$ $= -1$

Chapter 2 Section 3 **Question 4 Page 107**

a) $v(t) = 3t^2 - 6t + 1$ $\quad$ $a(t) = 6t - 6$

$v(2) = 3(2)^2 - 6(2) + 1$ $\quad$ $a(2) = 6(2) - 6$

$= 1$ $\quad$ $= 6$

The velocity 1 m/s. $\quad$ The acceleration is 6 m/s^2.

b) $v(t) = -9.8t + 15$ $\quad$ $a(t) = -9.8$

$v(2) = -9.8(2) + 15$ $\quad$ $a(2) = -9.8$

$= -4.6$

The velocity is –4.6 m/s. $\quad$ The acceleration is –9.8 m/s^2.

c) $s(t) = t(3t+5)(1-2t)$
$= (3t^2+5t)(1-2t)$
$= -6t^3 - 7t^2 + 5t$

$v(t) = -18t^2 - 14t + 5$

$v(2) = -18(2)^2 - 14(2) + 5$
$= -95$
The velocity is –95 m/s.

$a(t) = -36t - 14$

$a(2) = -36(2) - 14$
$= -86$
The acceleration is –86 m/s^2.

d) $s(t) = (t^2-2)(t^2+2)$
$= t^4 - 4$

$v(t) = 4t^3$

$v(2) = 4(2)^3$
$= 32$
The velocity is 32 m/s.

$a(t) = 12t^2$

$a(2) = 12(2)^2$
$= 48$
The acceleration is 48 m/s^2.

Chapter 2 Section 3 **Question 5 Page 107**

Explanations may vary. For example:

a) $s(t)$ is curve (2); $v(t)$ is line (1); $a(t)$ is line (3) **.**

Curve (2) represents the position function since it is a quadratic with the highest exponent. Curve (1) represents the velocity since its exponent is one less so it is a sloping line. Curve (3) is the acceleration since its exponent is one less than the velocity so it is just a constant function, which is the flat line.

b) $s(t)$ is curve (3); $v(t)$ is curve (1); $a(t)$ is line (2) **.**

Curve (3) represents the position function since it is a cubic with the highest exponent. Curve (1) represents the velocity since its exponent is one less so it is a quadratic function. Curve (3) is the acceleration since its exponent is one less than the velocity so it is a line.

Chapter 2 Section 3 **Question 6 Page 107**

a)

Interval	***v*(*t*)**	***a*(*t*)**	***v*(*t*) × *a*(*t*)**	**Motion of Object**	**Description of slope of slope of *s*(*t*)**
[0, 3)	+	–	–	forward slowing	+decreasing
(3, 6]	–	–	+	reverse accelerating	–decreasing

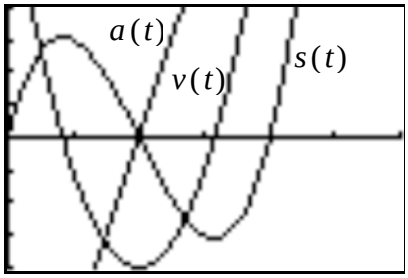
b)

Interval	$v(t)$	$a(t)$	$v(t) \times a(t)$	**Motion of Object**	**Description of slope of slope of $s(t)$**
[0, 1)	–	+	–	reverse slowing	–increasing
(1, 2)	+	+	+	forward accelerating	+increasing
(2, 3)	+	–	–	forward slowing	+decreasing
(3, infinity]	–	–	+	reverse accelerating	– decreasing

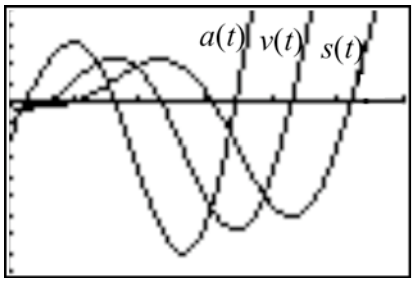
Chapter 2 Section 3 **Question 7 Page 107**

Answers may vary. For example:

a)



b)



Chapter 2 Section 3 **Question 7 Page 108**

Explanations may vary. For example:

a) i) $v(t)$ is increasing because the slope of the tangents are increasing.

ii) $a(t)$ is positive because the velocity is increasing.

b) i) $v(t)$ is increasing because the slope of the tangents are increasing.

ii) $a(t)$ is positive because the velocity is increasing.

c) i) $v(t)$ is decreasing because the slope of the tangents are decreasing.

ii) $a(t)$ is negative because the velocity is decreasing.

d) i) $v(t)$ is decreasing because the slope of the tangents are decreasing.

ii) $a(t)$ is negative because the velocity is decreasing.

e) i) $v(t)$ is constant because the slope of the tangents do not change.

ii) $a(t)$ is zero because the velocity is constant.

Chapter 2 Section 3 **Question 9 Page 108**

a) The initial velocity is zero.

b) The buses velocity at C and at F is zero.

c) The bus is going faster at A since the velocity is the slope of $s(t)$ and the slope is greater at A.

d) The bus is stopped since the velocity and acceleration of the bus are zero.

e) At A, the bus is speeding up. At B, the bus is slowing down. At D, the bus is speeding up.

f) At J, the bus returns to the starting point and slows to a stop.

Chapter 2 Section 3 **Question 10 Page 108**

a) The acceleration is positive.

b) The acceleration is negative.

c) The acceleration is positive.

d) The acceleration is negative.

e) The acceleration is positive.

Chapter 2 Section 3 **Question 11 Page 109**

a) i) The acceleration is positive.

ii) The acceleration is negative.

iii) The acceleration is positive.

iv) The acceleration is zero.

v) The acceleration is positive.

vi) At A the acceleration is positive and at D the acceleration is zero.

b) Justifications may vary. For example:

i) In all 3 intervals, the acceleration is positive as the velocity is increasing.

ii) In both intervals, the acceleration is negative as the velocity is decreasing.

iii) Both intervals have negative accelerations; The acceleration in the interval from B to C becomes more negative, while the acceleration in the interval from C to D becomes less negative.

Chapter 2 Section 3 **Question 12 Page 109**

a) Average velocity $= \frac{\Delta s}{\Delta t}$

$= \frac{s(3) - s(1)}{3 - 1}$

$= \frac{[80 - 4.9(3)^2] - [80 - 4.9(1)^2]}{3 - 1}$

$= -19.6$

The average velocity is –19.6 m/s.

b) The velocity of the bottle at 3 s is $v(3)$.

$v(t) = s'(t)$ $\quad$ $v(3) = -9.8(3)$

$= -9.8t$ $\quad$ $= -29.4$

The velocity of the bottle is –29.4 m/s.

c) The bottle will hit the ground when $s(t) = 0$.

$0 = 80 - 4.9t^2$

$t = \sqrt{\frac{80}{4.9}}$

$t \doteq 4.04$

The time is 4.04 s.

d) The impact velocity is $v(4.04)$.

$v(4.04) = -9.8(4.04)$

$= -39.6$

The impact velocity is –39.6 m/s.

Chapter 2 Section 3 **Question 13 Page 109**

a) $v(t) = h'(t)$ $a(t) = h''(t)$

$= -9.8t + 34.5$ $= -9.8$

$v(3) = -9.8(3) + 34.5$ $a(3) = -9.8$

$= 5.1$

The velocity of the rocket at 3 s is 5.1 m/s and the acceleration is -9.8 m/s^2.

b) When the rocket reaches its maximum height, $v(t) = 0$.

$0 = -9.8t + 34.5$

$t = 3.5$

The time is 3.5 s.

c) The starburst display occurs when $t = 3.5$ s so find $h(3.5)$ to get the height.

$h(3.5) = -4.9(3.5)^2 + 34.5(3.5) + 3.2$

$= 63.9$

The height is 63.9 m.

d) When the rocket is at ground level, $h(t) = 0$.

$0 = -4.9t^2 + 34.5t + 3.2$

$$t = \frac{-34.5 \pm \sqrt{34.5^2 - 4(-4.9)(3.2)}}{2(-4.9)}$$

$t \doteq 7.13$, $t \doteq -0.09$

Time cannot be negative, so it would take 7.13 s for the rocket to return to the ground.

e) $v(7.13) = -9.8(7.13) + 34.5$

$= -35.4$

The velocity is -35.4 m/s.

Chapter 2 Section 3 **Question 14 Page 109**

Justifications may vary. For example:

a) The velocity is positive because the truck is moving forward.

b) The velocity is decreasing because the truck is slowing down.

c) The acceleration is negative because the velocity is decreasing.

Chapter 2 Section 3 **Question 15 Page 109**

a) If the horizontal displacement is $s(t)$, then $s(t) = 48t$ where s is in kilometres and t is in hours.

b) $v(t) = s'(t)$ $\quad$ $a(t) = s''(t)$
$= 48$ $\quad$ $= 0$

c) Since the acceleration due to gravity is –9.8 m/s^2, $s''(t) = -9.8$. This means that $s'(t)$ should be $-9.8t$ and $s(t)$ should be $-4.9t^2$. But, when $t = 0$ the height needs to be $s(0) = 50$ m. This works for $s(0) = -4.9(0)^2 + 50$.

Therefore, the vertical displacement can be represented by $s(t) = 50 - 4.9t^2$, where s is the vertical height in metres and t is the time in seconds.

d) $v(t) = s'(t)$ $\quad$ $a(t) = s''(t)$
$= -9.8t$ $\quad$ $= -9.8$

e) When the prey hits the ground $s(t) = 0$ in the vertical displacement function.

$0 = 50 - 4.9t^2$

$t = \sqrt{\frac{50}{4.9}}$

$t = 3.19$

The vertical velocity when it hits the ground is $v(3.19)$.

$v(3.19) = -9.8(3.19)$

≈ -31.3

The velocity is about –31.3 m/s.

f) First, convert 48 km/h into metres per second.

The horizontal speed is $\frac{48 \times 1000}{1 \times 60 \times 60}$ or 13.33 m/s.

The vertical speed is $9.8t$.

When $9.8t = 13.3$, $t = 1.36$ s.

The vertical speed is greater than the horizontal speed when t is greater than 1.36 s.

g) Let V_T be the total velocity, V_V be the vertical velocity and V_H be the horizontal velocity.

$V_T = \sqrt{V_V^{\,2} + V_H^{\,2}}$

$= \sqrt{(9.8)^2 t^2 + (48)^2}$

$= \sqrt{2304 + 96.04t^2}$

h) Since the eagle and prey travel horizontally at a constant velocity, the horizontal acceleration is zero. Thus, the total acceleration is $a_{\text{tot}}(t) = -9.8$.

i) Since the acceleration due to gravity is constant, the prey's acceleration is –9.8 m/s^2.

Chapter 2 Section 3 **Question 16 Page 110**

a) Find $v(t)$ and $a(t)$ for $t = 1$ s and $t = 4$ s.

$v(t) = 6t^2 - 30t + 36$ $\quad a(t) = 12t - 30$

$v(1) = 12$ m/s; $v(4) = 12$ m/s $\quad a(1) = -18$ m/s^2; $a(4) = 18$ m/s^2

b) The object is stopped when $v(t) = 0$.

$0 = 6t^2 - 30t + 36$

$0 = t^2 - 5t + 6$

$0 = (t - 2)(t - 3)$

$t = 2$ or $t = 3$

The object is at rest at $t = 2$ s and $t = 3$ s.

The object's position at these times is $s(2) = 38$ m and $s(3) = 37$ m.

c) The object is moving in a positive direction for $v(t) > 0$ so when t belongs to the intervals $(-\infty, 2)$ and $(3, \infty)$.

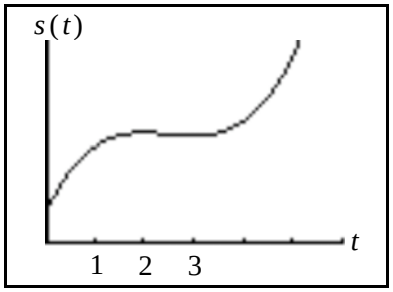
The object is moving in a negative direction when $v(t) < 0$ so when t belongs to the interval (2, 3).

d) Since the object travels in a negative direction from $t = 2$ s to $t = 3$ s, split up the intervals by the direction the object is moving and take the absolute values of the distances.

$$\text{Distance travelled} = |s(2) - s(0)| + |s(3) - s(2)| + |s(7) - s(3)|$$
$$= |28| + |-1| + |176|$$
$$= 205$$

The distance is 205 m.

e) Sketches may vary. For example:



Chapter 2 Section 3 **Question 17 Page 110**

a) $v(t) = h'(t)$ $\quad a(t) = h''(t)$

$= gt + v_0$ $\quad = g$

b) Set $v_0 = 17.4$ m/s, $s_0 = 4$ m, and $g = -9.8$ m/s^2.

$h(t) = -4.9t^2 + 17.5t + 4$

$v(t) = -9.8t + 17.5$

$a(t) = -9.8$

c) Find v_0 using the velocity function.

$v(t) = -9.8t + v_0$

$10.4 = -9.8(2) + v_0$

$v_0 = 30$

The original velocity is 30 m/s.

Find s_0 using the position function.

$h(t) = -4.9t^2 + v_0t + s_0$

$42.4 = -4.9(2)^2 + 30(2) + s_0$

$s_0 = 2$

The original position is 2 m.

The functions are $h(t) = -4.9t^2 + 30t + 2$, $v(t) = -9.8t + 30$, and $a(t) = -9.8$.

Chapter 2 Section 3 **Question 18 Page 110**

a) $v(t) = h'(t)$ $\quad$ $a(t) = h''(t)$

$= gt + v_0$ $\quad$ $= g$

The initial velocity is given in kilometres per hour so convert to metre per second.

$$v_0 = \frac{86.4}{1} \times \frac{1000}{60 \times 60}$$

$= 24$

The initial velocity is 24 m/s.

Set $v_0 = 24$ m/s, $s_0 = 0$ m, and $g = -12$ m/s^2.

$s(t) = -6t^2 + 24t$

$v(t) = -12t + 24$

$a(t) = -12$

b) The truck is stopped when $v(t) = 0$ so use this to find t.

$0 = -12t + 24$

$t = 2$

So the truck will take 2 s to stop.

Chapter 2 Section 3 **Question 19 Page 110**

E

Chapter 2 Section 3 **Question 20 Page 110**

D

Chapter 2 Section 4 **The Chain Rule**

Chapter 2 Section 4 **Question 1 Page 117**

i) a) $f'(x) = 3(2x)^2(2)$

$= 24x^2$

b) $g'(x) = 2(-4x^2)(-8x)$

$= 64x^3$

c) $p'(x) = \frac{1}{2}(9x^2)^{-\frac{1}{2}}(18x)$

$= \frac{9x}{\sqrt{9x^2}}$

$= \frac{3x}{\sqrt{x^2}}$

d) $f'(x) = \frac{3}{4}(-16x^2)^{-\frac{1}{4}}(-32x)$

$= \frac{-96x}{4(-16x^2)^{\frac{1}{4}}}$

$= \frac{-24x}{(-16x^2)^{\frac{1}{4}}}$

e) $q'(x) = \frac{2}{3}(8x)^{-\frac{1}{3}}(8)$

$= \frac{16}{3(8)^{\frac{1}{3}}(x)^{\frac{1}{3}}}$

$= \frac{8}{3x^{\frac{1}{3}}}$

ii) a) $f(x) = (2x)^3$

$= 8x^3$

$f'(x) = 24x^2$

b) $g(x) = (-4x^2)^2$

$= 16x^4$

$g'(x) = 64x^3$

c) $p(x) = \sqrt{9x^2}$

$= 3\sqrt{x^2}$

$p'(x) = 3\left(\frac{1}{2}(x^2)^{-\frac{1}{2}}(2x)\right)$

$= \frac{3x}{\sqrt{x^2}}$

d) $f(x) = (-16x^2)^{\frac{3}{4}}$

$= (-1)^{\frac{3}{4}}16^{\frac{3}{4}}x^{\frac{3}{2}}$

$= (-1)^{\frac{3}{4}}8x^{\frac{3}{2}}$

$f'(x) = \frac{3(-1)^{\frac{3}{4}}8x^{\frac{1}{2}}}{2}$

$= \frac{24(-1)^{\frac{3}{4}}x^{\frac{1}{2}}}{(2^4)^{\frac{1}{4}}} \cdot \frac{(-1)^{\frac{1}{4}}x^{\frac{1}{2}}}{(-1)^{\frac{1}{4}}x^{\frac{1}{2}}}$

$= \frac{-24x}{(-16x^2)^{\frac{1}{4}}}$

e) $q(x) = (8)^{\frac{2}{3}}(x^{\frac{2}{3}})$

$= 4x^{\frac{2}{3}}$

$q'(x) = 4\left(\frac{2}{3}\right)x^{-\frac{1}{3}}$

$= \frac{8}{3x^{\frac{1}{3}}}$

Chapter 2 Section 4 **Question 2 Page 117**

$f(x) = g[h(x)]$	$g(x)$	$h(x)$	$h'(x)$	$g'[h(x)]$	$f'(x) = g'[h(x)]\,h'(x)$
a) $(6x-1)^2$	x^2	$6x-1$	6	$2h(x)$	$12(6x-1)$
b) $(x^2+3)^3$	x^3	x^2+3	$2x$	$3[h(x)]^2$	$6x(x^2+3)^2$
c) $(2-x^3)^4$	x^4	$2-x^3$	$-3x^2$	$4[h(x)]^3$	$(12x^2)(x^3-2)^3$
d) $(-3x+4)^{-1}$	$x-1$	$-3x+4$	-3	$-[h(x)]^{-2}$	$\frac{3}{(3x-4)^2}$
e) $(7+x^2)^{-2}$	$x-2$	$7+x^2$	$2x$	$-2[h(x)]^{-3}$	$-\frac{4x}{(x^2+7)^3}$
f) $(x^4-3x^2)^{\frac{1}{2}}$	$x^{\frac{1}{2}}$	x^4-3x^2	$4x^3-6x$	$\frac{1}{2}(x^4-3x^2)^{-\frac{1}{2}}$	$\frac{2x^3-3x}{(x^4-3x^2)^{\frac{1}{2}}}$

Chapter 2 Section 4 **Question 3 Page 118**

a)

$$y' = 2(4x+1)(4)$$
$$= 8(4x+1)$$

b)

$$y' = 3(3x^2-2)^2(6x)$$
$$= 18x(9x^4-12x^2+4)$$
$$= 162x^5-216x^3+72x$$

c)

$$y' = -3(x^3-x)^{-4}(3x^2-1)$$
$$= \frac{-3(3x^2-1)}{(x^3-x)^4}$$
$$= \frac{-3(3x^2-1)}{x^4(x^2-1)^4}$$

d)

$$y' = -2(4x^2+3x)^{-3}(8x+3)$$
$$= \frac{-2(8x+3)}{(4x^2+3x)^3}$$
$$= \frac{-2(8x+3)}{x^3(4x+3)^3}$$

Chapter 2 Section 4 **Question 4 Page 118**

a) $y = (2x - 3x^5)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}(2x - 3x^5)^{-\frac{1}{2}}(2 - 15x^4)$$
$$= \frac{2 - 15x^4}{2(2x - 3x^5)^{\frac{1}{2}}}$$

b) $y = (-x^3 + 9)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}(-x^3 + 9)^{-\frac{1}{2}}(-3x^2)$$
$$= \frac{-3x^2}{2(-x^3 + 9)^{\frac{1}{2}}}$$

c) $y = (x - x^4)^{\frac{1}{3}}$

$$\frac{dy}{dx} = \frac{1}{3}(x - x^4)^{-\frac{2}{3}}(1 - 4x^3)$$
$$= \frac{1 - 4x^3}{3(x - x^4)^{\frac{2}{3}}}$$

d) $y = (2 + 3x^2 - x^3)^{\frac{1}{5}}$

$$\frac{dy}{dx} = \frac{1}{5}(2 + 3x^2 - x^3)^{-\frac{4}{5}}(6x - 3x^2)$$
$$= \frac{6x - 3x^2}{5(2 + 3x^2 - x^3)^{\frac{4}{5}}}$$

Chapter 2 Section 4 **Question 5 Page 118**

a) $y = (-x^3 + 1)^{-2}$

$$\frac{dy}{dx} = -2(-x^3 + 1)^{-3}(-3x^2)$$
$$= \frac{6x^2}{(-x^3 + 1)^3}$$

b) $y = (3x^2 - 2)^{-1}$

$$\frac{dy}{dx} = -(3x^2 - 2)^{-2}(6x)$$
$$= -\frac{6x}{(3x^2 - 2)^2}$$

c) $y = (x^2 + 4x)^{-\frac{1}{2}}$

$$\frac{dy}{dx} = -\frac{1}{2}(x^2 + 4x)^{-\frac{3}{2}}(2x + 4)$$
$$= -\frac{x + 2}{(x^2 + 4x)^{\frac{3}{2}}}$$

d) $y = (x - 7x^2)^{-\frac{1}{3}}$

$$\frac{dy}{dx} = -\frac{1}{3}(x - 7x^2)^{-\frac{4}{3}}(1 - 14x)$$
$$= -\frac{1 - 14x}{3(x - 7x^2)^{\frac{4}{3}}}$$

Chapter 2 Section 4 **Question 6 Page 118**

a) Use the chain rule.

$$f'(x) = \frac{1}{2}(25x^4)^{-\frac{1}{2}}(100x^3)$$
$$= \frac{50x^3}{5x^2}$$
$$= 10x$$

Alternately, simplify the function first and then differentiate.

$$f(x) = 5x^2$$
$$f'(x) = 10x$$

b) Answers may vary. For example:

I prefer the second method because simplifying the function first makes differentiating much easier.

c) No, the function cannot be simplified because I cannot take the square root of $25x^4 - 3$.

Chapter 2 Section 4 **Question 7 Page 118**

a) $f'(x) = 2(4x^2 - x + 1)(8x - 1)$

$f'(1) = 56$

b) $f'(x) = -2(3 - x + x^2)^{-3}(-1 + 2x)$

$f'(1) = \frac{-2}{27}$

c) $f(x) = (4x^2 + 1)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(4x^2 + 1)^{-\frac{1}{2}}(8x)$

$f'(1) = \frac{4}{\sqrt{5}}$

d) $f(x) = 5(2x - x^2)^{-\frac{1}{3}}$

$f'(x) = 5\left(-\frac{1}{3}(2x - x^2)^{-\frac{4}{3}}(2 - 2x)\right)$

$f'(1) = 0$

Chapter 2 Section 4 **Question 8 Page 118**

Use the chain rule in Leibniz notation: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

a)
$$\frac{dy}{dx} = (2u+3)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$
$$= (2\sqrt{x}+3)\left(\frac{1}{2\sqrt{x}}\right)$$
$$= 1+\frac{3}{2\sqrt{x}}$$
$$\left.\frac{dy}{dx}\right|_{x=4} = 1+\frac{3}{2\sqrt{4}}$$
$$= \frac{7}{4}$$

b)
$$\frac{dy}{dx} = \left(\frac{1}{2}u^{-\frac{1}{2}}\right)(4x+3)$$
$$= \left(\frac{1}{2}(2x^2+3x+4)^{-\frac{1}{2}}\right)(4x+3)$$
$$= \frac{4x+3}{2\sqrt{2x^2+3x+4}}$$
$$\left.\frac{dy}{dx}\right|_{x=-3} = \frac{4(-3)+3}{2\sqrt{2(-3)^2+3(-3)+4}}$$
$$= -\frac{9}{2\sqrt{13}} \text{ or } -\frac{9}{26}\sqrt{13}$$

c)
$$\frac{dy}{dx} = (-2u^{-3})(3x^2-5)$$
$$= -2(x^3-5x)^{-3}(3x^2-5)$$
$$= \frac{-2(3x^2-5)}{(x^3-5x)^{-3}}$$
$$\left.\frac{dy}{dx}\right|_{x=-2} = \frac{-2(3(-2)^2-5)}{((-2)^3-5(-2))^{-3}}$$
$$= -\frac{7}{4}$$

d) $\dfrac{dy}{dx} = (2-3u^2)(-x^{-2})$

$= (2-3(x^{-1})^2)(-x^{-2})$

$= \dfrac{3}{x^4} - \dfrac{2}{x^2}$

$\left.\dfrac{dy}{dx}\right|_{x=2} = \dfrac{3}{(2)^4} - \dfrac{2}{(2)^2}$

$= -\dfrac{5}{16}$

Chapter 2 Section 4 **Question 9 Page 118**

The slope m of the tangent will be $\left.\dfrac{dy}{dx}\right|_{x=3}$

$\dfrac{dy}{dx} = 3(x^3-4x^2)^2(3x^2-8x)$

$\left.\dfrac{dy}{dx}\right|_{x=3} = 3((3)^3-4(3)^2)^2(3(3)^2-8(3))$

$= 729$

The y-coordinate at $x = 3$ is $((3)^3 - 4(3)^2)^3 = -729$.
Use $m = 729$ and the point $(3, -729)$ to find b in the equation $y = mx + b$.

$-729 = 729(3) + b$

$b = -2916$

The equation of the tangent is $y = 729x - 2916$.

Chapter 2 Section 4 **Question 10 Page 118**

The slope m of the tangent will be $\left.\dfrac{dy}{dx}\right|_{x=2}$

$\dfrac{dy}{dx} = -\dfrac{1}{5}(5x^3-2x^2)^{-\frac{6}{5}}(15x^2-4x)$

$\left.\dfrac{dy}{dx}\right|_{x=2} = -\dfrac{1}{5}(5(2)^3-2(2)^2)^{-\frac{6}{5}}(15(2)^2-4(2))$

$= -\dfrac{13}{80}$

The y-coordinate at $x = 2$ is $(5(2)^3 - 2(2)^2)^{-\frac{1}{5}} = \dfrac{1}{2}$.

Use $m = -\dfrac{13}{80}$ and the point $\left(2, \dfrac{1}{2}\right)$ to find b in the equation $y = mx + b$.

$$\frac{1}{2} = -\frac{13}{80}(2) + b$$

$$b = \frac{33}{40}$$

The equation of the tangent is $y = -\frac{13}{80}x + \frac{33}{40}$.

Chapter 2 Section 4 **Question 11 Page 118**

$$v(t) = s'(t)$$

$$= \frac{1}{3}(t^5 - 750t^2)^{\frac{-2}{3}}(5t^4 - 1500t)$$

Substitute t = 5 s into the velocity function and factor the numbers using powers of 5 to find $v(5)$.

$$v(5) = \frac{1}{3}((5)^5 - 750(5)^2)^{-\frac{2}{3}}(5(5)^4 - 1500(5))$$

$$= \frac{1}{3}(-15625)^{\frac{-2}{3}}(-4375)$$

$$= \frac{1}{3}(-5^6)^{\frac{-2}{3}}(7)(-5^4)$$

$$= -\frac{1}{3}(-5^{-4})(-5^4)(7)$$

$$= -\frac{7}{3}$$

The velocity is $-\frac{7}{3}$ m/s.

Chapter 2 Section 4 **Question 12 Page 118**

Expand y and find the derivative.

$$y = (x^4(x^2 - 1)^2)$$

$$= x^4(x^4 - 2x^2 + 1)$$

$$= x^8 - 2x^6 + x^4$$

$$\frac{dy}{dx} = 8x^7 - 12x^5 + 4x^3$$

The points on the curve where the tangent line is horizontal are the points where $\frac{dy}{dx} = 0$.

$0 = 8x^7 - 12x^5 + 4x^3$

$0 = 4x^3(2x^4 - 3x^2 + 1)$

$0 = 4x^3(2x^2 - 1)(x^2 - 1)$

$x = 0,\ x = 1,\ x = -1,\ x = \frac{1}{\sqrt{2}},\ x = -\frac{1}{\sqrt{2}}$

Substituting the x-values into y gives the points $(-1, 0)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{16}\right)$, $(0, 0)$, $\left(\frac{1}{\sqrt{2}}, \frac{1}{16}\right)$, $(1, 0)$.

Chapter 2 Section 4 **Question 13 Page 118**

a)

$$N'(t) = 600\left(\frac{1}{2}\right)(16 + 3t^2)^{-\frac{3}{2}}(6t)$$

$$= \frac{1800t}{(16 + 3t^2)^{\frac{3}{2}}}$$

$N'(t)$ represents the rate at which the customers are being served.

b) $N(4) = 150 - \frac{600}{\sqrt{16 + 3(4)^2}}$

$= 75$

$N'(4) = \frac{1800(4)}{(16 + 3(4)^2)^{\frac{3}{2}}}$

$\doteq \frac{225}{16}$ or 14.06

After 4 h, 75 customers have been served at an instantaneous rate of change of 14.06 customers per hour.

c) $103 = 150 - \frac{600}{\sqrt{16 + 3t^2}}$

$16 + 3t^2 = \left(\frac{600}{47}\right)^2$

$t \doteq \pm 7$

This means that 103 customers have been served at time $t \doteq 7$ h.

d) $N'(7) = \dfrac{1800(7)}{(16+3(7)^2)^{\frac{3}{2}}}$

$= \dfrac{12600}{26569}\sqrt{163}$

$\doteq 6.05$

At $t = 7$ h, there are about 6.05 customers served per hour. At $t = 4$ h from before, there were 14.06 customers served per hour, which means that the customers are being served at a slower rate at 7 h.

Chapter 2 Section 4 **Question 14 Page 118**

$P'(t) = -1250(1+0.01t)^{-2}(0.01)$

$= \dfrac{-12.5}{(1+0.01t)^2}$

$P'(2) = \dfrac{-12.5}{(1+0.01(2))^2}$

$= -12.01$

The rate is –12.01 people/year.

$P'(4) = \dfrac{-12.5}{(1+0.01(4))^2}$

$= -11.56$

The rate is –11.56 people/year.

$P'(7) = \dfrac{-12.5}{(1+0.01(7))^2}$

$= -10.92$

The rate is –10.92 people/year.

Chapter 2 Section 4 **Question 15 Page 119**

$\dfrac{dV}{dx} = \dfrac{dV}{ds} \cdot \dfrac{ds}{dx}$

$= (3s^2)(6x-7)$

$= 3(3x^2-7x+1)^2(6x-7)$

$\left.\dfrac{dV}{dx}\right|_{x=3} = 3\left(3(3)^2-7(3)+1\right)^2(6(3)-7)$

$= 1617$

This value represents the rate of change of the volume of the cube with respect to x, when $x = 3$ m.

Chapter 2 Section 4 **Question 16 Page 119**

$$y = (4x - x^3)(3x^2 + 2)^{-2}$$

$$\begin{aligned}\frac{dy}{dx} &= (4x - x^3)(-2)(3x^2 + 2)^{-3}(6x) + (4 - 3x^2)(3x^2 + 2)^{-2} \\ &= \frac{(-48x^2 + 12x^4) + (4 - 3x^2)(3x^2 + 2)}{(3x^2 + 2)^3} \\ &= \frac{-48x^2 + 12x^4 + 6x^2 - 9x^4 + 8}{(3x^2 + 2)^3} \\ &= \frac{3x^4 - 42x^2 + 8}{(3x^2 + 2)^3}\end{aligned}$$

Chapter 2 Section 4 **Question 17 Page 119**

Solutions to the Achievement Checks are shown in the Teacher's Resource.

Chapter 2 Section 4 **Question 18 Page 119**

The slope of the tangent at x is $\frac{dy}{dx}$.

$$\begin{aligned}\frac{dy}{dx} &= (3x^2)(8x^2 + 1)^{\frac{1}{2}} + x^3\left(\frac{1}{2}(8x^2 + 1)^{\frac{-1}{2}}(16x)\right) \\ &= 3x^2\sqrt{8x^2 + 1} + \frac{8x^4}{\sqrt{8x^2 + 1}}\end{aligned}$$

Since all the exponents of x in the derivative are even, the slope at $x = -1$ and $x = 1$ will be the same.

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{x=1} &= 3(1)^2\sqrt{8(1)^2 + 1} + \frac{8(1)^4}{\sqrt{8(1)^2 + 1}} \\ &= \frac{35}{3}\end{aligned}$$

The slopes at both $x = -1$ and $x = 1$ are $m = \frac{35}{3}$.

When $x = -1$, the $y = 3$ so use m and the point $(-1, 3)$ to find b in the equation for the tangent.

$$3 = \frac{35}{3}(-1) + b$$

$$b = \frac{26}{3}$$

When $x = 1$, the $y = 3$ so use m and the point (1, 3) to find b in the equation for the second tangent.

$$3 = \frac{35}{3}(1) + b$$

$$b = -\frac{26}{3}$$

Therefore, the equations of the tangents are $y = \frac{35}{3}x + \frac{26}{3}$ and $y = \frac{35}{3}x - \frac{26}{3}$.

The tangents are related because they are parallel.
This is true because the curve is odd, so the tangents at $x = a$ and $x = -a$ will have the same slope for all values of x in the domain.

Chapter 2 Section 4 **Question 19 Page 119**

$$f'(x) = g'\left[h(x)\right] \times h'(x)$$

$$\begin{aligned} f'(2) &= g'\left[h(2)\right] \times h'(2) \\ &= g'(-6) \times 4 \\ &= -3 \times 4 \\ &= -12 \end{aligned}$$

Chapter 2 Section 4 **Question 20 Page 119**

$$\frac{dy}{dx} = \frac{1}{2}(2x+1)^{-\frac{1}{2}}(2)$$

$$= (2x+1)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}(2x+1)^{-\frac{3}{2}}(2)$$

$$= -\frac{1}{(2x+1)^{\frac{3}{2}}}$$

Chapter 2 Section 4 **Question 21 Page 119**

Answers may vary. For example:

a) Choose $f(x) = x^2$ and $g(x) = 2x$.

$$g \circ f(x) = 2x^2$$

$$\begin{aligned} f \circ g(x) &= (2x)^2 \\ &= 4x^2 \end{aligned}$$

$\frac{d}{dx} f \circ g(x) = 8x$ and $\frac{d}{dx} g \circ f(x) = 4x$ so they are not the same.

b) Choose $f(x) = x^2$ and $g(x) = x^2$.

$f \circ g(x) = (x^2)^2$ $\quad$ $g \circ f(x) = (x^2)^2$

$= x^4$ $\quad$ $= x^4$

Both compositions have the same derivative of $4x^3$.

Chapter 2 Section 4 **Question 22 Page 119**

a) $y = f\left[g\left[h(x)\right]\right]$

$$= \left(\frac{1}{\sqrt{x^2 + 2x}}\right)^2$$

$$= (x^2 + 2x)^{-1}$$

$$\frac{dy}{dx} = -(x^2 + 2x)^{-2}(2x + 2)$$

$$= \frac{-(2x + 2)}{(x^2 + 2x)^2}$$

$$= \frac{-2(x + 1)}{x^2(x + 2)^2}$$

b) $y = f\left[g\left[h(x)\right]\right]$

$$= (x^2 + 2x)^{-1}$$

$$\frac{dy}{dx} = \frac{-2(x + 1)}{x^2(x + 2)^2}$$

c) $y = g\left[h\left[f(x)\right]\right]$

$$= \frac{1}{\sqrt{x^4 + 2x^2}}$$

$$= (x^4 + 2x^2)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -\frac{1}{2}(x^4 + 2x^2)^{-\frac{3}{2}}(4x^3 + 4x)$$

$$= \frac{-2(x^3 + x)}{(x^2(x^2 + 2))^{\frac{3}{2}}}$$

$$= \frac{-2x(x^2 + 1)}{x^2(x^2 + 2)\sqrt{x^2(x^2 + 2)}}$$

$$= \frac{-2(x^2 + 1)}{x(x^2 + 2)\sqrt{x^2(x^2 + 2)}}$$

d) $y = g\left[h\left[f(x)\right]\right]$

$$= \sqrt{\left(\frac{1}{x^2}\right)^2 + 2\left(\frac{1}{x^2}\right)}$$

$$= (x^{-4} + 2x^{-2})^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(x^{-4} + 2x^{-2})^{-\frac{1}{2}}(-4x^{-5} - 4x^{-3})$$

$$= \frac{-2(x^{-5} + x^{-3})}{\left(\frac{1}{x^4} + \frac{2}{x^2}\right)^{\frac{1}{2}}}$$

$$= \frac{-2x^{-5}(1 + x^2)}{\sqrt{\frac{1}{x^4} + \frac{2}{x^2}}}$$

$$= \frac{-2(1 + x^2)}{x^5\sqrt{\frac{1 + 2x^2}{x^4}}}$$

Chapter 2 Section 4 **Question 23 Page 119**

$$\frac{dy}{dx} = f'(g \circ h(x)) \times (g \circ h(x))'$$

$$= f'(g[h(x)])(g'[h(x)])[h'(x)]$$

Chapter 2 Section 4 **Question 24 Page 119**

D

Chapter 2 Section 4 **Question 25 Page 119**

E

Chapter 2 Section 5 **Derivatives of Quotients**

Chapter 2 Section 5 **Question 1 Page 124**

a) $q(x) = (3x + 5)^{-1}$; domain: $x \neq \frac{-5}{3}$

b) $f(x) = -2(x - 4)^{-1}$; domain: $x \neq 4$

c) $g(x) = 6(7x^2 + 1)^{-1}$; domain: no restrictions

d) $r(x) = -2(x^3 - 27)^{-1}$; domain: $x \neq 3$

Chapter 2 Section 5 **Question 2 Page 124**

a) $q'(x) = -(3x + 5)^{-2}(3)$

b) $f'(x) = 2(x - 4)^{-2}$

c) $g'(x) = -6(7x^2 + 1)^{-2}(14x)$

d) $r'(x) = 2(x^3 - 27)^{-2}(3x^2)$

Chapter 2 Section 5 **Question 3 Page 124**

a) $q(x) = 3x(x + 1)^{-1}$; domain: $x \neq -1$

b) $f(x) = -x(2x + 3)^{-1}$; domain: $x \neq -\frac{3}{2}$

c) $g(x) = x^2(5x - 4)^{-1}$; domain: $x \neq \frac{4}{5}$

d) $r(x) = 8x^2(x^2 - 9)^{-1}$; domain: $x \neq \pm 3$

Chapter 2 Section 5 **Question 4 Page 124**

a) $q'(x) = 3(x + 1)^{-1} + 3x(-1)(x + 1)^{-2}$

b) $f'(x) = (-1)(2x + 3)^{-1} - x(-1)(2x + 3)^{-2}(2)$

c) $g'(x) = 2x(5x - 4)^{-1} + x^2(-1)(5x - 4)^{-2}(5)$

d) $r'(x) = 16x(x^2 - 9)^{-1} + 8x^2(-1)(x^2 - 9)^{-2}(2x)$

Chapter 2 Section 5 **Question 5 Page 125**

a) $\frac{dy}{dx} = (-x+3)(-1)(2x^2+5)^{-2}(4x) + (2x^2+5)^{-1}(-1)$

$= (2x^2+5)^{-2}[-4x(-x+3) - (2x^2+5)]$

$= \frac{2x^2-12x-5}{(2x^2+5)^2}$

b) $\frac{dy}{dx} = (4x+1)(-1)(x^3-2)^{-2}(3x^2) + (x^3-2)^{-1}(4)$

$= (x^3-2)^{-2}[-3x^2(4x+1) + 4(x^3-2)]$

$= -\frac{8x^3+3x^2+8}{(x^3-2)^2}$

c) $\frac{dy}{dx} = (9x^2-1)(-1)(1+3x)^{-2}(3) + (1+3x)^{-1}(18x)$

$= (1+3x)^{-2}[-3(9x^2-1) + 18x(1+3x)]$

$= 3(1+3x)^{-2}(1+3x)^2$

$= 3$

d) $\frac{dy}{dx} = x^4(-1)(x^2-x+1)^{-2}(2x-1) + (x^2-x+1)^{-1}(4x^3)$

$= (x^2-x+1)^{-2}[-x^4(2x-1) + 4x^3(x^2-x+1)]$

$= x^3(x^2-x+1)^{-2}[-x(2x-1) + 4(x^2-x+1)]$

$= \frac{x^3(2x^2-3x+4)}{(x^2-x+1)^2}$

Chapter 2 Section 5 **Question 6 Page 125**

a) $\frac{dy}{dx} = x^2(-1)(6x+2)^{-2}(6) + (6x+2)^{-1}(2x)$

$\left.\frac{dy}{dx}\right|_{x=-2} = (-2)^2(-1)(6(-2)+2)^{-2}(6) + (6(-2)+2)^{-1}2(-2)$

$= \frac{4}{25}$

The slope of the tangent at $x = 2$ is $\frac{4}{25}$.

b) $\frac{dy}{dx} = x^{\frac{1}{2}}(-1)(3x^2-1)^{-2}(6x) + (3x^2-1)^{-1}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$

$$\left.\frac{dy}{dx}\right|_{x=1} = 1^{\frac{1}{2}}(-1)(3(1)^2-1)^{-2}6(1) + (3(1)^2-1)^{-1}\left(\frac{1}{2}(1)^{-\frac{1}{2}}\right)$$

$$= -\frac{5}{4}$$

The slope of the tangent at $x = 1$ is $-\frac{5}{4}$.

c) $\frac{dy}{dx} = 4(x^2-1)^{-1} + (4x+1)(-1)(x^2-1)^{-2}(2x)$

$$\left.\frac{dy}{dx}\right|_{x=-3} = 4((-3)^2-1)^{-1} + (4(-3)+1)(-1)((-3)^2-1)^{-2}2(-3)$$

$$= -\frac{17}{32}$$

The slope of the tangent at $x = 1$ is $-\frac{17}{32}$.

d) $\frac{dy}{dx} = 2(x^2-x+1)^{-1} + 2x(-1)(x^2-x+1)^{-2}(2x-1)$

$$\left.\frac{dy}{dx}\right|_{x=-1} = 2((-1)^2-(-1)+1)^{-1} + 2(-1)(-1)((-1)^2-(-1)+1)^{-2}(2(-1)-1)$$

$$= 0$$

The slope of the tangent at $x = 1$ is 0.

e) $\frac{dy}{dx} = (3x^2)(x^2+x-1)^{-1} + (x^3-3)(-1)(x^2+x-1)^{-2}(2x+1)$

$$\left.\frac{dy}{dx}\right|_{x=2} = 3(2)^2(2^2+2-1)^{-1} + (2^3-3)(-1)(2^2+2-1)^{-2}(2(2)+1)$$

$$= \frac{7}{5}$$

The slope of the tangent at $x = 2$ is $\frac{7}{5}$.

Chapter 2 Section 5 **Question 7 Page 125**

Explanations may vary. For example:

a) Express the function as a product and use the product rule.

$q'(x) = (-4x^3 + 5x^2 - 2x + 6)(-3x^{-4}) + x^{-3}(-12x^2 + 10x - 2)$

$q'(x) = x^{-4}[-3(-4x^3 + 5x^2 - 2x + 6) + x(-12x^2 + 10x - 2)]$

$q'(x) = \dfrac{-5x^2 + 4x - 18}{x^4}$

Simplify the function and use the sum rule.

$q(x) = -4 + 5x^{-1} - 2x^{-2} + 6x^{-3}$

$q'(x) = -5x^{-2} + 4x^{-3} - 18x^{-4}$

I prefer simplifying the function first and using the sum rule because it is easier.

b) No, it is not possible to simplify and use the sum rule for the given function.

Chapter 2 Section 5 **Question 8 Page 125**

$y' = x^2(-1)(x+2)^{-2} + (x+2)^{-1}(2x)$

$= (x+2)^{-2}[-x^2(x+2)^{-2} + 2x(x+2)]$

$= \dfrac{x^2 + 4x}{(x+2)^2}$

The slope of the tangent is 3, so find x when $y' = -3$.

$$-3 = \frac{x^2 + 4x}{(x+2)^2}$$

$-3(x^2 + 4x + 4) = x^2 + 4x$

$4x^2 + 16x + 12 = 0$

$x^2 + 4x + 3 = 0$

$(x+1)(x+3) = 0$

$x = -1$ or $x = -3$

The points on the curve where the slope of the tangents are –3 are (–1, 1) and (–3, –9).

Chapter 2 Section 5 **Question 9 Page 125**

$y = \dfrac{(x^3 - 1)^2}{(x+2)^2}$

$= (x^3 - 1)^2(x+2)^{-2}$

$y' = (x^3 - 1)^2(-2)(x+2)^{-3} + (2)(x^3 - 1)(3x^2)(x+2)^{-2}$

The slope of the tangent at $x = -1$ is $y'(-1)$.

$y'(-1) = ((-1)^3 - 1)^2(-2)(-1+2)^{-3} + (2)((-1)^3 - 1)(3(-1)^2)(-1+2)^{-2}$

$= -20$

When $x = -1$, $y = 4$ so use this point and $m = -20$ to find b in the equation of the tangent, $y = mx + b$.
$4 = -20(-1) + b$ so $b = -16$.

The equation of the tangent at $x = -1$ is $y = -20x - 16$.

Chapter 2 Section 5 **Question 10 Page 125**

a) $v(t) = s'(t)$

$$= 5t(-1)(t^2 + 4)^{-2}(2t) + (t^2 + 4)^{-1}(5)$$
$$= (t^2 + 4)^{-2}[5t(-1)(2t) + (t^2 + 4)(5)]$$
$$= \frac{-5t^2 + 20}{(t^2 + 4)^2}$$

The hamster is moving at $v(1) = \frac{3}{5}$ m/s after 1 s.

b) First, find the time when the hamster stops, which is when $v(t) = 0$.

$$0 = \frac{-5t^2 + 20}{(t^2 + 4)^2}$$
$$0 = -5t^2 + 20$$
$$t = \pm 2$$

Since time cannot be negative, $t = 2$ s.
If $0 \le t < 2$, then $v(t) > 0$ and if $t > 2$, then $v(t) < 0$.
Therefore, the hamster changes direction when $t = 2$ s.

Chapter 2 Section 5 **Question 11 Page 125**

a) $C'(w) = 800w^2(-1)(200 + w^3)^{-2}(3w^2) + (200 + w^3)^{-1}(1600w)$

When $w = 1$, $C'(1) = 800(1)^2(-1)(200 + 1^3)^{-2}3(1)^2 + (200 + 1^3)^{-1}1600(1)$
$\doteq 7.9$

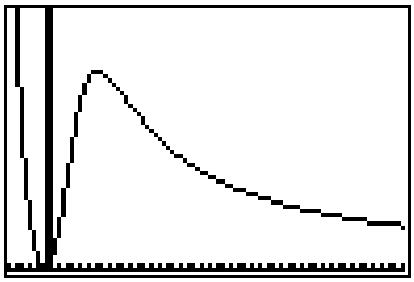
When $w = 3$, $C'(3) = 800(3)^2(-1)(200 + 3^3)^{-2}3(3)^2 + (200 + 3^3)^{-1}1600(3)$
$\doteq 17.4$

When $w = 5$, $C'(5) = 800(5)^2(-1)(200 + 5^3)^{-2}3(5)^2 + (200 + 5^3)^{-1}1600(5)$
$\doteq 10.4$

When $w = 8$, $C'(8) = 800(8)^2(-1)(200 + 8^3)^{-2}3(8)^2 + (200 + 8^3)^{-1}1600(8)$
$\doteq -1.4$

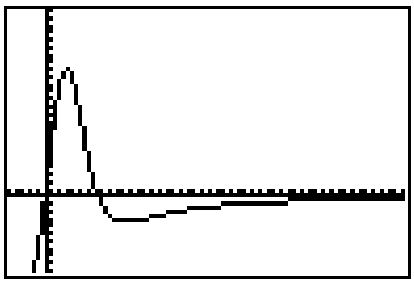
$C'(w)$ is the number of new clients in the fund per week.

b)



The slope of the curve indicates if $C'(w)$ is positive, negative, or zero.

c)



d)

$$C'(w) = 800w^2(-1)(200 + w^3)^{-2}(3w^2) + (200 + w^3)^{-1}(1600w)$$
$$= (200 + w^3)^{-2}[800w^2(-1)(3w^2) + (200 + w^3)(1600w)]$$
$$= \frac{-800w^4 + 320000w}{(200 + w^3)^2}$$

Find $C'(w) = 0$.

$$-800w^4 + 320000w = 0$$
$$-w^4 + 400w = 0$$
$$-w(w^3 - 400) = 0$$
$$w = 0,\ w = \sqrt[3]{400} \text{ or } 2\sqrt[3]{50}$$

$C'(w) = 0$ when $w = 0$, and $w = 2\sqrt[3]{50}$. $C'(w)$ is positive for $0 < w < 2\sqrt[3]{50}$ and is negative for $w > 2\sqrt[3]{50}$.

e) Answers may vary. For example:

The number of new clients per week increases from $w = 0$ to $w = 2\sqrt[3]{50}$ weeks, then declines for $w > 2\sqrt[3]{50}$.

Chapter 2 Section 5 **Question 12 Page 125**

a) $$N(8) = \frac{500(8)^2}{\sqrt{280 + 8^2}} + 10(8)$$
$$= 1805.3$$

The predicted number of new customers after 8 weeks is 1805.3 customers.

b) $N(1) = \dfrac{500(1)^2}{\sqrt{280+1^2}} + 10(1)$

$\doteq 39.8$

$N(6) = \dfrac{500(6)^2}{\sqrt{280+6^2}} + 10(6)$

$\doteq 1072.6$

$\text{Average} = \dfrac{N(6)-N(1)}{6-1}$

$= \dfrac{1072.6-39.8}{5}$

$\doteq 206.55$

The predicted average number of new customers between weeks 1 and 6 is 206.55 customers.

c) $N'(x) = 500x^2\left(-\dfrac{1}{2}\right)(280+x^2)^{-\frac{3}{2}}(2x) + (280+x^2)^{-\frac{1}{2}}(1000x) + 10$

$N'(1) = 500(1)^2\left(-\dfrac{1}{2}\right)(280+1^2)^{-\frac{3}{2}}2(1) + (280+1^2)^{-\frac{1}{2}}1000(1) + 10$

$\doteq 69.5$

$N'(6) = 500(6)^2\left(-\dfrac{1}{2}\right)(280+6^2)^{-\frac{3}{2}}2(6) + (280+6^2)^{-\frac{1}{2}}1000(6) + 10$

$\doteq 328.3$

The rate of change of the predicted number of new customers is 69.5 customers/week at week 1 and is 328.3 customers/week at week 6.

d) The rate of change, $N'(x)$, can never be zero, so the number of customers will always increase.

Chapter 2 Section 5 **Question 13 Page 126**

a) $V(0) = \dfrac{2500}{\sqrt{2}}$

$\doteq 1767.77$

The purchase price of the painting was \$1767.77.

b) Find $V'(t)$.

$$V'(t) = 0.2(1+t)(0.5t+2)^{-0.5} + (2500+0.2t)(0.5t+2)^{-0.5} + (2500+0.2t)(1+t)(-0.5)(0.5t+2)^{-1.5}(0.5)$$
$$= (0.5t+2)^{-1.5}[0.2(1+t)(0.5t+2) + (2500+0.2t)(0.5t+2) + (2500+0.2t)(1+t)(-0.5)(0.5)]$$
$$= (0.5t+2)^{-1.5}[(0.1t^2+0.5t+0.4) + (0.1t^2+1250.4t+5000) + (-0.05t^2-625.05t-625)]$$
$$= \frac{0.15t^2+625.85t+4375.4}{(0.5t+2)^{1.5}}$$
$$= \frac{0.6t^2+2503.4t+17501.6}{4(0.5t+2)^{1.5}} \text{ which can be transformed to}$$
$$= \frac{0.4(3t^2+12517t+87508)}{(2t+8)^{1.5}}$$

c) The value of the painting will always increase since $V'(t) > 0$ for all values of $t > 0$.

d) $V'(2) = \dfrac{0.4(3(2)^2+12517(2)+87508)}{(2(2)+8)^{1.5}}$ $\qquad$ $V'(22) = \dfrac{0.4(3(22)^2+12517(22)+87508)}{(2(22)+8)^{1.5}}$

$= 1083.05$ $\qquad$ $= 388.65$

These values represent the increase in value of the painting after 2 years and after 22 years. The painting gains value more quickly 2 years after purchase than 22 years after purchase.

Chapter 2 Section 5 **Question 14 Page 126**

a) $f'(x) = -(ax+b)^{-2}(a)$

$f''(x) = (-1)(-2)(ax+b)^{-3}(a)(a)$
$= 2(ax+b)^{-3}a^2$

$f'''(x) = (-1)(-2)(-3)(ax+b)^{-3}(a)(a)(a)$
$= -6(ax+b)^{-4}a^3$

By extrapolation:

$f^{(n)}(x) = (-1)^n(n!)(ax+b)^{-(n+1)}(a^n)$
$= \dfrac{(-1)^n(n!)(a^n)}{(ax+b)^{n+1}}$

b) $f^{(4)}(x) = (-1)^4(4!)(2x-3)^{-5}(2^4)$
$= \dfrac{384}{(2x-3)^5}$

Chapter 2 Section 5 **Question 15 Page 126**

a) Find $p'(x)$ and set it to zero to find the value of x.

$$p'(x) = (x^2 - 4)(-1)(x^2 + 4)^{-2}(2x) + 2x(x^2 + 4)^{-1}$$

$$0 = (x^2 + 4)^{-2}[(x^2 - 4)(-1)(2x) + 2x(x^2 + 4)]$$

$$0 = \frac{16x}{(x^2 + 4)^2}$$

$$0 = 16x$$

$$x = 0$$

When $x = 0$, $y = 1$ so the point is $(0, -1)$.

b) Find the points where $p''(x) = 0$.

$$p''(x) = 16x(-2)(x^2 + 4)^{-3}(2x) + 16(x^2 + 4)^{-2}$$

$$0 = (x^2 + 4)^{-3}[16x(-2)(2x) + 16(x^2 + 4)]$$

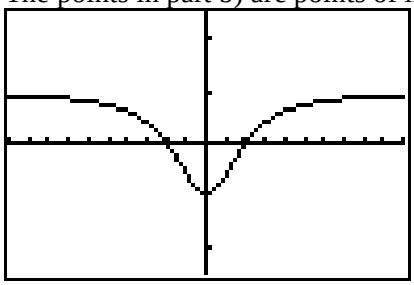
$$0 = \frac{64 - 48x^2}{(x^2 + 4)^3}$$

$$0 = 64 - 48x^2$$

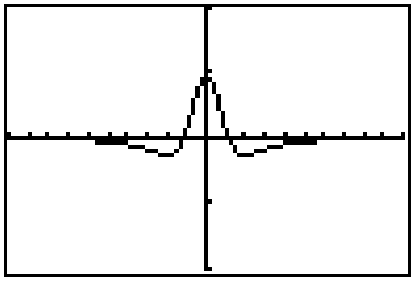
$$x = \pm\frac{2}{\sqrt{3}}$$

Substituting the x-values into $p(x)$ gives the points $\left(\frac{2}{\sqrt{3}}, -\frac{1}{2}\right)$ and $\left(-\frac{2}{\sqrt{3}}, -\frac{1}{2}\right)$.

c) The point in part a) represents the point on the curve with zero slope, which is the minimum of $p(x)$. The points in part b) are points of inflection.



d) The points in part b) represent the maximum and minimum points on the graph of $p'(x)$.



Chapter 2 Section 5 **Question 16 Page 126**

a) $y = \dfrac{\left(\dfrac{1}{x} + x\right)}{\sqrt{\left(\dfrac{1}{x} + x\right) - 1}}$

$= (x^{-1} + x)(x^{-1} + x - 1)^{-0.5}$

$\dfrac{dy}{dx} = (x^{-1} + x)(-0.5)(x^{-1} + x - 1)^{-1.5}(-x^{-2} + 1) + (-x^{-2} + 1)(x^{-1} + x - 1)^{-0.5}$

$= (x^{-1} + x - 1)^{-1.5}[(x^{-1} + x)(-0.5)(-x^{-2} + 1) + (-x^{-2} + 1)(x^{-1} + x - 1)]$

$= (x^{-1} + x - 1)^{-1.5}[(0.5x^{-3} - 0.5x^{-1} + 0.5x^{-1} - 0.5x) + (-x^{-3} - x^{-1} + x^{-2} + x^{-1} + x - 1)]$

$= \dfrac{-0.5x^{-3} + x^{-2} + 0.5x - 1}{(x^{-1} + x - 1)^{1.5}}$

$= \dfrac{-0.5 + x + 0.5x^4 - x^3}{x^3(x^{-1}(1 + x^2 - x))(x^{-1}(1 + x^2 - x))^{0.5}}$

$= \dfrac{x^4 - 2x^3 + 2x - 1}{2x^2(x^2 - x + 1)\sqrt{\dfrac{x^2 - x + 1}{x}}}$

domain: $x \neq 0$

b) $y = x^{-1}(x-1)^{\frac{1}{2}} + x(x-1)^{-\frac{1}{2}}$

$\dfrac{dy}{dx} = (x-1)^{\frac{1}{2}}(-1)x^{-2} + x^{-1}\left(\dfrac{1}{2}\right)(x-1)^{-\frac{1}{2}} + x\left(-\dfrac{1}{2}\right)(x-1)^{-\frac{3}{2}} + (x-1)^{-\frac{1}{2}}(1)$

$= x^{-2}(x-1)^{-\frac{3}{2}}[(x-1)^2(-1) + x\left(\dfrac{1}{2}\right)(x-1) + x^3\left(-\dfrac{1}{2}\right) + x^2(x-1)]$

$= x^{-2}(x-1)^{-\frac{3}{2}}[(-x^2 + 2x - 1) + \left(\dfrac{1}{2}x^2 - \dfrac{1}{2}x\right) + \left(-\dfrac{1}{2}x^3\right) + (x^3 - x^2)]$

$= \dfrac{x^3 - 3x^2 + 3x - 2}{2x^2(x-1)^{\frac{3}{2}}}$

domain: $x \neq 0,\ x \neq 1$

Chapter 2 Section 5 **Question 17 Page 126**

D

Chapter 2 Section 5 **Question 18 Page 126**

D

Chapter 2 Section 5 Extension **Question 1 Page 129**

a) $q'(x) = \dfrac{(x+1)(3) - 3x(1)}{(x+1)^2}$

$= \dfrac{3}{(x+1)^2}$

b) $f'(x) = \dfrac{(2x+3)(-1) - (-x)(2)}{(2x+3)^2}$

$= \dfrac{-3}{(2x+3)^2}$

c) $g'(x) = \dfrac{(5x-4)(2x) - (x^2)(5)}{(5x-4)^2}$

$= \dfrac{x(5x-8)}{(5x-4)^2}$

d) $r'(x) = \dfrac{(x^2-9)(16x) - (8x^2)(2x)}{(x^2-9)^2}$

$= \dfrac{-144x}{(x^2-9)^2}$

Chapter 2 Section 5 Extension **Question 2 Page 129**

a) $y' = \dfrac{(2x^2+5)(-1) - (-x+3)(4x)}{(2x^2+5)^2}$

$= \dfrac{2x^2 - 12x - 5}{(2x^2+5)^2}$

b) $y' = \dfrac{(x^3-2)(4) - (4x+1)(3x^2)}{(x^3-2)^2}$

$= -\dfrac{(8x^3 + 3x^2 + 8)}{(x^3-2)^2}$

c) $y' = \dfrac{(1+3x)(18x) - (9x^2-1)(3)}{(1+3x)^2}$

$= \dfrac{27x^2 + 18x + 3}{(1+3x)^2}$

$= \dfrac{3(1+3x)^2}{(1+3x)^2}$

$= 3$

d) $y' = \dfrac{(x^2 - x + 1)(4x^3) - x^4(2x - 1)}{(x^2 - x + 1)^2}$

$= \dfrac{2x^5 - 3x^4 + 4x^3}{(x^2 - x + 1)^2}$

$= \dfrac{x^3(2x^2 - 3x + 4)}{(x^2 - x + 1)^2}$

Chapter 2 Section 5 Extension **Question 3 Page 129**

a) $y' = \dfrac{(6x + 2)(2x) - x^2(6)}{(6x + 2)^2}$

$y'(-2) = \dfrac{(6(-2) + 2)(2(-2)) - (-2)^2(6)}{(6(-2) + 2)^2}$

$= \dfrac{4}{25}$

The slope of the tangent at $x = -2$ is $\dfrac{4}{25}$.

b) $y' = \dfrac{(3x^2 - 1)(0.5x^{-0.5}) - x^{0.5}(6x)}{(3x^2 - 1)^2}$

$y'(1) = \dfrac{(3(1)^2 - 1)(0.5(1)^{-0.5}) - (1)^{0.5}(6(1))}{(3(1)^2 - 1)^2}$

$= -\dfrac{5}{4}$

The slope of the tangent at $x = 1$ is $-\dfrac{5}{4}$.

c) $y' = \dfrac{(x^2 - 1)(4) - (4x + 1)(2x)}{(x^2 - 1)^2}$

$y'(-3) = \dfrac{((-3)^2 - 1)(4) - (4(-3) + 1)(2(-3))}{((-3)^2 - 1)^2}$

$= -\dfrac{17}{32}$

The slope of the tangent at $x = -3$ is $-\dfrac{17}{32}$.

d) $y' = \dfrac{(x^2 - x + 1)(2) - 2x(2x - 1)}{(x^2 - x + 1)^2}$

$y'(-1) = \dfrac{((-1)^2 - (-1) + 1)(2) - 2(-1)(2(-1) - 1)}{((-1)^2 - (-1) + 1)^2}$

$= 0$

The slope of the tangent at $x = -1$ is 0.

e)

$$y' = \frac{(x^2 + x - 1)(3x^2) - (x^3 - 3)(2x + 1)}{(x^2 + x - 1)^2}$$

$$y'(2) = \frac{((2)^2 + (2) - 1)(3(2)^2) - ((2)^3 - 3)(2(2) + 1)}{((2)^2 + (2) - 1)^2}$$

$$= \frac{7}{5}$$

The slope of the tangent at $x = 2$ is $\frac{7}{5}$.

Chapter 2 Section 5 Extension **Question 4 Page 129**

a) i) $\frac{dy}{dx} = \frac{(x^2 + 3x)^5(0) - (1)(5(x^2 + 3x)^4(2x + 3))}{(x^2 + 3x)^{10}}$

$$= \frac{-5(2x + 3)}{(x^2 + 3x)^6}$$

ii) $y = (x^2 + 3x)^{-5}$

$$\frac{dy}{dx} = -5(x^2 + 3x)^{-6}(2x + 3)$$

$$= \frac{-5(2x + 3)}{(x^2 + 3x)^6}$$

b) The power of a function rule is more efficient because the numerator is 1.

c) Answers may vary. For example:

I prefer the power of a functional rule because it is quicker.

Chapter 2 Section 5 Extension **Question 5 Page 129**

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \left(\frac{(u^2 + 1)(3u^2) - u^3(2u)}{(u^2 + 1)^2} \right)(3 - 2x)$$

$$= \frac{u^2(u^2 + 3)}{(u^2 + 1)^2}(3 - 2x)$$

When $x = 2$, $u = 3(2) - (2)^2$ or 2.

$$\left.\frac{dy}{dx}\right|_{x=2} = \frac{(2)^2((2)^2 + 3)}{((2)^2 + 1)^2}(3 - 2(2))$$

$$= -\frac{28}{25}$$

Chapter 2 Section 5 Extension **Question 6 Page 129**

$$y' = \frac{(2x+5)(2x)-(x^2)(2)}{(2x+5)^2}$$
$$= \frac{2x^2+10x}{(2x+5)^2}$$

Solve $y'(x) = 0$.

$$0 = \frac{2x^2+10x}{(2x+5)^2}$$
$$0 = 2x(x+5)$$
$$x = 0, x = -5$$

If $x = 0$, then $y = 0$ and if $x = -5$, then $y = -5$.
The points on the curve with horizontal tangent lines are the points (0, 0) and (–5, –5).

Chapter 2 Section 5 Extension **Question 7 Page 129**

No, the statement is not true.

$f'(x) = 75x^4 - 27x^2$ and $g'(x) = 6x$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{3x^2(75x^4-27x^2)-(15x^5-9x^3)(6x)}{(3x^2)^2}$$
$$= \frac{135x^6-27x^4}{9x^4}$$
$$= 15x^2-3$$
$$\frac{f'(x)}{g'(x)} = \frac{75x^4-27x^2}{6x}$$
$$= \frac{25}{2}x^3 - \frac{9}{2}x$$
$$\therefore \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \neq \frac{f'(x)}{g'(x)}$$

Chapter 2 Section 5 Extension **Question 8 Page 129**

a) $$n'(w) = \frac{(1+w^2)(600w)-300w^2(2w)}{(1+w^2)}$$
$$= \frac{600w}{(1+w^2)}$$

$$n'(1) = \frac{600(1)}{(1+1^2)}$$
$$= 150$$
$$n'(5) = \frac{600(5)}{(1+5^2)}$$
$$= \frac{750}{169}$$

b) No. The number of sales does not decrease since $n'(w)$ is not less than 0 for $0 \le w \le 10$.

Chapter 2 Section 5 Extension **Question 9 Page 129**

$$C'(t) = \frac{(3t^2+4)(4) - 4t(6t)}{(3t^2+4)^2}$$
$$C'(3) = \frac{(3(3)^2+4)(4) - 4(3)(6(3))}{(3(3)^2+4)^2}$$
$$= -\frac{92}{961}$$

$C'(3)$ represents the rate of change of concentration of the antibiotic in the blood at 3 h.

Chapter 2 Section 5 Extension **Question 10 Page 129**

a) $c'(t) = \frac{(2t^2+9)(6) - 6t(4t)}{(2t^2+9)^2}$
$= \frac{54 - 12t^2}{(2t^2+9)^2}$

$c'(1) = \frac{54 - 12(1)^2}{(2(1)^2+9)^2}$
$= \frac{42}{121}$

The rate is $\frac{42}{121}$ g/L per day.

$c'(4) = \frac{54 - 12(4)^2}{(2(4)^2+9)^2}$
$= -\frac{138}{1681}$

The rate is $-\frac{138}{1681}$ g/L per day.

$$c'(7) = \frac{54 - 12(7)^2}{(2(7)^2 + 9)^2}$$

$$= -\frac{534}{11\,449}$$

The rate is $-\frac{534}{11\,449}$ g/L per day.

b) Find t when $c'(t) = 0$.

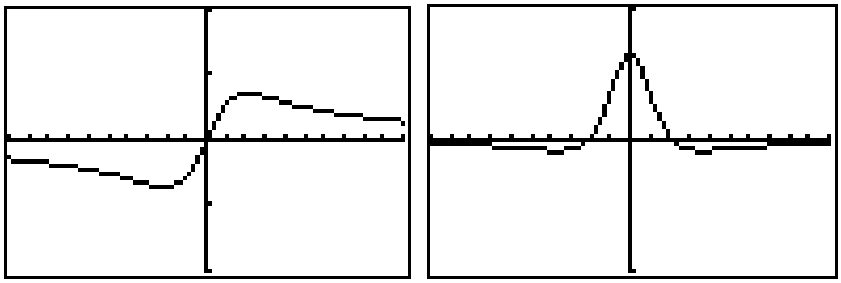
$$0 = \frac{54 - 12t^2}{(2t^2 + 9)^2}$$

$$0 = 54 - 12t^2$$

$$t = \frac{3}{\sqrt{2}}$$

The rate is positive for $0 \le t < \frac{3}{\sqrt{2}}$ days and negative for $t > \frac{3}{\sqrt{2}}$ days.

c)



d) The concentration of the cleaner rises to a maximum at $t = \frac{3}{\sqrt{2}}$ days and then starts to decrease.

e) $$c''(t) = \frac{(2t^2 + 9)^2(-24t) - (54 - 12t^2)(2)(2t^2 + 9)(4t)}{(2t^2 + 9)^4}$$

$$c''(4) = \frac{(2(4)^2 + 9)^2(-24(4)) - (54 - 12(4)^2)(2)(2(4)^2 + 9)(4(4))}{(2(4)^2 + 9)^4}$$

$$= \frac{480}{68\,921}$$

$c''(4)$ represents the rate of dissipation of the cleaner at 4 days.

Chapter 2 Section 6 **Rate of Change Problems**

Chapter 2 Section 6 **Question 1 Page 137**

a) $R(x) = xp(x)$

$= 575\sqrt{x} - 3x$

b) $R'(x) = 575(0.5)x^{-0.5} - 3$

$= \dfrac{575}{2\sqrt{x}} - 3$

c) $R'(200) = \dfrac{575}{2\sqrt{200}} - 3$

$= 17.33$

When 200 DVD's are sold, the marginal revenue is \$17.33 per DVD.

Chapter 2 Section 6 **Question 2 Page 137**

a) $P(x) = R(x) - C(x)$

$= (575\sqrt{x} - 3x) - (2000 + 150x - 0.002x^2)$

$= 0.002x^2 - 153x + 575\sqrt{x} - 2000$

b) $P'(x) = 0.004x + \dfrac{575}{2\sqrt{x}} - 153$

c) $P'(500) = 0.004(500) + \dfrac{575}{2\sqrt{500}} - 153$

$= -138.14$

When 500 DVD's are sold, the marginal profit is –\$138.14 per DVD.

Chapter 2 Section 6 **Question 3 Page 138**

a) $R(x) = xp(x)$

$= 17.5x$

b) $R'(x) = 17.5$

c) $P(x) = R(x) - C(x)$

$= (17.5x) - (-0.001x^3 + 0.025x^2 + 4x)$

$= 0.001x^3 - 0.025x^2 + 13.5x$

d) $P'(x) = 0.003x^2 - 0.05x + 13.5$

e) $R'(300) = 17.50$

$P'(300) = 0.003(300)^2 - 0.05(300) + 13.5$

$= 268.50$

When 300 large pizza combos are sold, the marginal revenue is \$17.50 per combo and the marginal profit is \$268.50 per combo.

Chapter 2 Section 6 **Question 4 Page 138**

a) Average linear density $= \dfrac{f(8) - f(1)}{8 - 1}$

$= \dfrac{\sqrt{2(8) - 1} - \sqrt{2(1) - 1}}{8 - 1}$

$= 0.41$

The average linear density is 0.41 g/m.

b) $f'(x) = 0.5(2x - 1)^{-0.5}(2)$

$= \dfrac{1}{\sqrt{2x - 1}}$

$f'(5) = \dfrac{1}{3}$, $f'(8) = \dfrac{1}{\sqrt{15}}$

The linear mass density of the wire is $\dfrac{1}{3}$ g/m at x = 5 m and $\dfrac{1}{\sqrt{15}}$ g/m at x = 8 m.

The density of the wire at x = 8 m is less than the density at x = 5 m, so the density of the wire decreases as the distance increases.

Chapter 2 Section 6 **Question 5 Page 138**

a) Let n represent the number of price decreases, and x represent the number of cans sold.

$x = 270 + 6n$

$p(x) = 32 - 1.2n$

$= 32 - 1.2\left(\dfrac{x - 270}{6}\right)$

$= 86 - 0.2x$

b) $R(x) = xp(x)$

$= 86x - 0.2x^2$

c) $R'(x) = 86 - 0.4x$

d) Solve $R'(x) = 0$.

$0 = 86 - 0.4x$

$x = 215$

When 215 cans per month are sold, revenue is at a maximum.

e) Solve for $p(215)$.

$$p(215) = 86 - 0.2(215)$$
$$= 43$$

The price in this case is \$43.00. This is the price of at which the revenue will be a maximum.

Chapter 2 Section 6 **Question 6 Page 138**

a) Average linear density $= \dfrac{f(3) - f(1)}{3 - 1}$

$$= \frac{(3 - 0.5)^3 + 5(3) - [(1 - 0.5)^3 + 5(1)]}{2}$$
$$= 12.75$$

The average linear density is 12.75 kg/m.

b) $f'(x) = 3(x - 0.5)^2 + 5$

$$f'(2) = 3(2 - 0.5)^2 + 5$$
$$= 11.75$$

The linear mass density of the rod at x = 2 m is 11.75 kg/m.

Chapter 2 Section 6 **Question 7 Page 138**

a) i) The CPI is increasing since the slope of the graph is positive.

ii) The rate of growth during this period is positive since the slope of the graph is positive.

b) i, ii, vi, v, iii, iv
I ordered the intervals from the least steep to the interval with the steepest slope.
A to B: 1951 – 1961; B to C: 1961 – 1971; C to D: 1971 – 1981; D to E: 1981 – 1983;
E to F: 1983 – 1991; F to G: 1991 – 2007

c) The rate of inflation is higher after 1975 because the slope of the graph is greater after 1975. I can conclude that the economy was doing well.

d) The rate of inflation has been decreasing since the slope of the graph begins to decrease at this time.

Chapter 2 Section 6 **Question 8 Page 139**

a) Find $C'(4000)$.

$$C'(x) = 0.0002x + 2$$
$$C'(4000) = 2.80$$

The marginal cost of producing 4000 containers of yogurt is \$2.80 per container.

b) Find $P'(4000)$.

$$P(x) = R(x) - C(x)$$
$$= 4.5x - (0.0001x^2 + 2x + 3200)$$
$$P'(x) = 2.5 - 0.0002x$$
$$P'(4000) = 1.70$$

The marginal profit from producing 4000 containers of yogurt is \$1.70 per container.

c) Since the revenue for selling x containers of yogourt is $R(x) = 4.5x$, the price of one container is \$4.50.

Chapter 2 Section 6 **Question 9 Page 139**

a) Determine $C'(5)$.

$$C'(5) = 5 + 40$$
$$= 45$$

The marginal cost of producing 5000 kitchen utensils is \$45 per utensil.

$$C(5.001) - C(5.000) = 0.5(5.001)^2 + 40(5.001) + 8000 - (0.5(5.000)^2 + 40(5.000) + 8000)$$
$$= 0.045$$

The cost of producing the 5001st item is \$0.045.

b) $$\text{Average cost} = \frac{C(5)}{5000}$$
$$= \frac{0.5(5)^2 + 200 + 8000}{5000}$$
$$= 1.64$$

The average cost of producing 5000 items is \$1.64 per item.

c) Let $A(x)$ represent the average cost, so find $A'(5)$.

$$A(x) = \frac{C(x)}{1000x}$$
$$= \frac{0.5x^2 + 40x + 8000}{1000x}$$
$$= \frac{x}{2000} + \frac{1}{25} + \frac{8}{x}$$
$$A'(x) = \frac{1}{2000} - \frac{8}{x^2}$$
$$A'(5) = -0.32$$

The rate of change of the average cost of producing 5000 items is –\$0.32 per item.
This is the change in cost of producing an additional item.

Chapter 2 Section 6 **Question 10 Page 139**

a) $P(0) = 12\,500$; The present population is 12 500 people.

b) $P'(x) = 320 - 0.75x^2$

$P'(3) = 313.25$; $P'(8) = 272$

The rate of change of the population in 3 years is 313.25 people/year and 272 people/year in 8 years.

c) Find x when $P(x) = 16\,294$.

$16\,294 = 12\,500 + 320x - 0.25x^3$

$0 = 0.25x^3 - 320x + 3794$

Using CAS or a graphing calculator, the positive solutions are $x = 14$, $x = (\sqrt{133} - 7)$ or 26.66.

The population will reach 16 294 after 14 and 26.66 years.

d) Set $P'(x) = 245$.

$245 = 320 - 0.75x^2$

$x = \pm 10$

The rate of growth of the population will be 245 people/year after 10 years.

e) First find when $P'(x) = 0$.

$0 = 320 - 0.75x^2$

$x = 20.66$

If $0 \le x < 20.66$ years, then $P'(x) > 0$ and if $x > 20.66$ years, then $P'(x) < 0$.

The population will increase up to 20.66 years and then will start to decrease.

Chapter 2 Section 6 **Question 11 Page 139**

a) Find $C'(750)$.

$C'(x) = 3450 - 2.04x$

$C'(750) = 1920$

The marginal cost at a production level of 750 hot tubs is \$1920 per hot tub. The marginal cost shows that the rate of change in cost of producing x items reduces for greater values of x.

b) $C(751) - C(750) = 3450(751) - 1.02(751^2) - (3450(750) - 1.02(750^2))$

$= 1918.98$

The cost of producing the 751st tub is \$1918.98.

c) The cost of the 751st hot tub is less than the marginal cost at 750 tubs.

d) $R(x) = 9200x$

e) Find $P'(750)$.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 9200x - (3450x - 1.02x^2) \\ &= 5750x + 1.02x^2 \\ P'(x) &= 5750 + 2.04x \\ P'(750) &= 7280 \end{aligned}$$

The rate of change of profit for the sale of 750 hot tubs is \$7280 per hot tub.

Chapter 2 Section 6 **Question 12 Page 139**

Find $I'(18)$.

$$I'(R) = -\frac{120}{R^2}$$

$$\begin{aligned} I'(18) &= -\frac{120}{18^2} \\ &= -\frac{10}{27} \end{aligned}$$

The rate of change of the current when the resistance is 18 Ω is $-\frac{10}{27}$ A/Ω.

Chapter 2 Section 6 **Question 13 Page 139**

$$\begin{aligned} F &= 1.8(10 + 4t) + 32 \\ F &= 7.2t + 50 \\ F'(t) &= 7.2 \end{aligned}$$

The rate of change of the temperature of the bar is 7.2°F/min for any t, so at t = 4 min too.

Chapter 2 Section 6 **Question 14 Page 139**

$$\begin{aligned} f'(x) &= \frac{(3x^{-0.5} + 18)(155(-0.5)x^{-1.5}) - (155x^{-0.5} + 85)(3)(-0.5)x^{-1.5}}{\left(3x^{-0.5} + 18\right)^2} \\ &= \frac{-3(155)(0.5)x^{-2} - 9(155)x^{-1.5} + 3(155)(0.5)x^{-2} + 85(3)(0.5)x^{-1.5}}{\left(3x^{-0.5} + 18\right)^2} \\ &= \frac{-1267.5x^{-1.5}}{\left(3x^{-0.5} + 18\right)^2} \\ &= -\frac{2535}{2x^{1.5}\left(3x^{-0.5} + 18\right)^2} \end{aligned}$$

This expression is always negative in value. The pupil gets smaller as it is exposed to more light.

Chapter 2 Section 6 **Question 15 Page 139**

a) Find $R'(1000)$ and $R'(5000)$.

$$R'(x) = \frac{(x^2+15)(15-2x)-(15x-x^2)(2x)}{(x^2+15)^2}$$

$$= \frac{-15x^2-30x+225}{(x^2+15)^2}$$

$R'(1000) = -0.000\,015$

$R'(5000) = -0.000\,000\,6$

The rate of change for the sale of 1000 items is \$–0.015 per item and is \$–0.0006 for 5000 items.

b) The rate of change of the revenue decreases with the increase in sales.

c) Find x for $R'(x) = 0$.

$$0 = \frac{-15x^2-30x+225}{(x^2+15)^2}$$

$0 = -15x^2-30x+225$

$0 = (x-3)(x+5)$

$x = 3$ or $x = -5$

Since x cannot be negative, 3 items must be sold to obtain a \$0 rate of change in revenue.

d) Find $R(3)$.

$$R(3) = \frac{15(3)-3^2}{3^2+15}$$

$$= 1.5$$

The revenue for 3 items sold is \$1500.

The number of items that need to be sold in order to maximize revenue is 3 items.

Chapter 2 Section 6 **Question 16 Page 140**

a) Find $V'(2.75)$.

$V'(r) = 5cr - 3cr^2$

$V'(2.75) = -8.9375c$

The rate of change of velocity of air when r = 2.75 cm is $8.9375c$.

b) Find r for $V'(r) = 0$.

$0 = 5cr - 3cr^2$

$0 = cr(5 - 3r)$

$r = 0$ or $r = \frac{5}{3}$

Since the radius of the windpipe is 2.5 cm when there is no cough, it cannot have a radius of $\frac{5}{3}$ cm so $r = 0$. This means that the rate of change of airflow is zero when the windpipe is closed.

Chapter 2 Section 6 **Question 17 Page 140**

a) Let n represent the number of price decreases, and x represent the number of lattes sold.

$x = 500 + 125n$

$p(x) = 4.75 - 0.25n$

$= 4.75 - 0.25\left(\frac{x - 500}{125}\right)$

$= 5.75 - 0.002x$

b) $R(x) = xp(x)$

$= 5.75x - 0.002x^2$

$R'(x) = 5.75 - 0.004x$

$R(350) = 5.75(350) - 0.002(350)^2$

$= 1767.50$

$R'(350) = 5.75 - 0.004(350)$

$= 4.35$

The revenue and marginal revenue from the monthly sales of 350 mocha lattes is \$1767.50 and \$4.35 per latte, respectively.

c) $C'(x) = -0.001x + 3.5$

$C'(350) = -0.001(350) + 3.5$

$= 3.15$

The marginal cost of producing 350 lattes is \$3.15 per latte.

d)

$C(351) - C(350) = -0.0005(351^2 - 350^2) + 3.5(1)$

$= -0.0005(701) + 3.5$

$= 3.1495$

The actual cost of producing the 351st latte is \$3.1495.

e)

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= (5.75x - 0.002x^2) - (-0.0005x^2 + 3.5x + 400) \\ &= -0.0015x^2 + 2.25x - 400 \end{aligned}$$

$$P(350) = 203.75$$

$$\begin{aligned} P'(x) &= -0.003x + 2.25 \\ P'(350) &= -0.003(350) + 2.25 \\ &= 1.20 \end{aligned}$$

The profit and marginal profit from selling 350 lattes is \$203.75 and \$1.20 per latte respectively.

f) Average revenue $= \dfrac{R(360)}{360}$

$$= \frac{5.75(360) - 0.002(360)^2}{360}$$

$$= 5.03$$

Average profit $= \dfrac{P(360)}{360}$

$$= \frac{-0.0015(360)^2 + 2.25(360) - 400}{360}$$

$$= 0.60$$

The profit is much lower at \$0.60 than the revenue of \$5.03 due to the cost of producing the mocha lattes.

Chapter 2 Section 6 **Question 18 Page 140**

a) Determine $M'(6)$.

$$\begin{aligned} M'(t) &= \frac{(t + 2.2)(6.3) - 6.3t(1)}{(t + 2.2)^2} \\ &= \frac{13.86}{(t + 2.2)^2} \end{aligned}$$

$$\begin{aligned} M'(6) &= \frac{13.86}{(6 + 2.2)^2} \\ &= 0.206 \end{aligned}$$

The rate of change of the mass after 6 s is 0.206 g/s.

b) No. The numerator and denominator of $M'(t)$ are always positive, so $M'(t)$ is always positive.

Chapter 2 Section 6 **Question 19 Page 140**

a) $R(x) = xp(x)$

$= 650\sqrt{x} - 4.5x$

b) $R'(x) = (0.5)650x^{-0.5} - 4.5$

$= \frac{325}{\sqrt{x}} - 4.5$

$R'(500) = \frac{325}{\sqrt{500}} - 4.5$

$= 10.03$

The marginal revenue for the sale of 500 PDAs is \$10.03 per PDA.

c) $P(x) = R(x) - C(x)$

$= (650\sqrt{x} - 4.5x) - 125x$

$= 650\sqrt{x} - 129.5x$

d) $P'(x) = (0.5)650x^{-0.5} - 129.5$

$= \frac{325}{\sqrt{x}} - 129.5$

$P'(500) = \frac{325}{\sqrt{500}} - 129.5$

$= -114.97$

The marginal profit for the sale of 500 PDAs is –\$114.97 per PDA.

Chapter 2 Section 6 **Question 20 Page 140**

$$\frac{dr}{dm} = \frac{2m}{a}\left(\frac{1}{b} - \frac{m}{c}\right) + \frac{m^2}{a}\left(-\frac{1}{c}\right)$$

$$= \frac{2m}{ab} - \frac{3m^2}{ac}$$

$$\left.\frac{dr}{dm}\right|_{m=15} = \frac{2(15)}{ab} - \frac{3(15)^2}{ac}$$

$$= \frac{15(2c - 45b)}{abc}$$

Chapter 2 Section 6 **Question 21 Page 141**

a) $\frac{du}{dM} = \frac{(M+m)(V(1+c)+vc)-[MV(1+c)+v(cM-m)(1)]}{(M+m)^2}$

$= \frac{MV(1+c)+Mvc+mV(1+c)+mvc-MV(1+c)-vcM+vm}{(M+m)^2}$

$= \frac{V(1+c)m+cvm+vm}{(M+m)^2}$ as required

b) i) Find $\frac{du}{dM}$ using $m = 150$ g, $c = 0.575$, $v = 40$ m/s, and $V = -35$ m/s.

$\frac{du}{dM} = \frac{-35(1+0.575)(150)+(150)(40)(0.575)+40(150)}{(M+150)^2}$

$= \frac{1181.25}{(M+150)^2}$

ii) Find $\frac{du}{dM}$, where M = 1050 g.

$\left.\frac{du}{dM}\right|_{M=1050} = \frac{1181.25}{(1050+150)^2}$

$= 0.000\,82$

The rate of change of the velocity of the ball is 0.000 82 m/s^2.

Chapter 2 Section 6 **Question 22 Page 141**

D

Chapter 2 Section 6 **Question 23 Page 141**

C

Chapter 2 Review

Chapter 2 Review Question 1 Page 142

a) $h'(t) = 3t^2 - 4t - \dfrac{2}{t^3}$

Use the sum rule and the power rule.

b) $p(n) = -n^5 + 5n^3 + n^{\frac{2}{3}}$

$p'(n) = -5n^4 + 15n^2 + \dfrac{2}{3\sqrt[3]{n}}$

Use the sum rule and the power rule.

c) $p(r) = r^6 - \dfrac{2}{5}r^{-\frac{1}{2}} + r - 1$

$p'(r) = 6r^5 + \dfrac{1}{5\sqrt{r^3}} + 1$

Use the sum rule, the power rule, and the constant rule.

Chapter 2 Review Question 2 Page 142

a) $V'(r) = 4\pi r^2$

$V'(1.5) = 4\pi(1.5)^2$

$= 9\pi$

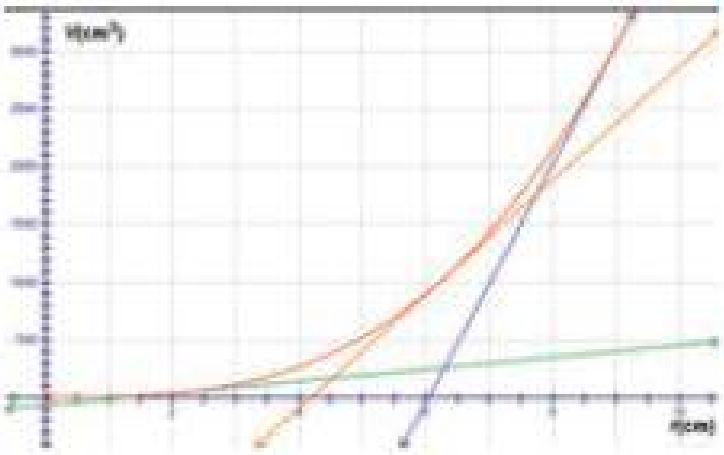
$V'(6) = 4\pi(6)^2$

$= 144\pi$

$V'(9) = 4\pi(9)^2$

$= 324\pi$

b)



c) If $r = 1.5$, then $V(1.5) = 4.5\pi$. Use $(1.5, 4.5\pi)$ and $m = 9\pi$ to find b.

$4.5\pi = 9\pi(1.5) + b$

$b = -9\pi$

So $y_{1.5} = 9\pi x - 9\pi$ or $y_{1.5} = 28.27x - 28.27$

If $r = 6$, then $V(6) = 288\pi$. Use $(6, 288\pi)$ and $m = 144\pi$ to find b.

$288\pi = 144\pi(6) + b$

$b = -576\pi$

So $y_6 = 144\pi x - 576\pi$ or $y_6 = 452.39x - 1809.56$

If $r = 9$, then $V(9) = 972\pi$. Use $(9, 972\pi)$ and $m = 324\pi$ to find b.

$972\pi = 324\pi(9) + b$

$b = -1944\pi$

So $y_9 = 324\pi x - 1944\pi$ or $y_9 = 1017.88x - 6107.26$

Chapter 2 Review **Question 3 Page 142**

a) $f(x) = (5x + 3)(2x - 11)$

$= 10x^2 - 49x - 33$

$f'(x) = 20x - 49$

b) $h(t) = 8t^3 - 10t^2 + 4t^{\frac{4}{3}} - 5t^{\frac{1}{3}}$

$h'(t) = 24t^2 - 20t + \frac{16}{3}t^{\frac{1}{3}} - \frac{5}{3}t^{-\frac{2}{3}}$

$= \frac{72t^{\frac{8}{3}} - 60t^{\frac{5}{3}} + 16t - 5}{3t^{\frac{2}{3}}}$

c) $g(x) = -4.5x^6 + 3 + 12x^7 - 8x$

$g'(x) = 84x^6 - 27x^5 - 8$

d) $p(n) = -55n + 33n^3 - 10 + 6n^2$

$p'(n) = 99n^2 + 12n - 55$

Chapter 2 Review **Question 4 Page 142**

a) $y' = (6x - 3)(-2x) + (6)(-x^2 + 2)$

$= -18x^2 + 6x + 12$

$y'(1) = -18(1)^2 + 6(1) + 12$

$= 0$

When $x = 1$, $y = 3$ so use the point $(1, 3)$ and $m = 0$ in $y = mx + b$ to find b.

$3 = 0(1) + b$ so $b = 3$.

The equation of the tangent to y at $x = 1$ is $y = 3$.

b) $y' = (-3x + 8)(3x^2) + (-3)(x^3 - 7)$

$= -12x^3 + 24x^2 + 21$

$y'(2) = -12(2)^3 + 24(2)^2 + 21$

$= 21$

When $x = 2$, $y = 2$ so use the point (2, 2) and $m = 21$ in $y = mx + b$ to find b.

$2 = 21(2) + b$ so $b = -40$.

The equation of the tangent to y at $x = 2$ is $y = 21x - 40$.

Chapter 2 Review **Question 5 Page 142**

$f'(x) = (-2x)(3x + 1) + 3(4 - x^2)$

$= -9x^2 - 2x + 12$

$f''(x) = -18x - 2$

$f''(-2) = 34$

Chapter 2 Review **Question 6 Page 142**

a) $h(2) = -4.9(2)^2 + 15(2) + 0.4$

$= 10.8$

The height of the missile after 2 s is 10.8 m.

b) Find $h'(1)$ and $h'(4)$.

$h'(t) = -9.8t + 15$

$h'(1) = 5.2$

$h'(4) = -24.2$

The rate of change of the height of the missile at 1 s is 5.2 m/s and –24.2 m/s at 4 s.

c) When the missile hits the ground $h(t) = 0$.

$0 = -4.9t^2 + 15t + 0.4 = 0$

$$t = \frac{-15 \pm \sqrt{15^2 - 4(-4.9)(0.4)}}{2(-4.9)}$$

$t = 3.088$ or $t = -0.27$

Since time cannot be negative, the toy missile returns to the ground when $t = 3.088$ s.

d) Find $h'(3.088)$.

$h'(3.088) = -9.8(3.088) + 15$

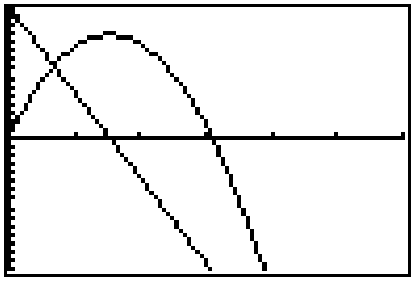
$= -15.26$

The missile was travelling –15.26 m/s when it hit the ground.

e) **i)** The missile reach it's maximum height when $t = 1.53$ s.

ii) The maximum height of the missile is 11.88 m.

iii) The velocity of the missile when it reaches its maximum height is 0 m/s. I can tell since the graph of the velocity is zero at this point.



Chapter 2 Review **Question 7 Page 142**

a) Find $p'(5)$.

$$p'(t) = \frac{1}{3}(16t + 50t^3)^{-\frac{2}{3}}(16 + 150t^2)$$

$$p'(5) = \frac{(16(5) + 50(5)^3)^{-\frac{2}{3}}(16 + 150(5)^2)}{3}$$

$$\doteq 3.67$$

The rate is 3.67 bushes/year.

b) Set $p(t) = 40$ and solve for t.

$$40 = (16t + 50t^3)^{\frac{1}{3}}$$

$$40^3 = 16t + 50t^3$$

$$0 = 50t^3 + 16t - 64000$$

Solving by using CAS or a graphing calculator, t = 10.85 years.

c) Solve for $p'(10.85)$.

$$p'(10.85) = \frac{(16(10.85) + 50(10.85)^3)^{-\frac{2}{3}}(16 + 150(10.85)^2)}{3}$$

$$= 3.68$$

The rate is 3.68 bushes/year.

Chapter 2 Review **Question 8 Page 142**

a) $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$= (2u + 3)\left(\frac{1}{2}(x-1)^{-\frac{1}{2}}\right)$$

When $x = 5$, $u = 2$.

$$\left.\frac{dy}{dx}\right|_{x=5} = (2(2)+3)\left(\frac{1}{2}(5-1)^{-\frac{1}{2}}\right)$$

$$= \frac{7}{4}$$

b) $$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{\sqrt{2}}{2}(u)^{-\frac{1}{2}}(-1)$$

When $x = -3$, $u = 9$.

$$\left.\frac{dy}{dx}\right|_{x=-3} = -\frac{\sqrt{2}}{2}(9)^{-\frac{1}{2}}$$

$$= -\frac{1}{3\sqrt{2}}$$

c) $$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= (8(1-u)-8u)(-x^{-2})$$

When $x = 4$, $u = \frac{1}{4}$.

$$\left.\frac{dy}{dx}\right|_{x=4} = (8(1-0.25)-8(0.25))(-(4)^{-2})$$

$$= -\frac{1}{4}$$

Chapter 2 Review **Question 9 Page 142**

a) $$y' = 2x^2(-1)(x+1)^{-2} + 4x(x+1)^{-1}$$

$$y'(2) = 2(2)^2(-1)(2+1)^{-2} + 4(2)(2+1)^{-1}$$

$$= \frac{16}{9}$$

The slope of the tangent at $x = 2$ is $\frac{16}{9}$.

b) $y' = (3x)^{0.5}(-1)(x^2-4)^{-2}(2x) + 3.5(3x)^{-0.5}(x^2-4)^{-1}$

$y'(3) = (3(3))^{0.5}(-1)(3^2-4)^{-2}(2(3)) + 1.5(3(3))^{-0.5}(3^2-4)^{-1}$

$= -\frac{31}{50}$

The slope of the tangent at $x = 3$ is $-\frac{31}{50}$.

c) $y' = (5x+3)(-1)(x^3+1)^{-2}(3x^2) + 5(x^3+1)^{-1}$

$y'(-2) = (5(-2)+3)(-1)((-2)^3+1)^{-2}(3(-2)^2) + 5((-2)^3+1)^{-1}$

$= 1$

The slope of the tangent at $x = -2$ is 1.

d) $y'(x) = (-4x+2)(-1)(3x^2-7x-1)^{-2}(6x-7) - 4(3x^2-7x-1)^{-1}$

$y'(1) = (-4(1)+2)(-1)(3(1)^2-7(1)-1)^{-2}(6(1)-7) - 4(3(1)^2-7(1)-1)^{-1}$

$= \frac{18}{25}$

The slope of the tangent at $x = 1$ is $\frac{18}{25}$.

Chapter 2 Review **Question 10 Page 143**

a) $q'(x) = \frac{(4x^2-3)^3(-7) - (-7x+2)(3)(4x^2-3)^2(8x)}{(4x^2-3)^6}$

$= \frac{(4x^2-3)(-7) - (-7x+2)(24x)}{(4x^2-3)^4}$

$= \frac{140x^2 - 48x + 21}{(4x^2-3)^4}$

b) $\frac{dy}{dx} = \frac{(3x-2)^{\frac{1}{2}}(24x^2) - (8x^3)\left(\frac{1}{2}\right)(3x-2)^{-\frac{1}{2}}(3)}{(3x-2)}$

$= \frac{(3x-2)(24x^2) - 12x^3}{(3x-2)^{\frac{3}{2}}}$

$= \frac{12x^2(5x-4)}{(3x-2)^{\frac{3}{2}}}$

c) $m'(x) = \dfrac{(3+5x)^4(2(-x+2))(-1)-(-x+2)^2(4)(3+5x)^3(5)}{(3+5x)^8}$

$= \dfrac{(3+5x)(2x-4)-20(-x+2)^2}{(3+5x)^5}$

$= \dfrac{-2(x-2)(5x-23)}{(3+5x)^5}$

d) $\dfrac{dy}{dx} = \dfrac{(4x+5)^{\frac{1}{2}}(2(x^2-3)(2x)-(x^2-3)^2\left(\frac{1}{2}\right)(4x+5)^{-\frac{1}{2}}(4)}{(4x+5)}$

$= \dfrac{(4x+5)(4x)(x^2-3)-2(x^2-3)^2}{(4x+5)^{\frac{3}{2}}}$

$= \dfrac{2(x^2-3)(7x^2+10x+3)}{(4x+5)^{\frac{3}{2}}}$

e) $\dfrac{dy}{dx} = \dfrac{(x^3-3x^2+1)^7(3)(2x^{\frac{1}{2}}+7)^2(x^{-\frac{1}{2}})-(2x^{\frac{1}{2}}+7)^3(7)(x^3-3x^2+1)^6(3x^2-6x)}{(x^3-3x^2+1)^{14}}$

$= \dfrac{(2x^{\frac{1}{2}}+7)^2[(x^3-3x^2+1)(3)(x^{-\frac{1}{2}})-7(2x^{\frac{1}{2}}+7)^3(3x^2-6x)]}{(x^3-3x^2+1)^8}$

$= \dfrac{3(2x^{\frac{1}{2}}+7)^2[x^3-3x^2+1-14x^3+28x^2-49x^{\frac{5}{2}}+98x^{\frac{3}{2}}]}{x^{\frac{1}{2}}(x^3-3x^2+1)^8}$

$= \dfrac{3(2x^{\frac{1}{2}}+7)^2[-13x^3+25x^2+1-49x^{\frac{5}{2}}+98x^{\frac{3}{2}}]}{x^{\frac{1}{2}}(x^3-3x^2+1)^8}$

Chapter 2 Review **Question 11 Page 143**

$y' = \dfrac{3(x^2-1)^2(2x)(4x+7)-(x^2-1)^3(3)(4x+7)^2(4)}{(4x+7)^6}$

$y'(-2) = \dfrac{3((-2)^2-1)^2(2(-2))(4(-2)+7)-((-2)^2-1)^3(3)(4(-2)+7)^2(4)}{(4(-2)+7)^6}$

$= -216$

When $x = -2$, $y = -27$ so use the point $(-2, -27)$ and $m = -216$ to find b in $y = mx + b$.
$-27 = -216(-2) + b$ so $b = -459$.

The equation of the tangent to the curve at $x = -2$ is $y = -216x - 459$.

Chapter 2 Review **Question 12 Page 143**

a) Let x be the number of CD's sold and n be the number of price decreases.

$x = 120 + 5n$

$n = \dfrac{x - 120}{5}$

$p(x) = 24 - 0.75n$

$= 24 - 0.75\left(\dfrac{x - 120}{5}\right)$

$= 42 - 0.15x$

b) $R(x) = xp(x)$

$= 42x - 0.15x^2$

$R'(x) = 42 - 0.3x$

$R'(150) = -3.00$

The marginal revenue from the weekly sales of 150 music CDs is –\$3.00 per CD.

c) Find $C'(150)$.

$C'(x) = -0.006x + 4.2$

$C'(150) = -0.006(150) + 4.2$

$= 3.30$

The marginal cost of producing 150 CDs is \$3.30 per CD.

d) $P(x) = R(x) - C(x)$

$= (42x - 0.15x^2) - (-0.003x^2 + 4.2x + 3000)$

$= -0.147x^2 + 37.8x - 3000$

$P'(x) = -0.294x + 37.8$

$P'(150) = -6.30$

The marginal profit from the sales of 150 CDs is –\$6.30 per CD.

Chapter 2 Review **Question 13 Page 143**

a) $V(t) = I(t)R(t)$

$= (4.85 - 0.01t^2)(15 + 0.11t)$

b) Find $V'(t)$.

$V'(t) = (4.85 - 0.01t^2)(0.11) + (15 + 0.11t)(-0.02t)$

$= -0.0033t^2 - 0.3t + 0.5335$

This expression represents the change in voltage over time.

c) Find $V'(2)$.

$V'(2) = -0.0033(2)^2 - 0.3(2) + 0.5335$

$= -0.0797$

The rate of change of voltage after 2 s is –0.0797 V/s.

d) Find $I'(2)$.

$I'(t) = -0.02t$

$I'(2) = -0.04$

The rate of change of current after 2 s is 0.04 A/s.

e) Find $R'(2)$.

$R'(t) = 0.11$

$R'(2) = 0.11$

The rate of change of resistance after 2 s is 0.11 Ω/s.

f) No. By the product rule: $V'(2) = I'(2)R(2) + I(2)R'(2) \neq I'(2)R'(2)$.

Chapter 2 Practice Test

Chapter 2 Practice Test **Question 1 Page 144**

B; Justifications may vary. For example:

$$\frac{d}{dx}\left(x^2+2x\right)^3 = 3(x^2+2x)^2(2x+2)$$
$$\frac{d}{dx}\left(x^3\right)\frac{d}{dx}\left(x^2+2x\right) = 3x^2(2x+2)$$
$$\frac{d}{dx}\left(x^2+2x\right)^3 \neq \frac{d}{dx}\left(x^3\right)\frac{d}{dx}\left(x^2+2x\right)$$

Chapter 2 Practice Test **Question 2 Page 144**

C; Answer may vary. For example:

In case A, if the velocity is positive and the acceleration is negative, then the product $a(t)v(t)$ is negative. However, an object with negative acceleration is slowing down so A is incorrect.

In case B, if the velocity is positive and the acceleration is positive, then the product $a(t)v(t)$ is positive. However, an object with positive acceleration is speeding up so B is incorrect.

In case D, an object with an acceleration of zero can be travelling at a constant speed so it does not have to be at rest. D is incorrect.

Chapter 2 Practice Test **Question 3 Page 144**

A and B are incorrect derivatives. Justifications may vary. For example:

In case A, the derivative of y is expressed as the derivative of the numerator divided by the derivative of the denominator, which is incorrect when taking the derivative of a quotient.

In case B, the quotient rule has been applied but the sign in front of the $4x(2x)$ should be a +.

Cases C and D are correct derivatives because one uses the quotient rule properly and the other uses the product rule.

Chapter 2 Practice Test **Question 4 Page 144**

$$f'(x) = 2(5x^2-3x)(10x-3)$$
$$f''(x) = 2[(5x^2-3x)(10)+(10x-3)(10x-3)]$$
$$f''(3) = 2[360+27^2]$$
$$= 2178$$

Chapter 2 Practice Test **Question 5 Page 144**

a) Use the power of a function rule.

$$\frac{dy}{dx} = \frac{1}{3}(3x^6)^{-\frac{2}{3}}(18x^5)$$
$$= \frac{6x^5}{(3x^6)^{\frac{2}{3}}}$$
$$= 2 \cdot 3^{\frac{1}{3}}x$$

Alternatively, evaluate the equation and use the power rule.

$$y = 3^{\frac{1}{3}}x^2$$
$$\frac{dy}{dx} = 2 \cdot 3^{\frac{1}{3}}x$$

b) Evaluate and use the sum rule.

$$y = 2x^3 - 8x + x^2 - 4$$
$$\frac{dy}{dx} = 6x^2 + 2x - 8$$

Alternatively, use the product rule.

$$\frac{dy}{dx} = (x^2 - 4)(2) + 2x(2x + 1)$$
$$= 6x^2 + 2x - 8$$

Chapter 2 Practice Test **Question 6 Page 144**

a) $\frac{dy}{dx} = -15x^2 - \frac{20}{x^6}$

$$= \frac{-5(3x^8 + 4)}{x^6}$$

b) $g'(x) = 3(8x^2 - 3x)^2(16x - 3)$

$$= 3x^2(8x - 3)^2(16x - 3)$$

c) $m'(x) = \frac{1}{2}(9-2x)^{-\frac{1}{2}}(-2)\left(x^2+\frac{2}{x^3}\right)+(9-2x)^{\frac{1}{2}}\left(2x-\frac{6}{x^4}\right)$

$$= \frac{-x^4\left(x^2+\frac{2}{x^3}+(9-2x)(2x^5-6)\right)}{x^4\sqrt{9-2x}}$$

$$= \frac{-x^6+2x+18x^5-54-4x^6-12x}{x^4\sqrt{9-2x}}$$

$$= -\frac{(5x^6-18x^5-10x+54)}{x^4\sqrt{9-2x}}$$

d) $f'(x) = \frac{(1-x^2)^{\frac{1}{2}}(3)-(3x-2)\left(\frac{1}{2}\right)(1-x^2)^{-\frac{1}{2}}(-2x)}{(1-x^2)}$

$$= \frac{(1-x^2)(3)+x(3x-2)}{(1-x)^{\frac{3}{2}}}$$

$$= \frac{3-2x}{(1-x)^{\frac{3}{2}}}$$

Chapter 2 Practice Test **Question 7 Page 144**

a) Determine $v(t)$ and $a(t)$ after 3 s.

$v(t) = -9.8t + 11$ and $a(t) = -9.8$

$v(3) = -18.4$ m/s and $a(3) = -9.8$ m/s^2

b) The arrow moves upward when $v(t) > 0$ and downwards when $v(t) < 0$ until it hits the ground.

The arrow changes direction when $v(t) = 0$, so when t = 1.12 s.

upward: $0 \le t < 1.12$; downward: $t > 1.12$

c) The arrow is at rest when $v(t) = 0$, when $t = 1.12$ s.

d) The height at t = 1.12 s is the maximum height.

$h(1.12) = -4.9(1.12)^2 + 11(1.12) + 2$

≈ 8.17

The maximum height is about 8.17 m.

e) The arrow hits the ground when $h(t) = 0$.

$$0 = -4.9t^2 + 11t + 2$$

$$t = \frac{-11 \pm \sqrt{11^2 - 4(-4.9)(2)}}{2(-4.9)}$$

$$= \frac{11 \pm 12.66}{9.8}$$

Since time cannot be negative, $t = 2.41$ s.

$$v(2.41) = -9.8(2.41) + 11$$

$$= -12.62$$

The arrow hits the ground at a velocity of 12.62 m/s.

Chapter 2 Practice Test **Question 8 Page 144**

$$y' = \frac{(3x+2)^3(-1) + x(3)(3x+2)^2(3)}{(3x+2)^6}$$

$$y'(-1) = \frac{(-1)^3(-1) + (-1)(3)(-1)^2(3)}{(-1)^6}$$

$$= -8$$

When $x = -1$, $y = -1$ so use $(-1, -1)$ and $m = -8$ to find b in $y = mx + b$.

$-1 = -8(-1) + b$ so $b = -9$.

The equation of the tangent to the curve at $x = -1$ is $y = -8x - 9$.

Chapter 2 Practice Test **Question 9 Page 144**

The slope of the line is –3, so the slope of the tangent must be $-\left(\frac{1}{-3}\right) = \frac{1}{3}$.

The slope of the tangent is also given by $f'(x)$ so set $f'(x) = \frac{1}{3}$.

$$f'(x) = \frac{1}{2}(2x+1)^{-\frac{1}{2}}(2)$$

$$\frac{1}{3} = \frac{1}{\sqrt{2x+1}}$$

$$3 = \sqrt{2x+1}$$

$$x = 4$$

When $x = 4$, $f(x) = 3$ so the point is (4, 3).

Chapter 2 Practice Test **Question 10 Page 145**

a) The vehicle is going faster at A; The vehicle is going faster at H.

b) The velocity at B and D is zero.

c) Between F and G the vehicle is stopped.

d) At C and I the vehicle is slowing down.

e) At J the vehicle has returned to its starting position.

f) **i)** The acceleration from 0-A is negative.

ii) The acceleration from B-C is negative.

iii) The acceleration from D-E is positive.

iv) The acceleration from F-G is zero.

v) The acceleration from I-J is negative.

Chapter 2 Practice Test **Question 11 Page 145**

a) Determine the demand function.
Let $p(x)$ be the price of one T-shirt, x the number of sales, and n the number of decreases in price.

$$x = 1500 + 20n$$

$$n = \frac{x - 1500}{20}$$

$$p = 12 - 0.50n$$

$$p = 12 - 0.5\left(\frac{x - 1500}{20}\right)$$

$$p(x) = 49.5 - 0.025x$$

b)

$$\begin{aligned} R(x) &= xp(x) \\ &= 49.5x - 0.025x^2 \\ R'(x) &= 49.5 - 0.05x \\ R'(1800) &= 49.5 - 0.05(1800) \\ &= -40.5 \end{aligned}$$

The marginal revenue is \$–40.5.

c) Find $C'(1800)$

$$\begin{aligned} C'(x) &= -0.001x + 7.5 \\ C'(1800) &= -0.001(1800) + 7.5 \\ &= 5.70 \end{aligned}$$

The marginal cost is \$5.70.

d)

$$\begin{aligned} C(1801) - C(1800) &= -0.0005(1801^2 - 1800^2) + 7.5(1) \\ &= -0.0005(3601) + 7.5 \\ &= 5.70 \end{aligned}$$

The cost of the 18th T-shirt is \$5.70.

e) $P(x) = R(x) - C(x)$

$= 49.5x - 0.025x^2 + 0.0005x^2 - 7.5x - 200$

$= -0.0245x^2 + 42x - 200$

$P(1800) = -3980$

$P'(x) = -0.049x + 42$

$P'(1800) = -46.20$

The profit for 1800 t-shirts is –$3980.00 and the marginal profit is –$46.20.

Chapter 2 Practice Test **Question 12 Page 145**

a) $V'(t) = \dfrac{(1 + 0.02t)(5) - (100\,000 + 5t)(0.02)}{(1 + 0.02t)^2}$

$V'(1) = \dfrac{5.1 - 100\,005(0.02)}{(1.02)^2}$

$= -1917.53$

$V'(3) = \dfrac{5.3 - 100\,015(0.02)}{(1.06)^2}$

$= -1775.54$

$V'(6) = \dfrac{5.6 - 100\,030(0.02)}{(1.12)^2}$

$= -1590.40$

The depreciation after 1 year is –$1917.53, 3 years is –$1175.54, and 6 years –$1590.40.

b) $v(0) = \$100\ 000$

c) It is more economical to purchase a used boat, since a new one depreciates much faster in the years just after the purchase was made and more slowly later on.

Chapter 2 Practice Test **Question 13 Page 145**

a) $C'(x) = 0.02x + 42$

$C'(250) = 0.02(250) + 42$

$= 47$

The marginal cost is $47.

b) $C(251) - C(250) = 0.01(251^2 - 250^2) + 42$

$= 47.01$

The cost is $47.01.

c) The cost of producing the 251st MP3 player only slightly exceeds the marginal cost of production at the production level of 250 MP3 players.

d) $R(x) = xp(x)$

$= 130x - 0.4x^2$

$P(x) = R(x) - C(x)$

$= 130x - 0.4x^2 - 0.01x^2 - 42x - 300$

$= -0.41x^2 + 88x - 300$

e) Determine the marginal revenue and marginal profit for $x = 250$ players.

$R'(x) = 130 - 0.8x$

$R'(250) = 130 - 0.8(250)$

$= -70.00$

$P'(x) = -0.82x + 88$

$P'(250) = -0.82(250) + 88$

$= -117.00$

The marginal revenue is –\$70.00 and the marginal profit is –\$117.00.

f) The reason that the marginal profit and marginal are negative is that the production level is too high. It costs more per unit as production is increased.

The profit would be maximized at

$x = \frac{88}{0.82}$ or 107.32 units.

Chapter 2 Practice Test **Question 14 Page 145**

a) $V(0) = \$5500$

b) Find $V'(t)$.

$$V'(t) = \frac{(0.002t^2 + 1)^{\frac{1}{2}}(18t^2) - (5500 + 6t^3)\left(\frac{1}{2}\right)(0.002t^2 + 1)^{-\frac{1}{2}}(0.004t)}{(0.002t^2 + 1)}$$

$$= \frac{(0.002t^2 + 1)(18t^2) - (0.002t)(5500 + 6t^3)}{(0.002t^2 + 1)^{\frac{3}{2}}} \times \frac{125000}{125000}$$

$$= \frac{1000(4.5t^4 + 2250t^2 - 1375t - 1.5t^4)}{(5t^2 + 2500)^{\frac{3}{2}}}$$

$$= \frac{1000t(3t^3 + 2250t - 1375)}{(5t^2 + 2500)^{\frac{3}{2}}}$$

c) The value of the dining set is increasing in value with time as $V'(t) > 0$ for $t > 0.611$ years.

d) Find $V(3)$ and $V(10)$.

$$V(3) = \frac{(5500 + 6(27))}{\sqrt{1.018}}$$
$$= 5611.72$$
$$V(10) = \frac{(5500 + 6(1000))}{\sqrt{1.2}}$$
$$= 10\,498.45$$

The value after 3 years is \$5611.72 and 10 years is \$10 498.45.

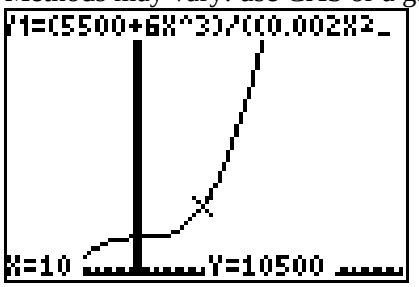
e) Compare $V'(3)$ and $V'(10)$.

$$V'(3) = \frac{3000(3(27) + 2250(3) - 1375)}{(5(9) + 2500)^{\frac{3}{2}}}$$
$$= 127.49$$

$$V'(10) = \frac{10000(3(1000) + 22500 - 1375)}{(500 + 2500)^{\frac{3}{2}}}$$
$$= 1468.20$$

The dining room set is appreciating in value much faster as time goes by.

f) Methods may vary: use CAS or a graphing calculator.



From parts d) and e), $V(10) = \$10\,500$ and $V'(10) = \$1468.20$ per year.

Chapter 3 **Curve Sketching**

Chapter 3 Prerequisite Skills

Chapter 3 Prerequisite Skills **Question 1 Page 148**

a) Let $f(x) = x^3 + 2x^2 + 2x + 1$.

$f(1) = 6$

$f(-1) = 0$ $(x+1)$ is a factor.

$x^3 + 2x^2 + 2x + 1 = (x+1)(x^2 + x + 1)$

b) Let $f(z) = z^3 - 6z - 4$.

$f(1) = -9$

$f(-1) = 1$

$f(2) = -8$

$f(-2) = 0$ $(z+2)$ is a factor.

$z^3 - 6z + 4 = (z+2)(z^2 - 2z - 2)$

c) Let $f(t) = t^3 + 6t^2 - 7t - 60$.

$f(1) = -60$

$f(-1) = -48$

$f(2) = -42$

$f(-2) = -30$

$f(3) = 0$ $(t-3)$ is a factor.

$$t^3 + 6t^2 - 7t - 60 = (t-3)(t^2 + 9t + 20)$$
$$= (t-3)(t+5)(t+4)$$

d) Let $f(b) = b^3 + 8b^2 + 19b + 12$.

$f(1) = 40$

$f(-1) = 0$ $(b+1)$ is a factor.

$$b^3 + 8b^2 + 19b + 12 = (b+1)(b^2 + 7b + 12)$$
$$= (b+1)(b+4)(b+3)$$

e) $$3n^3 - n^2 - 3n + 1 = n^2(3n-1) - (3n-1)$$
$$= (3n-1)(n^2 - 1)$$
$$= (3n-1)(n+1)(n-1)$$

f) Let $f(p) = 2p^3 - 9p^2 + 10p - 3$.

$f(1) = 0$ $(p-1)$ is a factor.

$$\begin{aligned} 2p^3 - 9p^2 + 10p - 3 &= (p-1)(2p^2 - 7p + 3) \\ &= (p-1)(2p-1)(p-3) \end{aligned}$$

g) $$\begin{aligned} 4k^3 + 3k^2 - 4k - 3 &= k^2(4k+3) - (4k+3) \\ &= (4k+3)(k^2 - 1) \\ &= (4k+3)(k+1)(k-1) \end{aligned}$$

h) Let $f(w) = 6w^3 - 11w^2 - 26w + 15$.

$f(1) = -16$

$f(-1) = 24$

$f(3) = 0$ $(w-3)$ is a factor.

$$\begin{aligned} 6w^3 - 11w^2 - 26w + 15 &= (w-3)(6w^2 + 7w - 5) \\ &= (w-3)(6w^2 + 10w - 3w - 5) \\ &= (w-3)(2w(3w+5) - (3w+5)) \\ &= (w-3)(3w+5)(2w-1) \end{aligned}$$

Chapter 3 Prerequisite Skills **Question 2 Page 148**

a) $x^2 - 7x + 12 = 0$

$(x-3)(x-4) = 0$

$x = 3, x = 4$

b) $4x^2 - 9 = 0$

$(2x+3)(2x-3) = 0$

$x = -\frac{3}{2}, x = \frac{3}{2}$

c) $18v^2 = 36v$

$18v^2 - 36v = 0$

$18v(v-2) = 0$

$v = 0, 2$

d) $a^2 + 5a = 3a + 35$

$a^2 + 2a - 35 = 0$

$(a+7)(a-5) = 0$

$a = -7, a = 5$

e) $4.9t^2 - 19.6t + 2.5 = 0$

$49t^2 - 196t + 25 = 0$

$$t = \frac{196 \pm \sqrt{(-196)^2 - 4(49)(25)}}{98}$$

$$t = \frac{196 \pm \sqrt{33516}}{98}$$

$$t = \frac{98 \pm \sqrt{8379}}{49}$$

$t \doteq 3.87$, $t \doteq 0.132$

f) $x^3 + 6x^2 + 3x - 10 = 0$

$(x-1)(x^2 + 7x + 10) = 0$ Use the factor theorem.

$(x-1)(x+2)(x+5) = 0$

$x = 1,\ x = -2,\ x = -5$

g) $\dfrac{x^2 - 5x - 14}{x^2 - 1} = 0$ $x \neq \pm 1$

$x^2 - 5x - 14 = 0$

$(x-7)(x+2) = 0$

$x = 7,\ x = -2$

Chapter 3 Prerequisite Skills **Question 3 Page 148**

a) $2x - 10 > 0$

$2x > 10$ Add 10.

$x > 5$ Multiply by $\frac{1}{2}$.

b) $x(x+5) < 0$

The roots of the related equation, $x(x+5) = 0$, are $x = 0$ and $x = -5$.

Test the inequality for arbitrary values in the intervals $(-\infty, -5)$, $(-5, 0)$, and $(0, \infty)$.

For the first interval, use $x = -8$.

L.S. $= -8(-8+5)$

$= 24$.

$\not< 0$

For the second interval, use $x = -1$.

$$\begin{aligned}\text{L.S.} &= -1(-1+5)\\ &= -4\\ &< 0\end{aligned}$$

For the third interval, use $x = 1$.

$$\begin{aligned}\text{L.S.} &= 1(1+5)\\ &= 5\\ &\not< 0\end{aligned}$$

Since only $x = -1$ satisfies the inequality, the solution is $-5 < x < 0$.

c) $x^2(x-4) > 0$

The roots of the related equation, $x^2(x-4) = 0$, are $x = 0$ and $x = 4$.
Test the inequality for arbitrary values in the intervals $(-\infty, 0), (0, 4),$ and $(4, \infty)$.

For the first interval, use $x = -1$.

$$\begin{aligned}\text{L.S.} &= (-1)^2(-1-4)\\ &= -5\\ &\not> 0\end{aligned}$$

For the second interval, use $x = 1$.

$$\begin{aligned}\text{L.S.} &= 1^2(1-4)\\ &= -3\\ &\not> 0\end{aligned}$$

For the third interval, use $x = 5$.

$$\begin{aligned}\text{L.S.} &= (5)^2(5-4)\\ &= 25\\ &> 0\end{aligned}$$

Since only $x = 5$ satisfies the inequality, the solution is $x > 4$.

d) $x^2 + 5x - 14 < 0$

Solve the related equation.

$$\begin{aligned}x^2 + 5x - 14 &= 0\\ (x+7)(x-2) &= 0\end{aligned}$$

$$x = -7,\ x = 2$$

The roots of the related equation, $x^2 + 5x - 14 = 0$ are $x = -7$ and $x = 2$.

Test the inequality for arbitrary values in the intervals $(-\infty, -7), (-7, 2),$ and $(2, \infty)$.

For the first interval, use $x = -10$.

$$\text{L.S.} = (-10)^2 + 5(-10) - 14$$
$$= 36$$
$$\not< 0$$

For the second interval, use $x = 0$.

$$\text{L.S.} = 0^2 + 5(0) - 14$$
$$= -14$$
$$< 0$$

For the third interval, use $x = 3$.

$$\text{L.S.} = (3)^2 + 5(3) - 14$$
$$= 10$$
$$\not< 0$$

Since only $x = 0$ satisfies the inequality, the solution is $-7 < x < 2$.

e) $(x-3)(x+2)(x-1) > 0$

The roots of the related equation, $(x-3)(x+2)(x-1) = 0$ are $x = -2$, $x = 1$, and $x = 3$.

Test the inequality for arbitrary values in the intervals $(-\infty, -2), (-2, 1), (1, 3),$ and $(3, \infty)$.

For the first interval, use $x = -3$.

$$\text{L.S.} = (-3-3)(-3-3)(-3+2)(-3-1)$$
$$= -24$$
$$\not> 0$$

For the second interval, use $x = 0$.

$$\text{L.S.} = (0-3)(0-3)(0+2)(0-1)$$
$$= 6$$
$$> 0$$

For the third interval, use $x = 2$.

$$\text{L.S.} = (2-3)(2+2)(2-1)$$
$$= -4$$
$$\not> 0$$

For the fourth interval, use $x = 4$.

$$\text{L.S.} = (4-3)(4+2)(4-1)$$
$$= 18$$
$$> 0$$

Since both $x = 0$ and $x = 4$ satisfy the inequality, the solution is $-2 < x < 1$ or $x > 3$.

f) The expression changes sign when either the numerator or the denominator is zero.
The possible values for change of sign are $x = 0$, $x = 1$, and $x = -1$.

Test the inequality for arbitrary values in the intervals $(-\infty, -1), (-1, 0), (0, 1)$, and $(1, \infty)$.

For the first interval, use $x = -2$

$$\text{L.S.} = \frac{-2}{(-2)^2 - 1}$$
$$= -\frac{2}{3}$$
$$\not> 0$$

For the second interval, use $x = -0.5$.

$$\text{L.S.} = \frac{-0.5}{(-0.5)^2 - 1}$$
$$= \frac{2}{3}$$
$$> 0$$

For the third interval, use $x = 0.5$.

$$\text{L.S.} = \frac{0.5}{(0.5)^2 - 1}$$
$$= -\frac{2}{3}$$
$$\not> 0$$

For the fourth interval, use $x = 2$.

$$\text{L.S.} = \frac{2}{(2)^2 - 1}$$
$$= \frac{2}{3}$$
$$> 0$$

Since both $x = -0.5$ and $x = 2$ satisfy the inequality, the solution is $-1 < x < 0$ or $x > 1$.
Restriction: The restriction on x occurs when the denominator is zero. Therefore, $x \neq \pm 1$.

Chapter 3 Prerequisite Skills Question 4 Page 148

a) To find the x-intercepts, let $f(x) = 0$.

$$5x - 15 = 0$$
$$x = 3$$

The x-intercept is 3.

b) To find the x-intercepts, let $g(x) = 0$.

$$x^2 - 3x - 28 = 0$$
$$(x-7)(x+4) = 0$$

$$x = 7,\ x = -4$$

The x-intercepts are –4 and 7.

c) To find the x-intercepts, let $h(x) = 0$.

$$x^3 + 6x^2 + 11x + 6 = 0$$

$$h(-1) = (-1)^3 + 6(-1)^2 + 11(-1) + 6 \qquad \text{Use the factor theorem.}$$
$$= 0 \qquad (x+1) \text{ is a factor.}$$

$$(x+1)(x^2 + 5x + 6) = 0$$
$$(x+1)(x+2)(x+3) = 0$$

$$x = -1,\ x = -2, x = -3$$

The x-intercepts are –1, –2, and –3.

d) To find the x-intercepts, let $y = 0$.

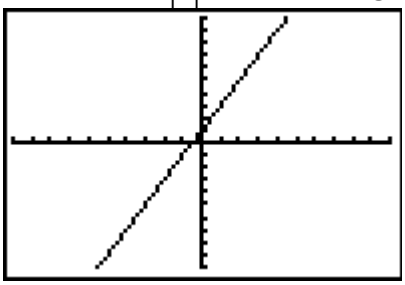
$$\frac{x^2 - 9}{x^2 + 1} = 0$$
$$x^2 - 9 = 0$$
$$(x+3)(x-3) = 0$$
$$x = -3, x = 3$$

The x-intercepts are –3 and 3.

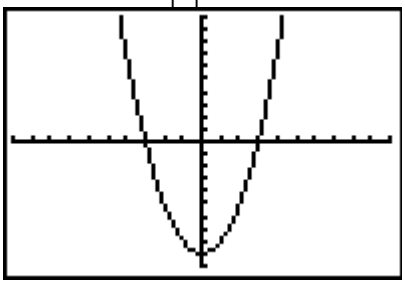
Chapter 3 Prerequisite Skills **Question 5 Page 148**

a) This is a line, not parallel to either axis.

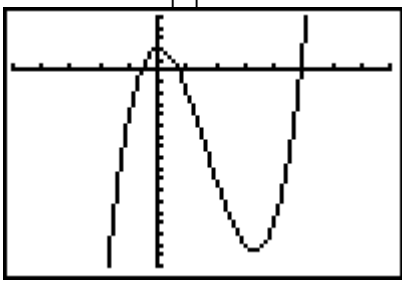
The domain is $\{x \in \mathbb{R}\}$. The range is $\{y \in \mathbb{R}\}$.



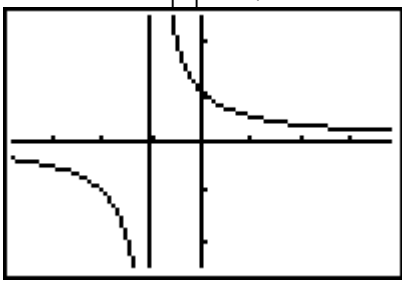
b) This is a parabola with vertex at (0, –9) and opening up.
The domain is $\{x \in \mathbb{R}\}$. The range is $\{y \in \mathbb{R} \mid y \geq -9\}$.



c) This is a cubic function.
The domain is $\{x \in \mathbb{R}\}$. The range is $\{y \in \mathbb{R}\}$.



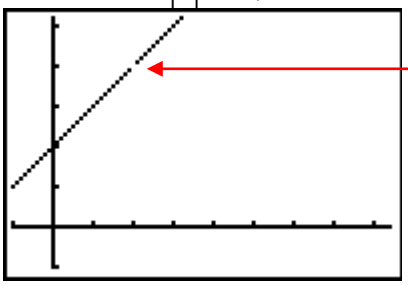
d) This is a rational function with vertical asymptote $x = -1$ and horizontal asymptote $y = 0$.
The domain is $\{x \in \mathbb{R} \mid x \neq -1\}$. The range is $\{y \in \mathbb{R} \mid y \neq 0\}$.



e) This function is linear since

$$\frac{x^2 - 4}{x - 2} = \frac{(x + 2)(x - 2)}{x - 2} \qquad (x \neq 2)$$
$$= x + 2$$

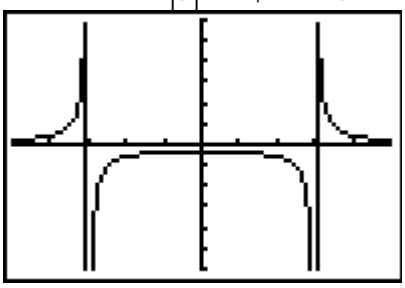
The domain is $\{x \in \mathbb{R} \mid x \neq 2\}$. The range is $\{y \in \mathbb{R} \mid y \neq 4\}$.



The value of y does not exist at $x = 2$.
This is indicated by a break in the graph.

f) This is a rational function with vertical asymptotes $x = -3$ and $x = 3$ and horizontal asymptote $y = 0$.

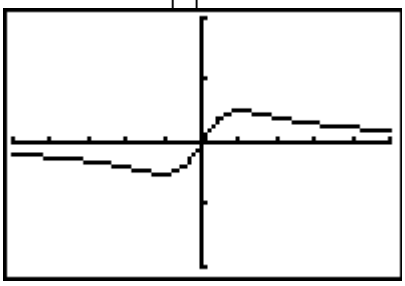
The domain is $\{x \in \mathbb{R} \mid x \neq \pm 3\}$. The range is $\{y \in \mathbb{R} \mid y \neq 0\}$.



g) This is a rational function without vertical asymptotes since $x^2 + 1 \neq 0$ for any values of x.

A graph is required to determine the domain and range.

The domain is $\{x \in \mathbb{R}\}$. The range is $\{y \in \mathbb{R} \mid -0.5 \leq y \leq 0.5\}$.



Chapter 3 Prerequisite Skills **Question 6 Page 148**

See question 5 above.

a) None

b) None

c) None

d) $x = 1$, $y = 0$

e) None

f) $x = 3$, $x = -3$, $y = 0$

g) $y = 0$

Chapter 3 Prerequisite Skills **Question 7 Page 148**

a) The function is increasing on the intervals $(-\infty, 0)$ and $(2, \infty)$.

The function is decreasing on the interval $(0, 2)$.

b) The function is increasing on the intervals $(-2,0)$ and $(2,\infty)$.

The function is decreasing on the intervals $(-\infty,-2)$ and $(0,2)$.

Chapter 3 Prerequisite Skills **Question 8 Page 149**

a) $f(x)=5x^2-7x+12$

$f'(x)=10x-7$

b) $y=x^3-2x^2+4x-8$

$\frac{dy}{dx}=3x^2-4x+4$

c) $f(x)=\frac{1}{x}$

$=x^{-1}$

$f'(x)=(-1)x^{-2}$

$=-\frac{1}{x^2}$

d) $y=\frac{x^2-9}{x^2+1}$

$=(x^2-9)(x^2+1)^{-1}$

$\frac{dy}{dx}=(x^2-9)(-1)(x^2+1)^{-2}(2x)+2x(x^2+1)^{-1}$

$=(x^2+1)^{-2}\left[2x(-x^2+9)+2x(x^2+1)\right]$

$=(x^2+1)^{-2}\left[-2x^3+18x+2x^3+2x\right]$

$=\frac{20x}{(x^2+1)^2}$

Chapter 3 Prerequisite Skills **Question 9 Page 149**

$V(x)=lwh$

$=(60-2x)(40-2x)(x)$

$=4x^3-200x^2+2400x$

There is a restriction on the possible values for x since its value must be positive and cannot exceed half the value of the width of the sheet of tin; i.e., $0<x<20$.

Chapter 3 Prerequisite Skills **Question 10 Page 149**

$V = \pi r^2 h$

$= 1000$ ①

$\text{S.A.} = 2^1 r^2 + 2^1 rh$ ②

You want a formula for S.A., only in terms of r, so eliminate the variable h.

Substitute $h = \frac{1000}{^1 r^2}$ from ① in formula ②

$\text{S.A.} = 2^1 r^2 + 2^1 r\left(\frac{1000}{^1 r^2}\right)$

$= 2^1 r^2 + \frac{2000}{r}$

Chapter 3 Prerequisite Skills **Question 11 Page 149**

a) Since this graph is symmetric about the origin, the function is odd.

b) Since this graph is symmetric about the y-axis, the function is even.

c) Since this graph is symmetric about the y-axis, the function is even.

d) Since this graph is neither symmetric about the y-axis nor the origin, the function is neither even nor odd.

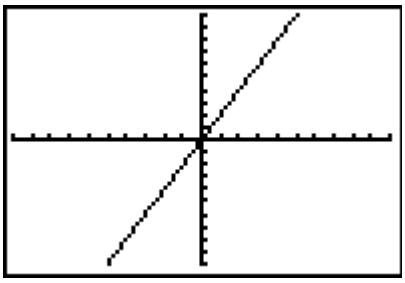
Chapter 3 Prerequisite Skills **Question 12 Page 149**

a) $f(-x) = 2(-x)$

$= -2x$

$= -f(x)$

This is an odd function.



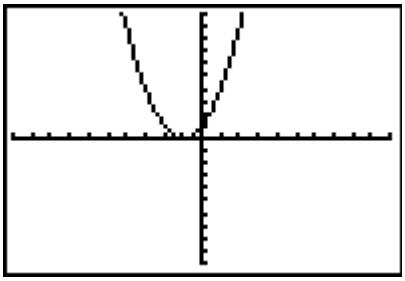
b) $r(-x) = (-x)^2 + 2(-x) + 1$

$= x^2 - 2x + 1$

$r(x) = x^2 + 2x + 1$

$-r(x) = -x^2 - 2x - 1$

The function is neither even nor odd.

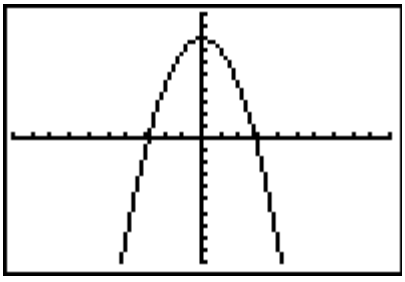


c) $f(-x) = -(-x)^2 + 8$

$= -x^2 + 8$

$= f(x)$

This is an even function.



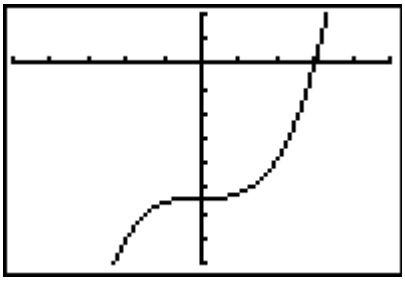
d) $s(-t) = (-t)^3 - 27$

$= -t^3 - 27$

$s(t) = t^3 - 27$

$-s(t) = -t^3 + 27$

The function is neither even nor odd.



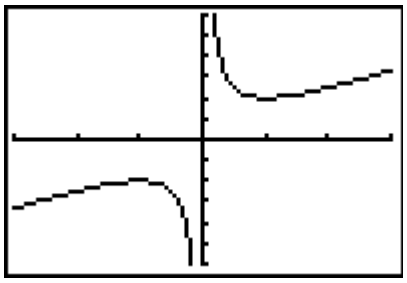
(Yscl = 5)

e) $h(-x) = (-x) + \dfrac{1}{(-x)}$

$= -x - \dfrac{1}{x}$

$= -h(x)$

This is an odd function.

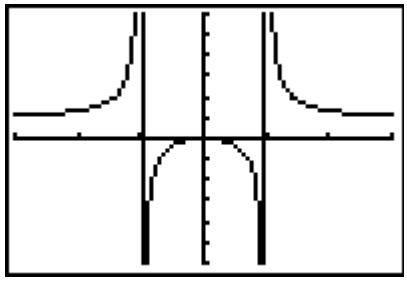


f) $f(-x) = \dfrac{(-x)^2}{(-x)^2 - 1}$

$= \dfrac{x^2}{x^2 - 1}$

$= f(x)$

This is an even function.



Chapter 3 Section 1 **Increasing and Decreasing Functions**

Chapter 3 Section 1 **Question 1 Page 156**

a) $15-5x=0$

$-5x=-15$

$x=3$

b) $x^2+8x-9=0$

$(x+9)(x-1)=0$

$x=-9,\ x=1$

c) $3x^2-12=0$

$3x^2=12$

$x^2=4$

$x=\pm 2$

d) $x^3-6x^2=0$

$x^2(x-6)=0$

$x=0,\ x=6$

e) $x^2+2x-4=0$

$x=\dfrac{-2\pm\sqrt{(2)^2-4(1)(-4)}}{2(1)}$ Use the quadratic formula.

$x=\dfrac{-2\pm\sqrt{20}}{2}$ $\sqrt{20}=2\sqrt{5}$

$x=-1\pm\sqrt{5}$

f) $x^3-3x^2-18x+40=0$

$f(1)=1-3-18+40=20$

$\neq 0$

$f(-1)=-1-3+18+40$

$=54$

$\neq 0$

$f(2)=8-12-36+40$

$=0$ $(x-2)$ is a factor.

$(x-2)(x^2-x-20)=0$ Use the factor theorem.

$(x-2)(x-5)(x+4)=0$

$x=2,\ x=5,\ x=-4$

g) $x^3 + 3x^2 - 4x - 12 = 0$

$x^2(x+3) - 4(x+3) = 0$

$(x+3)(x^2 - 4) = 0$

$(x+3)(x+2)(x-2) = 0$

$x = -3, x = -2, x = 2$

h) $x^4 - x^3 - x^2 + x = 0$

$x^3(x-1) - x(x-1) = 0$

$(x-1)(x^3 - x) = 0$

$(x-1)(x)(x^2 - 1) = 0$

$x(x-1)(x+1)(x-1) = 0$

$x = 0,\ x = 1,\ x = -1$

Chapter 3 Section 1 Question 2 Page 156

a) The function f is increasing when $f'(x) = 15 - 5x > 0$.

Test the inequality for arbitrary values in the intervals $(-\infty, 3)$ and $(3, \infty)$.

For the first interval, use $x = 0$.

$\text{L.S.} = 15 - 5(0)$

$= 15$

> 0

For the second interval, use $x = 4$.

$\text{L.S.} = 15 - 5(4)$

$= -5$.

$\not> 0$

f is increasing on the interval $(-\infty, 3)$.

f is decreasing on the interval $(3, \infty)$.

b) The function h is increasing when $h'(x) = x^2 + 8x - 9 > 0$.

Test the inequality for arbitrary values in the intervals $(-\infty, -9), (-9, 1)$, and $(1, \infty)$.

For the first interval, use $x = -10$.

$\text{L.S.} = (-10)^2 + 8(-10) - 9$

$= 11$

> 0

For the second interval, use $x = 0$.

$$\begin{aligned}\text{L.S.} &= (0)^2 + 8(0) - 9 \\ &= -9 \\ &\not> 0\end{aligned}$$

For the third interval, use $x = 2$.

$$\begin{aligned}\text{L.S.} &= (2)^2 + 8(2) - 9 \\ &= 11 \\ &> 0\end{aligned}$$

h is increasing on the intervals $(-\infty, -9)$ and $(1, \infty)$.
h is decreasing on the interval $(-9, 1)$.

c) The function g is increasing when $g'(x) = 3x^2 - 12 > 0$.

Test the inequality for arbitrary values in the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.
For the first interval, use $x = -3$.

$$\begin{aligned}\text{L.S.} &= 3(-3)^2 - 12 \\ &= 15 \\ &> 0\end{aligned}$$

For the second interval, use $x = 0$.

$$\begin{aligned}\text{L.S.} &= 3(0)^2 - 12 \\ &= -12 \\ &\not> 0\end{aligned}$$

For the third interval, use $x = 3$.

$$\begin{aligned}\text{L.S.} &= 3(3)^2 - 12 \\ &= 15 \\ &> 0\end{aligned}$$

g is increasing on the intervals $(-\infty, -2)$ and $(2, \infty)$.
g is decreasing on the interval $(-2, 2)$.

d) The function f is increasing when $f'(x) = x^3 - 6x^2 > 0$.

Test the inequality for arbitrary values in the intervals $(-\infty, 0), (0, 6)$, and $(6, \infty)$.
For the first interval, use $x = -3$.

$$\begin{aligned}\text{L.S.} &= (-3)^3 - 6(-3)^2 \\ &= -81 \\ &\not> 0\end{aligned}$$

.

For the second interval, use $x = 1$.

$$\begin{aligned}\text{L.S.} &= (1)^3 - 6(1)^2 \\ &= -5 \\ &\not> 0\end{aligned}$$

.

For the third interval, use $x = 7$.

$$\begin{aligned}\text{L.S.} &= (7)^3 - 6(7)^2 \\ &= 49 \\ &> 0\end{aligned}$$

f is increasing on the interval $(6, \infty)$.

f is decreasing on the intervals $(-\infty, 0)$ and $(0, 6)$.

e) The function d is increasing when $d'(x) = x^2 + 2x - 4 > 0$.

Test the inequality for arbitrary values in the intervals $\left(-\infty, -1-\sqrt{5}\right)\left(-1-\sqrt{5}, -1+\sqrt{5}\right)$ and $\left(-1+\sqrt{5}, \infty\right)$.

For the first interval, use $x = -4$.

$$\begin{aligned}\text{L.S.} &= (-4)^2 + 2(-4) - 4 \\ &= 4 \\ &> 0\end{aligned}$$

For the second interval, use $x = 0$.

$$\begin{aligned}\text{L.S.} &= (0)^2 + 2(0) - 4 \\ &= -4 \\ &\not> 0\end{aligned}$$

For the third interval, use $x = 2$.

$$\begin{aligned}\text{L.S.} &= (2)^2 + 2(2) - 4 \\ &= 4 \\ &> 0\end{aligned}$$

d is increasing on the intervals $\left(-\infty, -1-\sqrt{5}\right)$ and $\left(-1+\sqrt{5}, \infty\right)$.

d is decreasing on the interval $\left(-1-\sqrt{5}, -1+\sqrt{5}\right)$.

f) The function k is increasing when $k'(x) = x^3 - 3x^2 - 18x + 40 > 0$.

Test the inequality for arbitrary values in the intervals $(-\infty, -4)$, $(-4, 2)$, $(2, 5)$, and $(5, \infty)$.
For the first interval, use $x = -5$.

$$\begin{aligned} \text{L.S.} &= (-5)^3 - 3(-5)^2 - 18(-5) + 40 \\ &= -70 \\ &\not> 0 \end{aligned}$$

For the second interval, use $x = 0$.

$$\begin{aligned} \text{L.S.} &= (0)^3 - 3(0)^2 - 18(0) + 40 \\ &= 40 \\ &> 0 \end{aligned}$$

For the third interval, use $x = 3$.

$$\begin{aligned} \text{L.S.} &= (3)^3 - 3(3)^2 - 18(3) + 40 \\ &= -14 \\ &\not> 0 \end{aligned}$$

For the fourth interval, use $x = 6$.

$$\begin{aligned} \text{L.S.} &= (6)^3 - 3(6)^2 - 18(6) + 40 \\ &= 40 \\ &> 0 \end{aligned}$$

k is increasing on the intervals $(-4, 2)$ and $(5, \infty)$.
k is decreasing on the intervals $(-\infty, -4)$ and $(2, 5)$.

g) The function b is increasing when $b'(x0 = x^3 + 3x^2 - 4x - 12 > 0$.

Test the inequality for arbitrary values in the intervals $(-\infty, -3)$, $(-3, -2)$, $(-2, 2)$, and $(2, \infty)$.
For the first interval, use $x = -4$.

$$\begin{aligned} \text{L.S.} &= (-4)^3 + 3(-4)^2 - 4(-4) - 12 \\ &= -12 \\ &\not> 0 \end{aligned}$$

For the second interval, use $x = -2.5$.

$$\begin{aligned} \text{L.S.} &= (-2.5)^3 + 3(-2.5)^2 - 4(-2.5) - 12 \\ &= 1.125 \\ &> 0 \end{aligned}$$

For the third interval, use $x = 0$.

$$\begin{aligned} \text{L.S.} &= (0)^3 + 3(0)^2 - 4(0) - 12 \\ &= -12 \\ &\not> 0 \end{aligned}$$

For the fourth interval, use $x = 3$.

$$\begin{aligned}\text{L.S.} &= (3)^3 + 3(3)^2 - 4(3) - 12 \\ &= 30 \\ &> 0\end{aligned}$$

b is increasing on the intervals $(-3, -2)$ and $(2, \infty)$.
b is decreasing on the intervals $(-\infty, -3)$ and $(-2, 2)$.

h) The function f is increasing when $f'(x) = x^4 - x^3 - x^2 + x > 0$.

Test the inequality for arbitrary values in the intervals $(-\infty, -1), (-1, 0)$, $(0,1)$, and $(1, \infty)$.
For the first interval, use $x = -2$.

$$\begin{aligned}\text{L.S.} &= (-2)^4 - (-2)^3 - (-2)^2 + (-2) \\ &= 18 \\ &> 0\end{aligned}$$

For the second interval, use $x = -0.5$.

$$\begin{aligned}\text{L.S.} &= (-0.5)^4 - (-0.5)^3 - (-0.5)^2 + (-0.5) \\ &= -0.5625 \\ &\not> 0\end{aligned}$$

For the third interval, use $x = 0.5$.

$$\begin{aligned}\text{L.S.} &= (0.5)^4 - (0.5)^3 - (0.5)^2 + (0.5) \\ &= 0.1875 \\ &> 0\end{aligned}$$

For the fourth interval, use $x = 2$.

$$\begin{aligned}\text{L.S.} &= (2)^4 - (2)^3 - (2)^2 + (2) \\ &= 6 \\ &> 0\end{aligned}$$

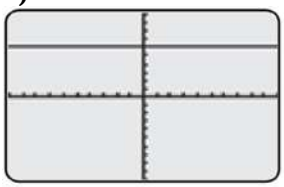
f is increasing on the intervals $(-\infty, -1)$, $(0, 1)$, and $(1, \infty)$.
f is decreasing on the interval $(-1, 0)$.

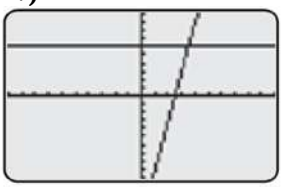
Chapter 3 Section 1 Question 3 Page 156

a) i) $f(x) = 6x - 15$

$f'(x) = 6$

ii) **iv)**





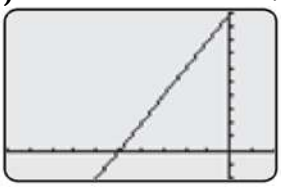
iii) f is always increasing.
f is never decreasing.

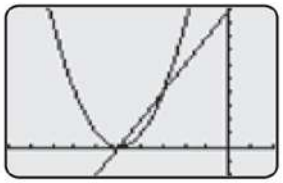
b) i) $f(x) = (x+5)^2$

$= x^2 + 10x + 25$

$f'(x) = 2x + 10$

ii) **iv)**



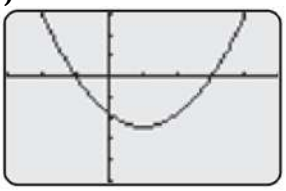


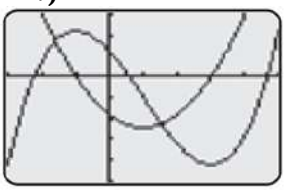
iii) f is increasing on the interval $(-5, \infty)$.
f is decreasing on the interval $(-\infty, -5)$.

c) i) $f(x) = x^3 - 3x^2 - 9x + 6$

$f'(x) = 3x^2 - 6x - 9$

ii) **iv)**





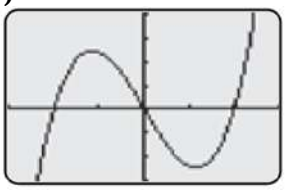
iii) f is increasing on the intervals $(-\infty, -1)$ and $(3, \infty)$.
f is decreasing on the interval $(-1, 3)$.

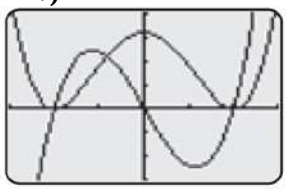
d) i) $f(x) = (x^2 - 4)^2$

$= x^4 - 8x^2 + 16$

$f'(x) = 4x^3 - 16x$

ii) **iv)**

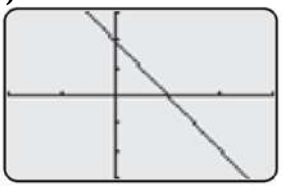


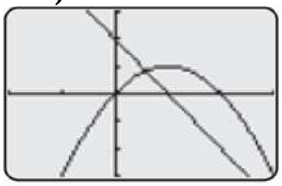


iii) f is increasing on the intervals $(-2, 0)$ and $(2, \infty)$.
f is decreasing on the intervals $(-\infty, -2)$ and $(0, 2)$.

e) i) $f(x) = 2x - x^2$
$f'(x) = 2 - 2x$

ii) **iv)**

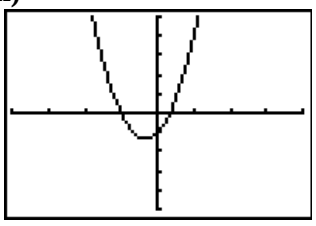


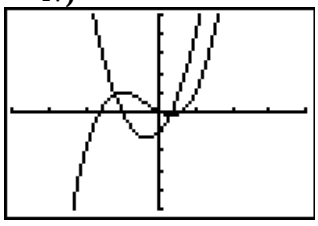


iii) f is increasing on the interval $(-\infty, 1)$.
f is decreasing on the interval $(1, \infty)$.

f) i) $f(x) = x^3 + x^2 - x$
$f'(x) = 3x^2 + 2x - 1$

ii) **iv)**



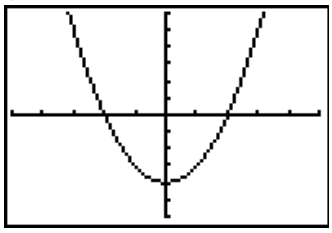


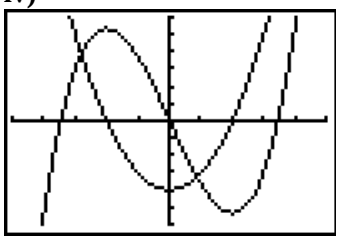
iii) f is increasing on the intervals $(-\infty, -1)$ and $\left(\frac{1}{3}, \infty\right)$.

f is decreasing on the interval $\left(-1, \frac{1}{3}\right)$.

g) i) $f(x) = \frac{1}{3}x^3 - 4x$
$f'(x) = x^2 - 4$

ii) **iv)**





iii) f is increasing on the intervals $(-\infty, -2)$ and $(2, \infty)$.
f is decreasing on the interval $(-2, 2)$.

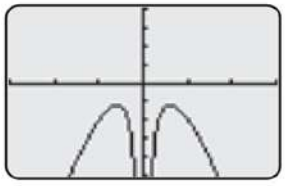
h) i) $f(x) = \frac{1}{x} - 3x^3$

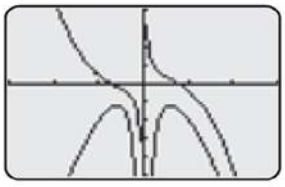
$= x^{-1} - 3x^3$

$f'(x) = -1x^{-2} - 9x^2$

$= -\frac{1}{x^2} - 9x^2$

ii) **iv)**





iii) f is always decreasing (except for $x = 0$ when the function is not defined).

Chapter 3 Section 1 Question 4 Page 156

a) f is increasing on the interval $(3, \infty)$.
f is decreasing on the interval $(-\infty, 3)$.

b) f is increasing on the interval $(-\infty, 8)$.
f is decreasing on the interval $(8, \infty)$.

c) f is increasing on the intervals $(-\infty, -1)$ and $(2, \infty)$.
f is decreasing on the interval $(-1, 2)$.

d) f is increasing on the interval $(-2, 2)$.
f is decreasing on the intervals $(-\infty, -2)$ and $(2, \infty)$.

e) f is increasing on the intervals $(-\infty, -5)$ and $(1, \infty)$.
f is decreasing on the interval $(-5, 1)$.

f) f is increasing on the intervals $(-2, 0)$ and $(0, \infty)$.
f is decreasing on the interval $(-\infty, -2)$.

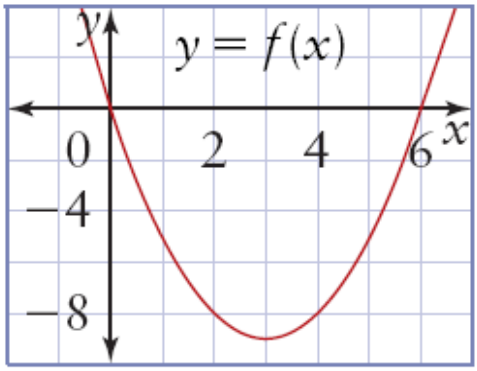
g) f is increasing on the intervals $(-4, 0)$ and $(2, \infty)$.
f is decreasing on the intervals $(-\infty, -4)$ and $(0, 2)$.

h) f is increasing on the interval $(1, \infty)$.
f is decreasing on the interval $(-\infty, 1)$.

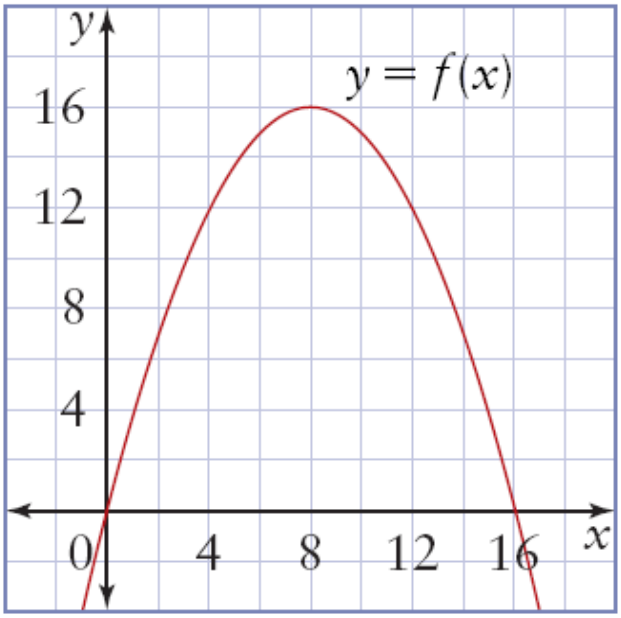
Chapter 3 Section 1 Question 5 Page 156

Many function graphs are possible. They need only satisfy the conditions for intervals of increase and decrease from question 4.

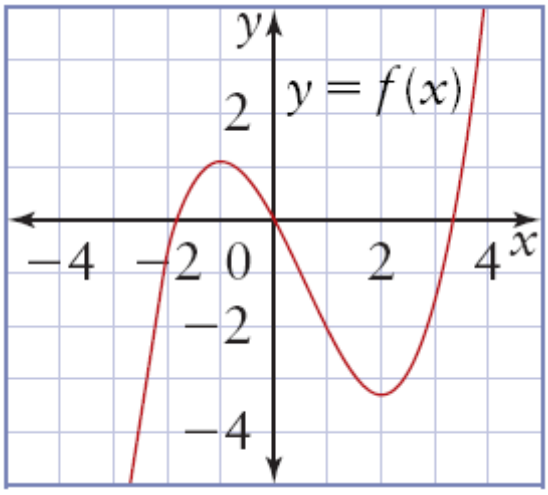
a)



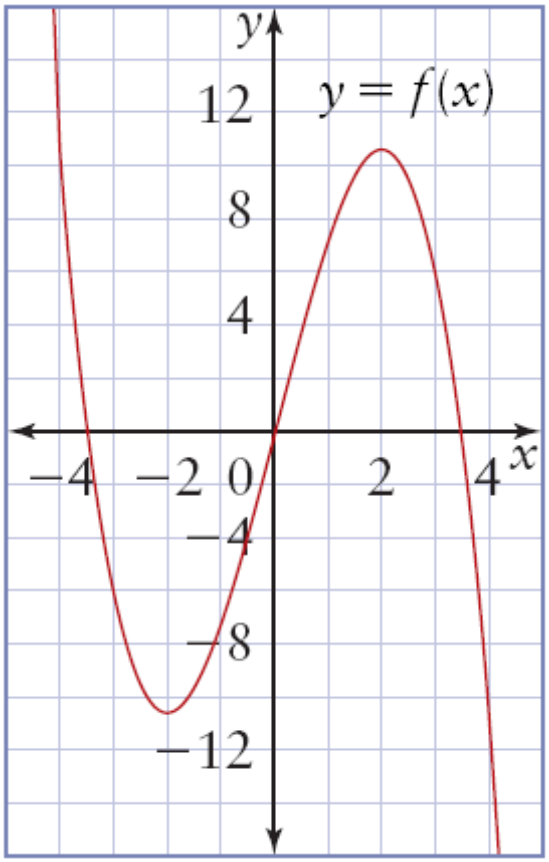
b)



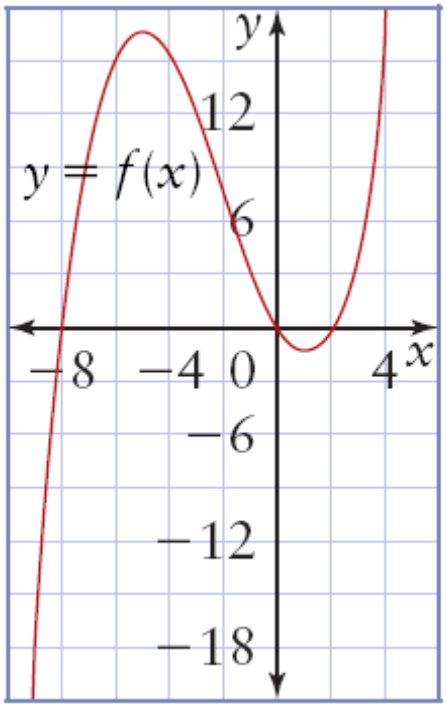
c)



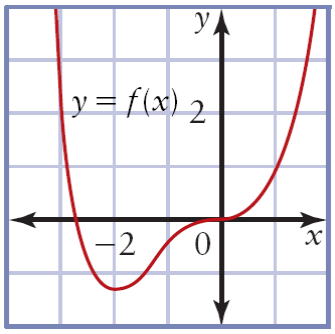
d)



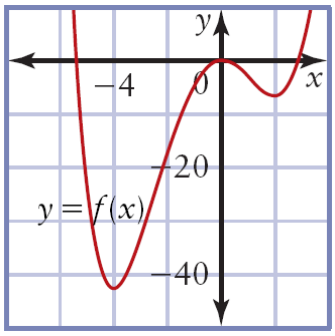
e)



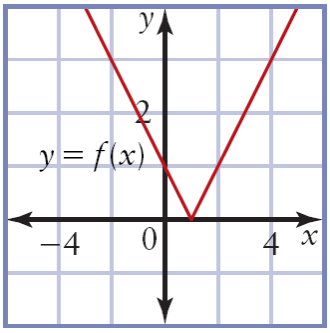
f)



g)



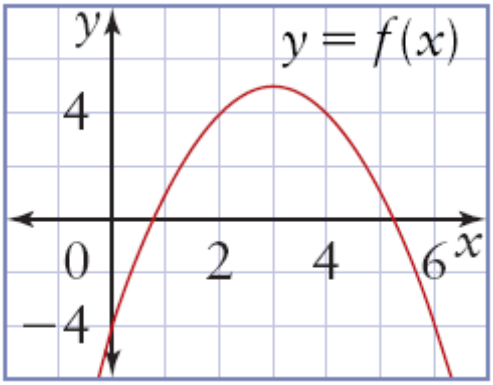
h) Note that there is no derivative at $x = 1$.



Chapter 3 Section 1 **Question 6 Page 157**

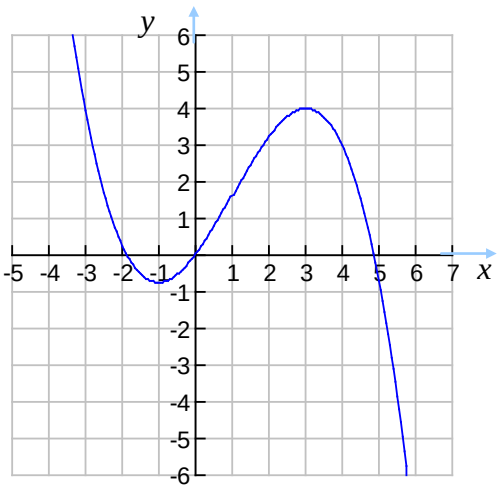
Many function graphs are possible. They need only satisfy the conditions for intervals of increase (when $f'(x) > 0$) and decrease (when $f'(x) < 0$) and pass through the specified points given.

a)

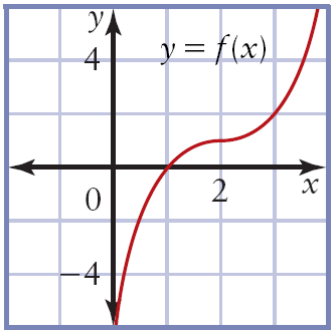


b) <Graph is not quite correct and needs to be revised in text answers.

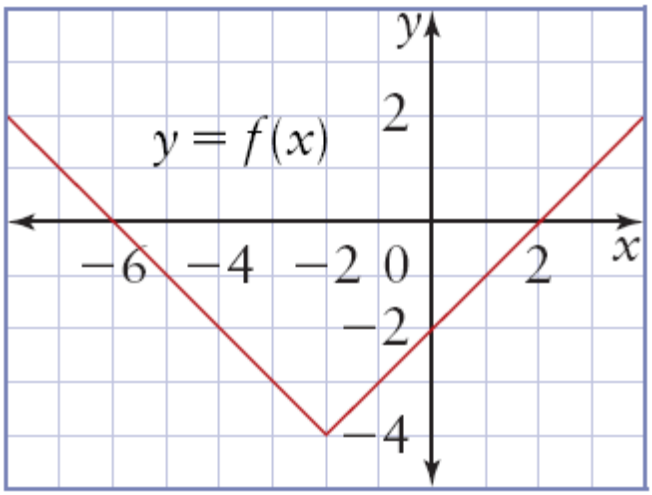
Use $f(x) = \left(-\frac{4}{27}\right)(x^3 - 3x^2 - 9x)$ as shown.



c)



d)



Chapter 3 Section 1 **Question 7 Page 157**

a) $k(x)$ is decreasing in the interval (0, 6) since $k'(x) < 0$ in this interval. The values $x = 3$ and $x = 5$ are both in this interval. Therefore $k(3) > k(5)$.

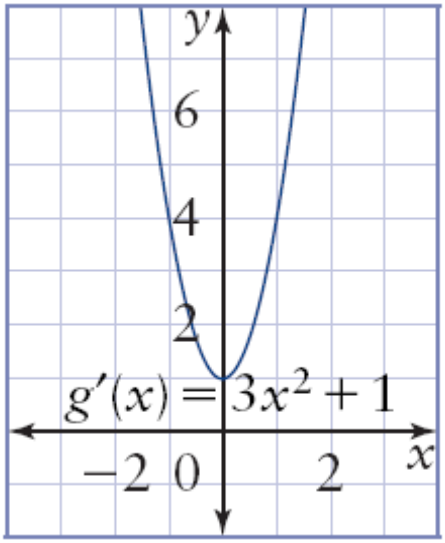
b) $k(x)$ is increasing in the interval $(6, \infty)$ since $k'(x) > 0$ in this interval. The values $x = 8$ and $x = 12$ are both in this interval. Therefore $k(12) > k(8)$.

c) $k(x)$ is decreasing from x = 5 to x = 6 and increasing from x = 6 to x = 9. Also, the (average) rate of increase (from 6 to 9) is greater than the average rate of decrease (from 5 to 6). Therefore $k(9) > k(5)$.

d) $k(x)$ is increasing at both x = –2 and x = 10. Between x = –2 and x = 10, $k(x)$ is increasing for 6 units along the x-axis and decreasing for 6 units as well. However, the average rate of increase is greater than the average rate of decrease in these intervals. Therefore $k(10) > k(-2)$.

Chapter 3 Section 1 Question 8 Page 157

a) Draw the graph of $y = g'(x)$ or $3x^2 + 1$.



The graph of $g'(x)$ is always above the x-axis and so is always positive. This implies the function $g(x)$ is always increasing.

b) Use the rules for manipulating inequalities.

$x^2 > 0$ For all $x \in \mathbb{R}$.

$3x^2 > 0$ Multiply boths sides of the inequality by 3.

$3x^2 + 1 > 0$ Add 1 to both sides of the inequality.

Therefore, $g'(x) = 3x^2 + 1 > 0$. For all $x \in \mathbb{R}$.

Chapter 3 Section 1 Question 9 Page 157

a) $h(x) = (x^2 + 2x - 3) + (x + 5)$

$= x^2 + 3x + 2$

$h'(x) = 2x + 3$

The function is increasing when $h'(x) = 2x + 3 > 0$.

First solve the related equation.

$2x + 3 = 0$

$x = -\frac{3}{2}$

Test the inequality for arbitrary values in the intervals $(-\infty, -1.5)$ and $(-1.5, \infty)$.

For the first interval, use $x = -2$.

L.S. $= 2(-2) + 3$

$= -1$

$\not> 0$

For the second interval, use $x = 0$.

$$\begin{aligned} \text{L.S.} &= 2(0)+3 \\ &= 3 \\ &> 0 \end{aligned}$$

h is increasing on the interval $(-1.5, \infty)$.
h is decreasing on the interval $(-\infty, -1.5)$.

b) $h(x) = f(g(x)) = (x+5)^2 + 2(x+5) - 3$

$$= x^2 + 12x + 32$$

$$h'(x) = 2x + 12$$

The function is increasing when $h'(x) = 2x + 12 > 0$.

First solve the related equation.

$$\begin{aligned} 2x + 12 &= 0 \\ x &= -6 \end{aligned}$$

Test the inequality for arbitrary values in the intervals $(-\infty, -6)$ and $(-6, \infty)$.
For the first interval, use $x = -7$.

$$\begin{aligned} \text{L.S.} &= 2(-7)+12 \\ &= -2 \not> 0 \end{aligned}$$

For the second interval, use $x = 0$.

$$\begin{aligned} \text{L.S.} &= 2(0)+12 \\ &= 12 \\ &> 0 \end{aligned}$$

h is increasing on the interval $(-6, \infty)$.
h is decreasing on the interval $(-\infty, -6)$.

c) $h(x) = f(x) - g(x) + 2$

$$\begin{aligned} &= (x^2 + 2x - 3) - (x + 5) + 2 \\ &= x^2 + x - 6 \end{aligned}$$

$$h'(x) = 2x + 1$$

The function is increasing when $h'(x) = 2x + 1 > 0$.

First solve the related equation.

$$\begin{aligned} 2x + 1 &= 0 \\ x &= -0.5 \end{aligned}$$

Test the inequality for arbitrary values in the intervals $(-\infty, -0.5)$ and $(-0.5, \infty)$.

For the first interval, use $x = -1$.

$$\begin{aligned}\text{L.S.} &= 2(-1)+1\\ &= -1\\ &\not> 0\end{aligned}$$

For the second interval, use $x = 0$.

$$\begin{aligned}\text{L.S.} &= 2(0)+1\\ &= 1\\ &> 0\end{aligned}$$

h is increasing on the interval $(-0.5, \infty)$.

h is decreasing on the interval $(-\infty, -0.5)$.

d) $h(x) = f(x) \times g(x)$

$$\begin{aligned}&= (x^2 + 2x - 3)(x + 5)\\ &= x^3 + 7x^2 + 7x - 15\end{aligned}$$

$$h'(x) = 3x^2 + 14x + 7$$

The function is increasing when $h'(x) = 3x^2 + 14x + 7 > 0$.

First solve the related equation.

$$3x^2 + 14x + 7 = 0$$

$$x = \frac{-14 \pm \sqrt{14^2 - 84}}{6}$$

$$x = \frac{-14 \pm \sqrt{112}}{6}$$

$$x = \frac{-14 \pm 4\sqrt{7}}{6}$$

$$x = \frac{-7 \pm 2\sqrt{7}}{3}$$

$x \approx -0.6$, $x \approx -4.1$

Test the inequality for arbitrary values in the intervals $\left(-\infty, \frac{-7-2\sqrt{7}}{3}\right)$, $\left(\frac{-7-2\sqrt{7}}{3}, \frac{-7+2\sqrt{7}}{3}\right)$, and $\left(\frac{-7+2\sqrt{7}}{3}, \infty\right)$.

For the first interval, use $x = -5$.

$$\begin{aligned}\text{L.S.} &= 3(-5)^2 + 14(-5) + 7\\ &= 12\\ &> 0\end{aligned}$$

For the second interval, use $x = -1$.

$$\begin{aligned}\text{L.S.} &= 3(-1)^2 + 14(-1) + 7 \\ &= -4 \\ &\not> 0\end{aligned}$$

For the third interval, use $x = 0$.

$$\begin{aligned}\text{L.S.} &= 3(0)^2 + 14(0) + 7 \\ &= 7 \\ &> 0\end{aligned}$$

h is increasing on the intervals $\left(-\infty, \frac{-7-2\sqrt{7}}{3}\right)$ and $\left(\frac{-7+2\sqrt{7}}{3}, \infty\right)$.

h is decreasing on the interval $\left(\frac{-7-2\sqrt{7}}{3}, \frac{-7+2\sqrt{7}}{3}\right)$.

Chapter 3 Section 1 Question 10 Page 157

a) The function is increasing when $f'(x) = x(x-1)(x+2) > 0$.

First solve the related equation.

$$x(x-1)(x+2) = 0$$

$$x = 0,\ x = 1,\ x = -2$$

Test the inequality for arbitrary values in the intervals $(-\infty, -2)$, $(-2, 0)$, $(0, 1)$, and $(1, \infty)$.

For the first interval, use $x = -3$.

$$\begin{aligned}\text{L.S.} &= -3(-3-1)(-3+2) \\ &= -12 \\ &\not> 0\end{aligned}$$

For the second interval, use $x = -1$.

$$\begin{aligned}\text{L.S.} &= -1(-1-1)(-1+2) \\ &= 2 \\ &> 0\end{aligned}$$

For the third interval, use $x = 0.5$.

$$\begin{aligned}\text{L.S.} &= 0.5(0.5-1)(0.5+2) \\ &= -0.625 \\ &\not> 0\end{aligned}$$

For the fourth interval, use $x = 2$.

$$\begin{aligned}\text{L.S.} &= 2(2-1)(2+2) \\ &= 8 \\ &> 0\end{aligned}$$

f is increasing on the intervals $(-2, 0)$ and $(1, \infty)$.

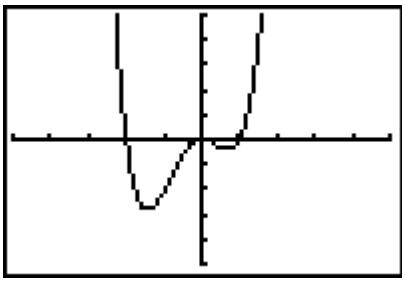
f is decreasing on the intervals $(-\infty, -2)$ and $(0,1)$.

b) The additional factor in the derivative polynomial would alter the pattern of alternating increasing and decreasing intervals. The (positive) quartic derivative polynomial will start with a decreasing interval. Typically the intervals alternate between increasing and decreasing but the appearance of a repeated root ($x = 0$) alters the pattern. You can imagine an additional interval (0, 0) in the sequence of alternating intervals. The previous sequence of –, +, –, + becomes +, –, +, –, + where the second '+' belongs to the (imaginary) interval (0, 0).

f is increasing on the intervals $(-\infty, -2)$ and $(1, \infty)$.

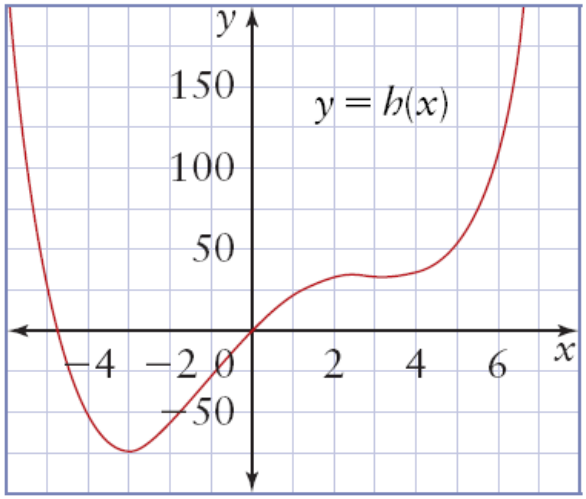
f is decreasing on the intervals $(-2, 0)$ and $(0, 1)$.

This can be confirmed by technology. Below is the graph of $f'(x)$.



Chapter 3 Section 1 Question 11 Page 157

a) A possible sketch is shown.



b) The sketch above can be translated vertically without changing the intervals of increase or decrease.

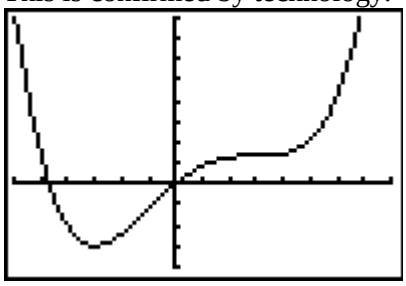
c) Since the function has turning points at –3 and 3, the derivative could have the form $h'(x) = (x+3)(x-3)^2$. Note the inclusion of a repeated factor since the curve does not change direction at $x = 3$.

$$h'(x) = (x+3)(x-3)^2$$
$$= (x+3)(x^2 - 6x + 9)$$
$$= x^3 - 3x^2 - 9x + 27$$

What function could have this as its derivative?

$h(x) = \frac{1}{4}x^4 - x^3 - \frac{9}{2}x^2 + 27x$ would be a possible function.

This is confirmed by technology.



$(-11 < y < 200,\ \text{Yscl} = 25)$

Chapter 3 Section 1 Question 12 Page 157

a) $A(0) = \left(\frac{4+1}{4^1}\right)(0)^2 - 10(0) + 100$

$= 100$

This represents the area of the gardens if the quarter circle is not used. The 20 m of edging could be used totally for the two sides of the square garden. Each side would be 10 m and the resulting area would be 100 m^2. Therefore, an answer of 100 is sensible.

b) $A'(x) = 2\left(\frac{4+1}{4^1}\right)x - 10$

$= \left(\frac{4+1}{2^1}\right)x - 10$

The function is increasing when $A'(x) = \left(\frac{4+\pi}{2\pi}\right)x - 10 > 0$.

First solve the related equation.

$\left(\frac{4+1}{2^1}\right)x - 10 = 0$

$x = 10\left(\frac{2^1}{4+1}\right)$

$x = \frac{20^1}{4+1}$

□ x B8.8

Test the inequality for arbitrary values in the intervals $\left(5, \frac{20^1}{4+^1}\right)$ and $\left(\frac{20^1}{4+^1}, 15\right)$.

Note the restrictions on x. Since each piece of edging must be at least 5 m long, $x \in [5, 15]$.

For the first interval, use $x = 5$.

$$\text{L.S.} = \left(\frac{4+^1}{2^1}\right)(5) - 10$$

B –4.3

$\sqcap \not> 0$

For the second interval, use $x = 10$.

$$\text{L.S.} = \left(\frac{4+^1}{2^1}\right)(10) - 10$$

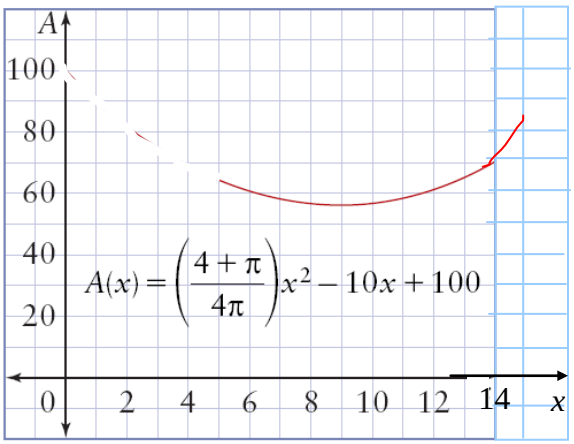
B1.4

$\sqcap > 0$

A is increasing on the interval $\left(\frac{20^1}{4+^1}, 15\right)$.

A is decreasing on the interval $\left(5, \frac{20^1}{4+^1}\right)$.

c) The graph clearly shows that the curve changes from decreasing to increasing at about $x = 9$.



Chapter 3 Section 1 Question 13 Page 157

a) $n(t) = 100 + 32t^2 - t^4$

$n'(t) = 64t - 4t^3$

The function is increasing when $n'(x) = 64t - 4t^3 > 0$.

First solve the related equation.

$$64t - 4t^3 = 0$$
$$4t(16 - t^2) = 0$$
$$4t(4 + t)(4 - t) = 0$$

$t = -4, t = 0, t = 4$

However, note that the domain is restricted to $0 < t < 5$.

Test the inequality for arbitrary values in the intervals $(0, 4)$ and $(4, 5)$.

For the first interval, use $x = 1$.

$$\text{L.S.} = 64(1) - 4(1)^3$$
$$= 60$$
$$> 0$$

For the second interval, use $x = 5$.

$$\text{L.S.} = 64(5) - 4(5)^3$$
$$= -180$$
$$\not> 0$$

The results can be summarized in a table.

	$t = 0$	$0 < t < 4$	$t = 4$	$t > 4$
$n'(t)$	0	Positive	0	Negative
$n(t)$	Horizontal	Increasing	Horizontal	Decreasing

b) If it were possible to have t-values for $t < 0$, the population would be increasing for $x < -4$.
The function would be horizontal at $x = -4$.
The function would be decreasing for $-4 < t < 0$.
There would be no change after $t = 5$; the function would continue to decrease.

Chapter 3 Section 1 Question 14 Page 158

a) $R(s) = -\frac{1}{2000}s^2 + 2s - 1200$

$R(2100) = -\frac{1}{2000}(2100)^2 + 2(2100) - 1200$

$= 795$

The range of the aircraft is 795 mi. when the engine speed is 2100 rpm.

b) $R'(s) = -\frac{1}{2000}2s + 2$

$= -\frac{1}{1000}s + 2$

The function is increasing when $R'(s) = -\frac{1}{1000}s + 2 > 0$.

First solve the related equation.

$-\frac{1}{1000}s + 2 = 0$

$s = 2000$

Test the inequality for arbitrary values in the intervals $(1000, 2000)$ and $(2000, 3100)$.
Note the restrictions on s. $(1000 < s < 3100)$
For the first interval, use x = 1500.

$\text{L.S.} = -\frac{1}{1000}(1500) + 2$

$= 0.5$

> 0

For the second interval, use x = 3000.

$\text{L.S.} = -\frac{1}{1000}(3000) + 2$

$= -1$

$\not> 0$

A is increasing on the interval $(1000, 2000)$.
A is decreasing on the interval $(2000, 3100)$.

c)

($x \in [0, 3100]$, Xscl = 500; $y \in [0, 1000]$, Yscl = 150)

d) An airplane has a peak efficiency engine speed (here about 2000 rpm). If that engine speed is exceeded, the engine will start to use more fuel per mile and hence the range will be shorter since there is a limited amount of fuel in the tank.

"There is no question indicated in the textbook for part d)."

Chapter 3 Section 1 Question 15 Page 158

The graph of f is a parabola opening up since the coefficient of x^2 is positive (+3).
Such a curve will be decreasing to the left of the vertex and increasing to the right of it.
The interval $a < x < \infty$ must include some part of the right half of the parabola and so must include an increasing section. The function cannot be strictly decreasing on this interval.

Chapter 3 Section 1 Question 16 Page 158

$$f(x) = x^3 + bx^2 + 12x - 3$$
$$f'(x) = 3x^2 + 2bx + 12$$

f is increasing when $f'(x) = 3x^2 + 2bx + 12 > 0$.

$$3x^2 + 2bx + 12 > 0$$
$$x^2 + \frac{2}{3}bx + 4 > 0 \qquad \text{Divide both sides by 4.}$$
$$\left(x^2 + \frac{2}{3}bx + \left(\frac{b}{3}\right)^2\right) - \left(\frac{b}{3}\right)^2 + 4 > 0 \qquad \text{Complete the square.}$$
$$\left(x + \frac{b}{3}\right)^2 - \frac{b^2}{9} + 4 > 0$$
$$\left(x + \frac{b}{3}\right)^2 > \frac{b^2}{9} - 4 \qquad \text{Add/subtract same amount on each side.}$$

The last inequality will be true for **all** values of x if and only if the R.S. is less than zero.

$$\frac{b^2}{9} - 4 < 0$$

$$\frac{b^2}{9} < 4$$

$$b^2 < 36$$

Therefore, $-6 < b < 6$ is the necessary condition for the function f to be increasing for all values of x.

Chapter 3 Section 1 Question 17 Page 158

D is the correct answer.

$$y = x^{2n+1} + x^{2n-1} + \text{K} + x^3 + x$$

$$\frac{dy}{dx} = (2n+1)x^{2n} + (2n-1)x^{2n-2} + \text{K} + 3x^2 + 1$$

This expression is always positive since the exponents on x are always even and the coefficients of each term are positive (i.e., the odd numbers from 1 to $2n+1$).
Since the derivative is always positive, the function is always increasing.

Chapter 3 Section 1 Question 18 Page 158

C is the correct answer.

If f is even, then $f(-x) = f(x)$.
Differentiate both sides of this equation using the chain rule.
$-f'(-x) = f'(x)$
Hence f' is odd.

If f is odd, then $f(-x) = -f(x)$.
Differentiate both sides of this equation using the chain rule.
$-f'(-x) = -f'(x)$
Hence f' is even.

Chapter 3 Section 2 Maxima and Minima

Chapter 3 Section 2 Question 1 Page 163

a) The absolute maximum value is 10 and the absolute minimum value is –3. (Assume the function is restricted to the domain shown.)

b) The absolute maximum value is 0.5 and the absolute minimum value is 0. (Assume the function is restricted to the domain shown.)

Chapter 3 Section 2 Question 2 Page 163

a) Since the function is linear, there are no critical numbers.
Examine the endpoints.
At $x = -10$,
$y = -(-10) + 7$
$= 17$

At $x = 10$,
$y = -(10) + 7$
$= -3$

The absolute maximum value is 17, occurring at the left endpoint.
The absolute minimum value is –3, occurring at the right endpoint.
There are no local extreme values.

b) Find the critical numbers.

$$f(x) = 3x^2 - 12x + 7$$
$$f'(x) = 6x - 12$$
$$6x - 12 = 0$$
$$x = 2$$

Examine the local extremum at $x = 2$ and the endpoints.

$$f(0) = 3(0)^2 - 12(0) + 7$$
$$= 7$$
$$f(2) = 3(2)^2 - 12(2) + 7$$
$$= -5$$
$$f(4) = 3(4)^2 - 12(4) + 7$$
$$= 7$$

The absolute maximum value is 7 and occurs twice, at both endpoints.
The absolute minimum value is –5 and occurs when $x = 2$.

c) Find the critical numbers.

$$f(x) = 2x^3 - 3x^2 - 12x + 2$$
$$f'(x) = 6x^2 - 6x - 12$$
$$6x^2 - 6x - 12 = 0$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$

$$x = 2, x = -1$$

Examine the local extrema at $x = 2$ and $x = -1$ and the endpoints.

$$f(-3) = 2(-3)^3 - 3(-3)^2 - 12(-3) + 2$$
$$= \check{G}43$$
$$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 2$$
$$= 9$$
$$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 2$$
$$= -18$$
$$f(3) = 2(3)^3 - 3(3)^2 - 12(3) + 2$$
$$= -7$$

The absolute maximum value is 9.
The absolute minimum value is –43.
There is a local minimum value of –18 when $x = 2$.

d) Find the critical numbers.

$$f(x) = x^3 + x$$
$$f'(x) = 3x^2 + 1$$

$$3x^2 + 1 = 0$$
$$x^2 = -\frac{1}{3}$$

There are no solutions to this equation and so there are no critical points.

Examine the endpoints.

$$f(0) = (0)^3 + (0)$$
$$= 0$$
$$f(10) = (10)^3 + (10)$$
$$= 1010$$

The absolute maximum value is 1010.
The absolute minimum value is 0.
There are no local extrema.

e) Find the critical numbers.

$$f(x) = (x-3)^2 - 9$$
$$f'(x) = 2(x-3)(1)$$
$$= 2x - 6$$

$$2x - 6 = 0$$
$$x = 3$$

This critical point is not within the given interval.

Examine the endpoints.

$$f(-8) = ((-8)-3)^2 - 9$$
$$= 112$$
$$f(-3) = ((-3)-3)^2 - 9$$
$$= 27$$

The absolute maximum value is 112.
The absolute minimum value is 27.
There are no local extrema.

Chapter 3 Section 2 Question 3 Page 163

a) $f(x) = -x^2 + 6x + 2$

$$f'(x) = -2x + 6$$

$$-2x + 6 = 0$$
$$x = 3$$

The critical number is 3.

b) $f(x) = x^3 - 2x^2 + 3x$

$$f'(x) = 3x^2 - 4x + 3$$

$$3x^2 - 4x + 3 = 0$$
$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(3)(3)}}{2(3)}$$
$$= \frac{4 \pm \sqrt{-20}}{6}$$

There are no critical numbers for this function.

c) $f(x) = x^4 - 3x^3 + 5$

$f'(x) = 4x^3 - 9x^2$

$4x^3 - 9x^2 = 0$

$x^2(4x - 9) = 0$

$x = 0, x = \frac{9}{4}$

The critical numbers are 0 and 2.25.

d) $g(x) = 2x^3 - 3x^2 - 12x + 5$

$g'(x) = 6x^2 - 6x - 12$

$6x^2 - 6x - 12 = 0$

$6(x^2 - x - 2) = 0$

$6(x - 2)(x + 1) = 0$

$x = -1,\ x = 2$

The critical numbers are –1 and 2.

e) $f(x) = x - \sqrt{x}$

$= x - x^{\frac{1}{2}}$

$f'(x) = 1 - \left(\frac{1}{2}\right) x^{-\frac{1}{2}}$

$= 1 - \frac{1}{2\sqrt{x}}$

$1 - \frac{1}{2\sqrt{x}} = 0$

$2\sqrt{x} - 1 = 0$

$\sqrt{x} = \frac{1}{2}$

$x = \frac{1}{4}$

The critical number is 0.25.

Note that $x = 0$ is not a critical number since that number is not in the domain of the function.

Chapter 3 Section 2 **Question 4 Page 164**

a) $y = f(x)$

$= 4x - x^2$

$f'(x) = 4 - 2x$

$4 - 2x = 0$

$x = 2$

Use a table to show increasing and decreasing intervals for the function.

	$x < 2$	$x = 2$	$x > 2$
Test Value	0	2	4
$f'(x)$	$4 - 2(0) = 4$ Positive	0	$4 - 2(4) = -4$ Negative
$f(x)$	Increasing	(2, 4)	Decreasing

The critical point (2, 4) is a local maximum.

b) $f(x) = (x-1)^4$

$f'(x) = 4(x-1)^3$

$4(x-1)^3 = 0$

$x = 1$

Use a table to show increasing and decreasing intervals for the function.

	$x < 1$	$x = 1$	$x > 1$
Test Value	0	1	4
$f'(x)$	$4((0)-1)^3 = -4$ Negative	0	$4((4)-1)^3 = 108$ Positive
$f(x)$	Decreasing	(1, 0)	Increasing

The critical point (1, 0) is a local minimum.

c) $g(x) = 2x^3 - 24x + 5$

$g'(x) = 6x^2 - 24$

$6x^2 - 24 = 0$

$x^2 = 4$

$x = \pm 2$

Use a table to show increasing and decreasing intervals for the function.

	$x < -2$	$x = -2$	$-2 < x < 2$	$x = 2$	$x > 2$
Test Value	–3	–2	0	2	3
$g'(x)$	$g'(-3) = 30$ Positive	0	$g'(0) = -24$ Negative	0	$g'(3) = 30$ Positive
$g(x)$	Increasing	(–2, 37)	Decreasing	(2, –27)	Increasing

The critical point (–2, 37) is a local maximum and the critical point (2, –27) is a local minimum.

d) $h(x) = x^5 + x^3$

$h'(x) = 5x^4 + 3x^2$

$5x^4 + 3x^2 = 0$

$x^2(5x^2 + 3) = 0$

$x = 0$ There are no roots associated with the second factor.

Use a table to show increasing and decreasing intervals for the function.

	$x < 0$	$x = 0$	$x > 0$
Test Value	–1	0	1
$h'(x)$	$h'(-1) = 8$ Positive	0	$h'(1) = 8$ Positive
$h(x)$	Increasing	(0, 0)	Increasing

The critical point (0, 0) is neither a local minimum nor a local maximum.

Chapter 3 Section 2 Question 5 Page 164

When you stop to rest, your elevation is not changing for a period of time. The rate of change of elevation, $f'(t)$, is zero.

You would be at a local maximum if the trial from the rest point went down in all directions.
You would be at a local minimum if the trial from the rest point went up in all directions.
You would be at neither a maximum nor a minimum if the trail went up in one direction but down in the other, or if you were riding on level ground.

Chapter 3 Section 2 Question 6 Page 164

a) Find the critical numbers.

$f(x) = 2x^3 - 3x^2 - 12x + 5$

$f'(x) = 6x^2 - 6x - 12$

$6x^2 - 6x - 12 = 0$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = -1, x = 2$

The critical numbers are –1 and 2.

b) To find local extrema, use a table to show increasing and decreasing intervals for the function.

	$x < -1$	$x = -1$	$-1 < x < 2$	$x = 2$	$x > 2$
Test Value	–2	–1	0	2	3
$f'(x)$	$f'(-2) = 24$ Positive	0	$f'(0) = -12$ Negative	0	$f'(3) = 24$ Positive
$f(x)$	Increasing	(–1, 12)	Decreasing	(2, –15)	Increasing

The critical point (–1, 12) is a local maximum and the critical point (2, –15) is local minimum.

c) Check the function values at the endpoints of the interval.

$f(-2) = 1$

$f(4) = 37$

The absolute minimum is (2, –15) and the absolute maximum is (4, 37) for the interval $[-2, 4]$.

Chapter 3 Section 2 Question 7 Page 164

a) Find the critical values.

$f(x) = 7 + 6x - x^2$

$f'(x) = 6 - 2x$

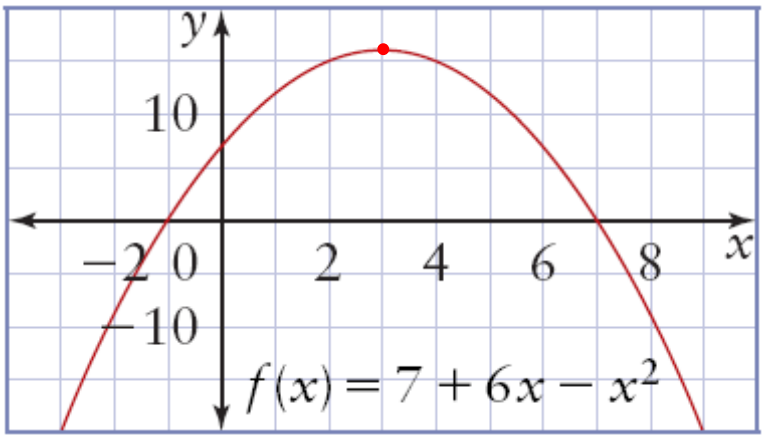
$6 - 2x = 0$

$x = 3$

To decide the nature of this critical value, evaluate $f'(x)$. This information can be summarized in a table.

	$x < 3$	$x = 3$	$x > 3$
Test value	0	3	4
$f'(x)$	$f'(0) = 6$ Positive	0	$f'(4) = -2$ Negative
$f(x)$	Increasing	(3, 16)	Decreasing

Sketch a curve with an absolute maximum at (3, 16).



b) Find the critical values.

$g(x) = x^4 - 8x^2 - 10$

$g'(x) = 4x^3 - 16x$

$4x^3 - 16x = 0$

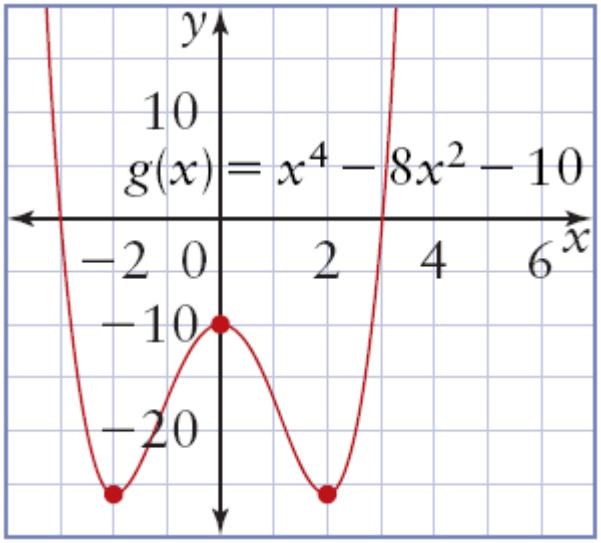
$4x\left(x^2 - 4\right) = 0$

$x = 0,\ x = -2,\ x = 2$

To decide the nature of these critical values, evaluate $g'(x)$. This information can be summarized in a table.

	$x < -2$	$x = -2$	$-2 < x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$x > 2$
Test Value	–3	–2	–1	0	1	2	3
$g'(x)$	$g'(-3) = -60$ Negative	0	$g'(-1) = 12$ Positive	0	$g'(1) = -12$ Negative	0	$g'(3) = 60$ Positive
$g(x)$	Decreasing	(–2, –26)	Increasing	(0, –10)	Decreasing	(2, –26)	Increasing

Sketch a curve with an absolute minima at (–2, –26) and (2, –26) and a local maximum at (0, –10).



c) Find the critical values.

$$\begin{aligned} y &= f(x) \\ &= x(x+2)^2 \\ &= x^3 + 4x^2 + 4x \end{aligned}$$

$$f'(x) = 3x^2 + 8x + 4$$

$$3x^2 + 8x + 4 = 0$$

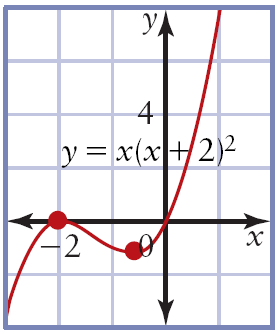
$$(3x+2)(x+2) = 0$$

$$x = -\frac{2}{3},\ x = -2$$

To decide the nature of these critical values, evaluate $f'(x)$. This information can be summarized in a table.

	$x < -2$	$x = -2$	$-2 < x < -\frac{2}{3}$	$x = -\frac{2}{3}$	$x > -\frac{2}{3}$
Test Value	–3	–2	–1	$-\frac{2}{3}$	0
$f'(x)$	$f'(-3) = 7$ Positive	0	$f'(-1) = -1$ Negative	0	$f'(0) = 4$ Positive
$f(x)$	Increasing	(–2, 0)	Decreasing	$\left(-\frac{2}{3}, -\frac{32}{27}\right)$	Increasing

Sketch a curve with a local maximum at (–2, 0) and a local minimum at $\left(-\frac{2}{3}, -\frac{32}{27}\right)$.



d) Find the critical values.

$h(x) = 27x - x^3$

$h'(x) = 27 - 3x^2$

$27 - 3x^2 = 0$

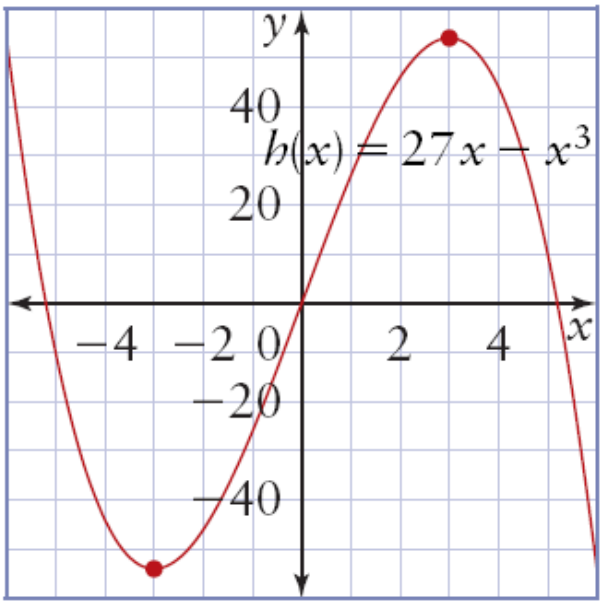
$x^2 = 9$

$x = -3,\ x = 3$

To decide the nature of these critical values, evaluate $h'(x)$. This information can be summarized in a table.

	$x < -3$	$x = -3$	$-3 < x < 3$	$x = 3$	$x > 3$
Test Value	–4	–3	0	3	4
$h'(x)$	$h'(-4) = -21$ Negative	0	$h'(0) = 27$ Positive	0	$h'(4) = -21$ Negative
$h(x)$	Decreasing	(–3, –54)	Increasing	(3, 54)	Decreasing

Sketch a curve with a local minimum at (–3, –54) and a local maximum at (3, 54).



Chapter 3 Section 2 Question 8 Page 164

No, the statement is generally false. There could be any number of local minima and maxima between $x = a$ and $x = b$. The function can have many sections of increase and decrease in the specified interval provided that the absolute maximum is at $x = a$ and the absolute minimum is at $x = b$.

Chapter 3 Section 2 Question 9 Page 164

a) The vertex of the parabola $f(x)$ is (3, 0) and the direction of opening is up.

b) $f'(x) = 2(x-3)(1)$

$$2(x-3) = 0$$
$$x = 3$$

Check the value of $f'(x)$ in the specified interval.

$f'(4) = 2$

Since this value is positive, $f(x)$ is increasing for $3 < x < 6$.

Therefore $x = 3$ is a global minimum value and $x = 6$ is a global maximum value.

c) The function is a parabola opening up, having vertex (3, 0).

The interval $3 < x < 6$ only involves the right half of the graph which is increasing only.

The global minimum value must occur at the start of the interval while the global maximum must occur at the end of the interval.

Chapter 3 Section 2 Question 10 Page 164

a) $x^3 - 2x^2 = 0$

$x^2(x-2) = 0$

b) The critical points create 3 intervals. Test the derivative in each interval.

	$x < 0$	$0 < x < 2$	$x > 2$
Test Value	–1	1	3
$f'(x)$	$f'(1) = -3$ Negative	$f'(1) = -1$ Negative	$f'(3) = 9$ Positive
$f(x)$	Decreasing	Decreasing	Increasing

The function is decreasing for $x < 0$ and $0 < x < 2$ and is increasing for $x > 2$.

c) Polynomial functions usually have a turning point at each critical value. That is because the function changes from increasing to/from decreasing at these points. The 'double root' $x = 0$ leads to the function not changing from increase/decrease at that point and therefore there is no turning point there. The only turning point occurs when $x = 2$.

Chapter 3 Section 2 Question 11 Page 164

a) Find the critical numbers.

$f(x) = x^3 - 6x^2 + 11x$

$f'(x) = 3x^2 - 12x + 11$

$3x^2 - 12x + 11 = 0$

$$x = \frac{12 \pm \sqrt{(-12)^2 - 4(3)(11)}}{2(3)}$$

$$= \frac{12 \pm \sqrt{12}}{6}$$

$$= \frac{6 \pm \sqrt{3}}{3}$$

$x \approx 2.6,\ x \approx 1.4$

The critical numbers are $\frac{6+\sqrt{3}}{3}$ and $\frac{6-\sqrt{3}}{3}$.

b) Test the function at the critical points and at the endpoints of the interval.

$f(0) = 0$

$f(1.4) \approx 6.4$

$f(2.6) \approx 5.6$

$f(4) = 12$

The absolute minimum value is 0; the absolute maximum value is 4.

Chapter 3 Section 2 **Question 12 Page 164**

a) $A(x) = \left(\frac{4+\pi}{4}\right)x^2 - 10x + 100$

$A'(x) = \left(\frac{4+\pi}{2}\right)x - 10$

$\left(\frac{4+\pi}{2}\right)x - 10 = 0$

$x = \frac{20}{4+\pi}$

There is only one critical number: $\frac{20\pi}{4+\pi}$.

b) $\frac{20}{4+\pi} \approx 8.8$

Use $x = 0$ and $x = 10$ for test values in the intervals.

	$x < \frac{20\pi}{4+\pi}$	$x = \frac{20\pi}{4+\pi}$	$x > \frac{20\pi}{4+\pi}$
$A'(x)$	Negative	0	Positive

c) The critical point is a local minimum because the function is decreasing on the interval before it and increasing in the interval after it.

d) Since the critical point is local minimum, check the endpoints for a maximum.

$A(5) \approx 64.2$

$A(15) \approx 77.9$

The maximum area for the interval $5 \le x \le 15$ is about 77.9 m^2.

Chapter 3 Section 2 **Question 13 Page 164**

a) Find the critical points.

$f(x) = -x^3 - 2x^2 + x + 15$

$f'(x) = -3x^2 - 4x + 1$

$-3x^2 - 4x + 1 = 0$

$3x^2 + 4x - 1 = 0$

$x = \dfrac{-4 \pm \sqrt{4^2 - 4(3)(-1)}}{6}$

$x = \dfrac{-4 \pm \sqrt{28}}{6}$

$x \doteq 0.22, \; x = -1.55$

To decide the nature of these critical values, evaluate $f'(x)$. This information can be summarized in a table.

" In the below table the $\sqrt{26}$ part should actually be $\sqrt{28}$. Please adjust all the tabular values accordingly."

	$x < \dfrac{-4-\sqrt{28}}{6}$	$x = \dfrac{-4-\sqrt{28}}{6}$	$\dfrac{-4-\sqrt{28}}{6} < x < \dfrac{-4+\sqrt{28}}{6}$	$x = \dfrac{-4+\sqrt{28}}{6}$	$x > \dfrac{-4+\sqrt{28}}{6}$
Test Value	-2	$\dfrac{-4-\sqrt{28}}{6}$	0	$\dfrac{-4+\sqrt{28}}{6}$	1
$h'(x)$	$f'(-2) = -3$ Negative	0	$f'(0) = 1$ Positive	0	$f'(1) = -6$ Negative
$h(x)$	Decreasing	$(-1.55, 12.37)$	Increasing	$(0.22, 15.11)$	Decreasing

The coaster starts down a hill from $x = -2$, reaching a local minimum at the bottom of a hill at $(-1.55, 12.37)$. It then increases height until it reaches a local maximum at the top of a hill at $(0.22, 15.11)$. It then continues downward until $x = 2$.

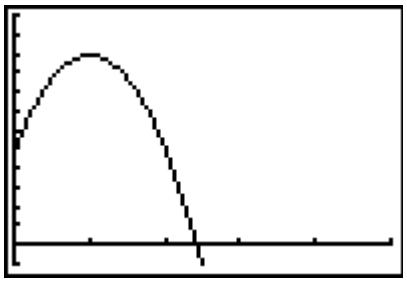
b) Check the values at the endpoints.

$f(-2) = 13$

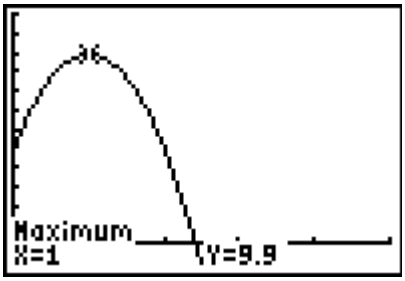
$f(2) = 1$

The highest point of the ride is at the critical point (0.22, 15.11), not at either endpoint.

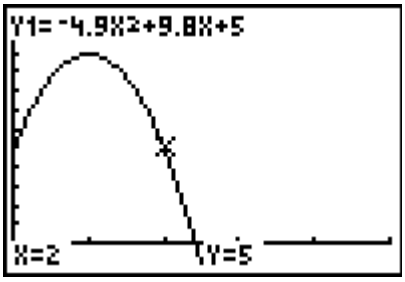
Chapter 3 Section 2 Question 14 Page 164

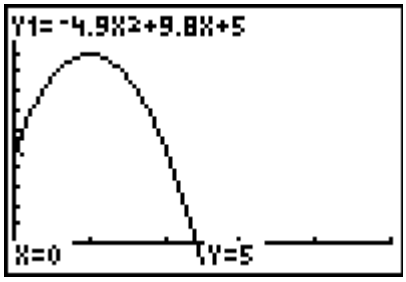


a) Use the **maximum** operation on the graphing calculator.



The minimum value occurs at an endpoint. Use the **value** operation to determine this value.





The minimum height is 5 m and the maximum height is 9.9 m.

b) One method is shown in part a) of the question. Some others include:

- Use the **trace** operation to trace the curve at its highest and lowest points.
- Use the [TABLE] feature to examine a table of values of the function.
- Graph $h'(t)$ and find the **zeroes**.
- Graph $h(t)$; then [ZOOM] in to find the coordinates of the extrema.

c) Answers will vary. Some possibilities include:

- Fill in random values of t until you converge on the maximum or minimum.
- Set $h(t) = 5$ and solve for t. The t-value that is half way between your two values will be the t-value of the maximum.
- Complete the square to determine the vertex. The curve is a parabola that will have a maximum value at its vertex.

Chapter 3 Section 2 Question 15 Page 165

a) Find the critical points for this function.

$$d(v) = 4.8v^3 - 28.8v^2 + 52.8v$$
$$d'(v) = 14.4v^2 - 57.6v + 52.8$$

$$14.4v^2 - 57.6v + 52.8 = 0$$
$$3v^2 - 12v + 11 = 0$$

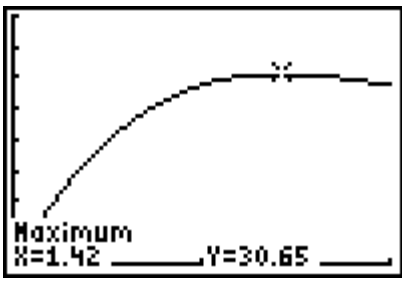
$$v = \frac{12 \pm \sqrt{12}}{6}$$
$$v = \frac{6 \pm \sqrt{3}}{3}$$
$$v \approx 2.58, \sqrt{v} = 1.42$$

Only one of these critical points is in the interval. Check for the maximum distance by substituting that value and also for the endpoints in the function.

$$d(0) = 0$$
$$d\left(\frac{6-\sqrt{3}}{3}\right) \approx 30.65$$
$$d(2) = 28.8$$

A speed of about 1.42 m/s will result in a maximum distance that the diver can swim.

b) The graphing calculator verifies our algebraic result.



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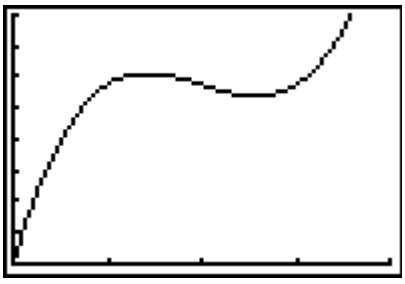
c) If you extend the domain of the graph, the function increases without bound after $x = 2$. This is not reasonable since a swimmer has a limit to how fast the person can swim. A speed of 2m/s is already quite fast and sustainable only over very short distances.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Chapter 3 Section 2 Question 16 Page 165

Solutions for Achievement Checks are shown in the Teacher's Resource.

Chapter 3 Section 2 Question 17 Page 165

$f(x) = ax^4 + bx^2 + cx + d$

$f'(x) = 4ax^3 + 2bx + c$

Since there are critical value at $x = 0$ and $x = 1$, the cubic derivative function must be of the form $f'(x) = kx(x-1)(x-p)$ where p is a third critical number and k must be $4a$.

$4ax^3 + 2bx + c = 4ax(x-1)(x-p)$

$4ax^3 + 2bx + c = 4ax^3 + 4a(p-1)x^2 - 4apx$

Comparing coefficients of like terms,

$0 = 4a(p-1)$ x^2 terms

$p = 1$

$2b = -4ap$

$b = -2a$ x terms ($p = 1$)

$c = 0$ constant terms

Therefore the function is now of the form $f(x) = ax^4 - 2ax^2 + d$.

Since (0, –6) is on the curve,

$a(0)^4 - 2a(0)^2 + d = -6$

$d = -6$

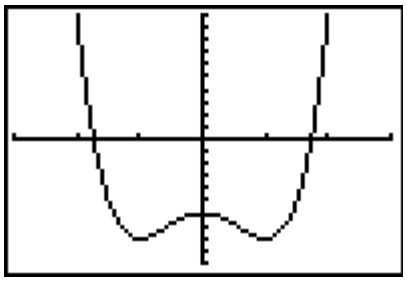
Since (1, –8) is on the curve,

$a(1)^4 - 2a(1)^2 - 6 = -8$

$a = 2$

Therefore $f(x) = 2x^4 - 4x^2 - 6$; that is, $a = 2$, $b = -4$, $c = 0$, and $d = -6$.

A graph of $f(x) = 2x^4 - 4x^2 - 6$ confirms the solution.



Chapter 3 Section 2 Question 18 Page 165

a) Find the critical numbers.

$f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c$

$3ax^2 + 2bx + c = 0$

$$x = \frac{-2b \pm \sqrt{(2b)^2 - 4(3a)(c)}}{6a}$$

There will be no critical numbers and hence no extrema if the discriminant is negative.

$4b^2 - 12ac < 0$

$b^2 - 3ac < 0$ or $b^2 < 3ac$

There are no extrema when $b^2 < 3ac$ or $b < \sqrt{3ac}$.

b) There will be exactly two critical numbers and hence two extrema if the discriminant is strictly positive.

$4b^2 - 12ac > 0$

$b^2 - 3ac > 0$ or $b^2 > 3ac$

There are exactly two extrema when $b^2 > 3ac$ or $b > \sqrt{3ac}$.

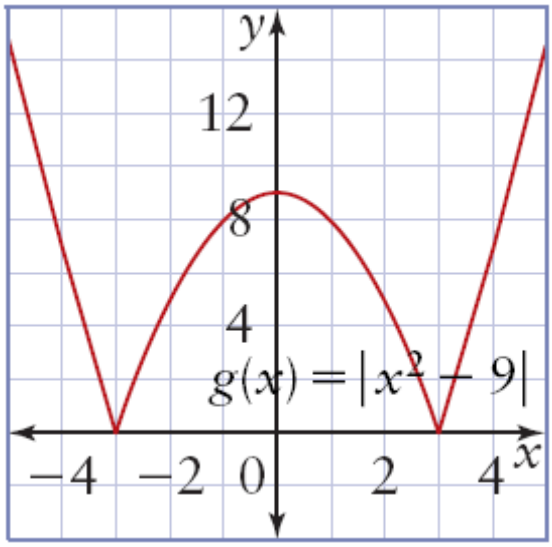
Chapter 3 Section 2 Question 19 Page 165

To find possible extrema, we examine the derivative of the function.
The derivative of a cubic function is quadratic.
This derivative will have either
i) no (real) roots,
ii) two identical (real) roots, or
iii) two different (real) roots.

In the first case, there will be no extrema since there are no roots.
In the third case, there will be exactly two extrema.
In the second case, there will also be no extrema. At each critical point, a polynomial function changes from increasing to decreasing or decreasing to increasing. At a double root, a polynomial function changes twice and so does not change. Such a critical number does not lead to an extremum.

Chapter 3 Section 2 Question 20 Page 165

a) Graph $y = x^2 - 9$. Any points below the x-axis will be reflected in the x-axis to produce the graph of $g(x) = |x^2 - 9|$.



b) Critical points occur when the derivative is zero or undefined.

Find the zeros of $y = x^2 - 9$.

$y' = 2x$

$2x = 0$

$x = 0$

One critical point occurs when $x = 0$.

Examine values of y' before and after 0. The point (0, –9) is an (absolute) minimum point for $y = x^2 - 9$. However, reflection in the x-axis reverses this to become a (local) maximum point (0, 9) for $g(x) = |x^2 - 9|$.

The derivative of $g(x) = |x^2 - 9|$ is undefined where $y = x^2 - 9$ has an x-intercept, i.e., at $x = -3$ and $x = 3$. Since $g(x) = |x^2 - 9|$ is always above the x-axis, (–3, 0) and (3, 0) must be (absolute) minimum points.

c) Consider $g(x)$ as a piecewise function.

$$g(x) = \begin{cases} x^2 - 9 & \text{if } x < -3 \\ -x^2 + 9 & \text{if } -3 \le x \le 3 \\ x^2 - 9 & \text{if } x > 3 \end{cases}$$

Since each of the pieces is a polynomial, the derivative is easily computed. At the boundaries for the intervals, the derivative will be undefined since the left and right hand derivatives are not equal.

$$g'(x) = \begin{cases} 2x & \text{if} \quad x < -3 \\ \text{undefined} & \text{if} \quad x = -3 \\ -2x & \text{if} \quad -3 \le x \le 3 \\ \text{undefined} & \text{if} \quad x = 3 \\ 2x & \text{if} \quad x > 3 \end{cases}$$

Chapter 3 Section 2 Question 21 Page 165

B is the correct answer.

$y = x^n - nx$

$\frac{dy}{dx} = nx^{n-1} - n$

For all positive integers, $n \ge 2$, $\frac{dy}{dx} = 0$ when $x = 1$.

Note that $\frac{dy}{dx} = 0$ when $x = -1$ only for odd *n*.
Hence, depending on the parity of *n*, the function does not necessarily have a critical point at $x = -1$.

For $x = 1^-$, $\frac{dy}{dx} < 0$; for $x = 1^+$, $\frac{dy}{dx} > 0$.
Therefore, there is a local minimum at $x = 1$.

Chapter 3 Section 2 Question 22 Page 165

D is the correct answer.

The function $f(x)$ has a local minimum at $x = a$ because $f'(x)$ changes from negative to positive when passing through $x = a$.

Chapter 3 Section 2 Question 23 Page 165

E is the correct answer.

The function $f(x)$ has a horizontal tangent at $x = b$ because $f'(b) = 0$.
Note that $f(x)$ is increasing on either side of $x = b$.

Chapter 3 Section 3 Concavity and the Second Derivative Test

Chapter 3 Section 3 Question 1 Page 173

a) Graph is concave up when $x > -2$.
Graph is concave down when $x < -2$.

b) Graph is concave up when $x < -2$ and $x > 0$.
Graph is concave down when $-2 < x < 0$.

Chapter 3 Section 3 Question 2 Page 173

a) The function $f(x)$ is concave up for $x \in \mathbb{R}$. There are no points of inflection.

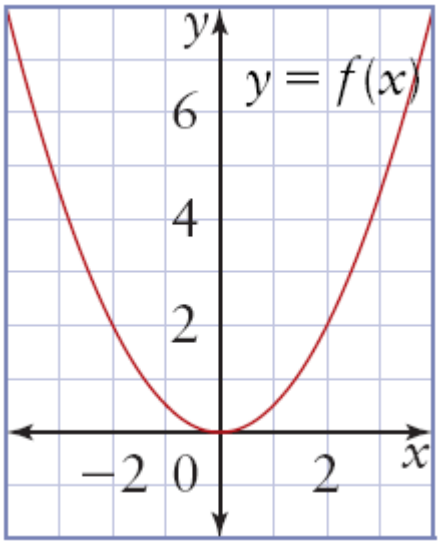
b) The function $f(x)$ is concave up for $x > 2$ and concave down for $x < 2$. There is a point of inflection when $x = 2$.

c) The function $f(x)$ is concave up for $-1 < x < 2$ and concave down for $x < -1$ and $x > 2$. There are points of inflection when $x = -1$ and $x = 2$.

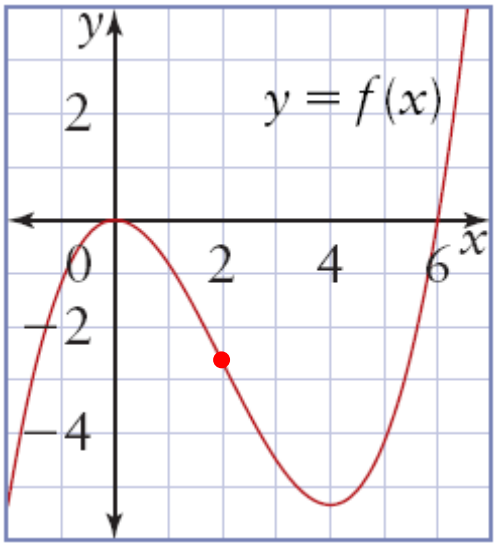
d) The function $f(x)$ is concave up for $x > 2$ and concave down for $x < -1$ and $-1 < x < 2$. There is a point of inflection when $x = 2$.

Chapter 3 Section 3 Question 3 Page 174

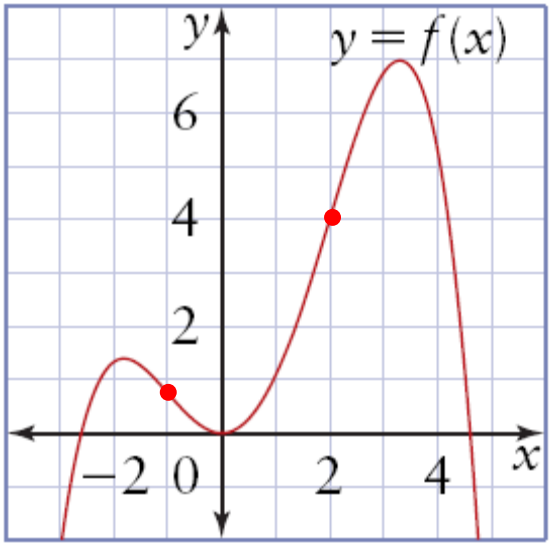
a)



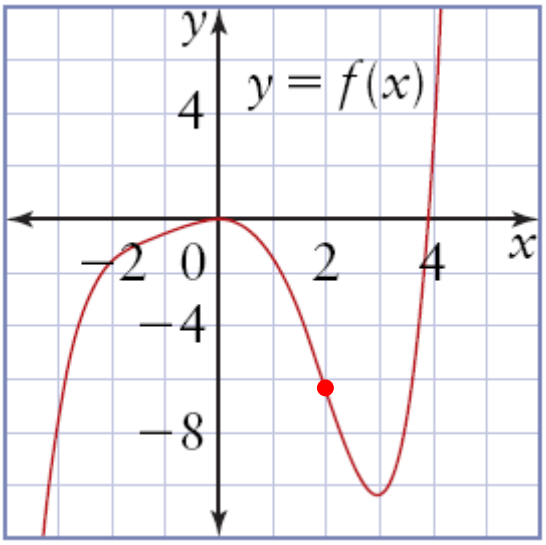
b)



c)



d)



Chapter 3 Section 3 Question 4 Page 174

a) $y = 6x^2 - 7x + 5$

$y' = 12x - 7$

$y'' = 12$

b) $f(x) = x^3 + x$

$f'(x) = 3x^2 + 1$

$f''(x) = 6x$

c) $g(x) = -2x^3 + 12x^2 - 9$

$g'(x) = -6x^2 + 24x$

$g''(x) = -12x + 24$

d) $y = x^6 - 5x^4$

$y' = 6x^5 - 20x^3$

$y'' = 30x^4 - 60x^2$

Chapter 3 Section 3 Question 5 Page 174

a) $y'' = 12$

This function is always concave up (i.e., for $x \in$ °). There are no points of inflection.

b) $f''(x) = 6x$

$6x = 0$

$x = 0$

This value divides the domain into two intervals. Test the value of $f''(x)$ in each interval and summarize in a table.

	$x < 0$	$x = 0$	$x > 0$
Test Value	–1	0	1
$f''(x)$	$f''(-1) = -6$ Negative	0	$f''(1) = 6$ Positive
$f(x)$	Concave down ⌒	Point of inflection (0, 0)	Concave up ⌣

c) $g''(x) = -12x + 24$

$-12x + 24 = 0$

$x = 2$

This value divides the domain into two intervals. Test the value of $f''(x)$ in each interval and summarize in a table.

	$x < 2$	$x = 2$	$x > 2$
Test Value	0	2	3
$g''(x)$	$g''(0) = 24$ Positive	0	$g''(3) = -12$ Negative
$g(x)$	Concave up ⌣	Point of inflection (2, 23)	Concave down ⌒

d) $y'' = 30x^4 - 60x^2$

$30x^4 - 60x^2 = 0$

$30x^2(x^2 - 2) = 0$

$x = 0,\ x = -\sqrt{2},\ x = \sqrt{2}$

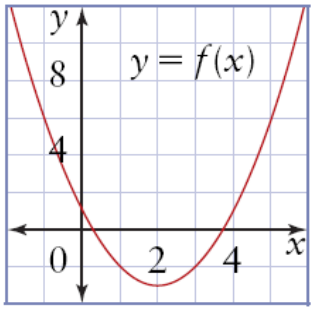
These values divide the domain into four intervals.

Test the value of $f''(x)$ in each interval and summarize in a table.

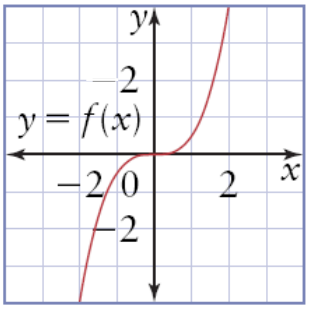
	$x < -\sqrt{2}$	$x = -\sqrt{2}$	$-\sqrt{2} < x < 0$	$x = 0$	$0 < x < \sqrt{2}$	$x = \sqrt{2}$	$x > \sqrt{2}$
Test Value	-2	$-\sqrt{2}$	-1	0	1	$\sqrt{2}$	2
$y'' = f''(x)$	$f''(-2) = 240$ Positive	0	$f''(-1) = -30$ Negative	0	$f''(1) = -30$ Negative	0	$f''(3) = 240$ Positive
$y = f(x)$	Concave up ⌣	Point of Inflection $(-\sqrt{2}, -12)$	Concave down ⌢	No Point of Inflection	Concave down ⌢	Point of Inflection $(\sqrt{2}, -12)$	Concave up ⌣

Chapter 3 Section 3 **Question 6 Page 174**

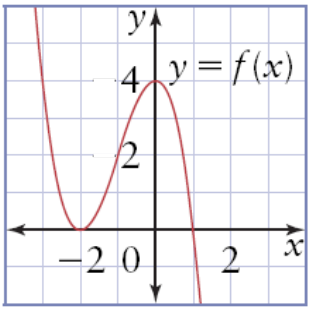
a) This function is always concave up.
It passes through (2, –3) where the slope is 0 (horizontal).



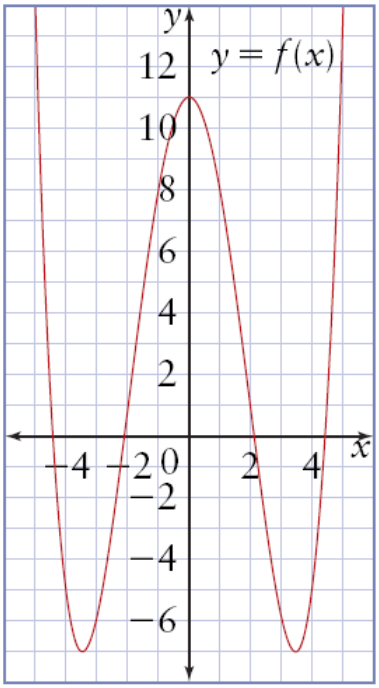
b) This function is concave down when $x < 0$ and concave up when $x > 0$.
It passes through the origin where it has a horizontal slope (point of inflection).



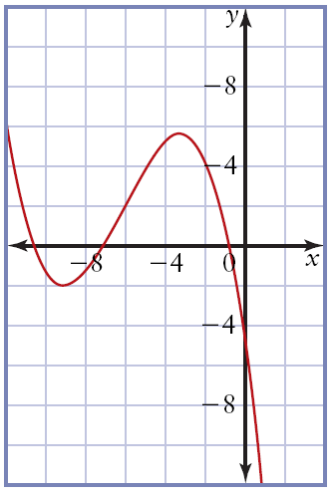
c) This function is concave up when $x < -1$ and concave down when $x > -1$.
It passes through (–1, 2) where it has a slope of 1.



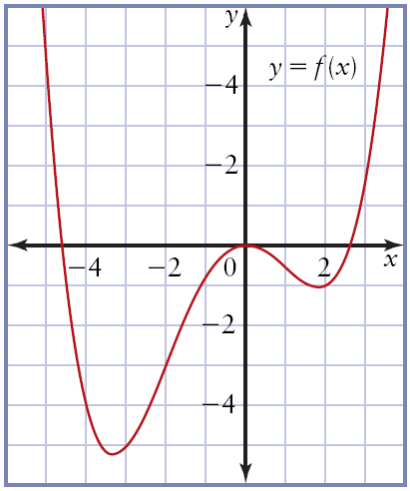
d) This function is concave down between –2 and 2 and concave up when $x < -2$ or $x > 2$.
It passes through (2, 1). Since the function is even, it is symmetric about the y-axis. For instance, it passes through (–2, 1).



e) This function is concave up when $x < -6$ and concave down when $x > -6$.
It passes through (–6, 2) where it has a slope of 2.



f) This function is concave down between –2 and 1 and concave up when $x < -2$ or $x > 1$.
It passes through (–2, –3) and the origin.



Chapter 3 Section 3 Question 7 Page 174

a) Begin as in section 2.2. Find the critical values.

$$y = f(x)$$
$$= x^2 + 10x - 11$$

$$f'(x) = 2x + 10$$

$$2x + 10 = 0$$
$$x = -5$$

Use a table to show increasing and decreasing intervals for the function.

	$x < -5$	$x = -5$	$x > -5$
Test Value	-6	-5	0
$f'(x)$	$2(-6)+10=-2$ Negative	0	$2(0)+10=10$ Positive
$f(x)$	Decreasing	$(-5, -36)$	Increasing

The critical point $(-5, -36)$ is an absolute minimum.

$f''(x)=2$, which is always positive. This means the curve is always concave up and any critical point must be a minimum.

b) Find the critical values.

$g(x)=3x^5-5x^3-5$

$g'(x)=15x^4-15x^2$

$15x^4-15x^2=0$

$15x^2(x^2-1)=0$

$x=0,\ x=-1,\ x=1$

Use a table to show increasing and decreasing intervals for the function.

	$x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$x > 1$
Test Value	-2	-1	-0.5	0	0.5	1	2
$g'(x)$	$g'(-2)=180$ Positive	0	$g'(-0.5)=-2.8$ Negative	0	$g'(0.5)=-2.8$ Negative	0	$g'(2)=180$ Positive
$g(x)$	Increasing	$(-1, -3)$	Decreasing	$(0, 0)$	Decreasing	$(1, -7)$	Increasing

The critical point $(-1, -3)$ is a local maximum and the critical point $(1, -7)$ is a local minimum.

Now use the second derivative test to verify the results.

$g''(x)=60x^3-30x$

Test the critical points.

$g''(-1)=-30$

$g''(0)=0$

$g''(1)=30$

The second derivative test indicates that there is a maximum at $x=-1$ and a minimum at $x=1$.
Since the sign of the second derivative changes at $x=0$, there is a point of inflection there.

c) Find the critical values.

$f(x) = x^4 - 6x^2 + 10$

$f'(x) = 4x^3 - 12x$

$4x^3 - 12x = 0$

$4x(x^2 - 3) = 0$

$x = 0,\ x = -\sqrt{3},\ x = \sqrt{3}$

Use a table to show increasing and decreasing intervals for the function.

	$x < -\sqrt{3}$	$x = -\sqrt{3}$	$-\sqrt{3} < x < 0$	$x = 0$	$0 < x < \sqrt{3}$	$x = \sqrt{3}$	$x > \sqrt{3}$
Test Value	-2	$-\sqrt{3}$	-1	0	1	$\sqrt{3}$	$2\sqrt{}$
$f'(x)$	$f'(-2) = -8$ Negative	0	$f'(-1) = 8$ Positive	0	$f'(1) = -8$ Negative	0	$f'(2) = 8$ Positive
$f(x)$	Decreasing	$(-\sqrt{3}, 1)$	Increasing	(0, 10)	Decreasing	$(\sqrt{3}, 1)$	Increasing

The critical point $(-\sqrt{3}, 1)$ is a local minimum, (0, 10) is a local maximum and $(\sqrt{3}, 1)$ is a local minimum.

Now use the second derivative test to verify the results.

$f''(x) = 12x^2 - 12$

Test the critical points.

$f''(-\sqrt{3}) = 24$

$f''(0) = 0$

$f''(\sqrt{3}) = 24$

The second derivative test indicates that there are minimums at $x = -\sqrt{3}$ and $x = \sqrt{3}$ and a maximum at $x = 0$.

d) Find the critical values.

$h(t) = -4.9t^2 + 39.2t + 2$

$h'(t) = -9.8t + 39.2$

$-9.8t + 39.2 = 0$

$t = 4$

Use a table to show increasing and decreasing intervals for the function.

	$t < 4$	$t = 4$	$t > 4$
Test Value	0	4	5
$h'(t)$	$h'(0) = 39.2$ Positive	0	$h'(5) = -9.8$ Negative
$h(t)$	Increasing	(4, 80.4)	Decreasing

The critical point (4, 80.4) is an absolute maximum.

Now use the second derivative test to verify the results.
$h''(t) = -9.8$ which is always negative.
The second derivative test indicates that there is a maximum at $t = 4$.

Chapter 3 Section 3 Question 8 Page 174

a) $h(x) = 0.01x^3 - 0.3x^2 + 60$

$h'(x) = 0.03x^2 - 0.6x$

$h''(x) = 0.06x - 0.6$

$0.06x - 0.6 = 0$

$x = 10$

Use a table to show concavity of the intervals for the function.

	$0 < x < 10$	$x = 10$	$10 < x < 22$
Test Value	1	10	20
$h''(x)$	$h''(1) = -0.54$ Negative	0	$h''(20) = 0.6$ Positive
$h(x)$	Concave down 	Point of Inflection (10, 40)	Concave up

b) The steepest slope occurs when $h'(x)$ has a maximum. This will occur when its derivative, $h''(x)$, equals zero. This occurs when $x = 10$, the inflection point.
The steepest point on the ski ramp is at (10, 40).

Chapter 3 Section 3 Question 9 Page 174

a) For a polynomial function, this is sometimes true. The exception is that if the minimum or maximum occurs at an endpoint of an interval, the derivative will not be defined for that point.
This statement may not be true for other types of functions, such as broken line functions like $f(x) = |x|$ where the derivative is undefined at some local extrema.

b) This is always true. The definition of inflection point requires that the curve be concave up and down on opposite sides of the point. This requires the second derivative to be zero at the point of inflection.

Chapter 3 Section 3 Question 10 Page 174

a) The curve is concave up for all values of x since this is a parabola with the coefficient of x^2 positive.

b) The curve will have a local minimum since it is a parabola opening up. The minimum is located at the vertex. You can determine this point using calculus.

$$A(x) = \left(\frac{4+1}{4^1}\right)x^2 - 10x + 100$$

$$A'(x) = \left(\frac{4+1}{2^1}\right)x - 10$$

$$\left(\frac{4+1}{2^1}\right)x - 10 = 0$$

$$x = \frac{20^1}{4+1}$$

The (absolute) minimum point is $\left(\frac{20\pi}{4+\pi}, \frac{400}{4+\pi}\right)$.

c) The maximum area must occur at one of the endpoints of the interval since the curve is concave up throughout the interval.

$A(0) = 100$

$A(20) \approx 127.3$

The maximum occurs when $x = 20$.

Chapter 3 Section 3 Question 11 Page 174

The car is starting out at A and then accelerating over the interval AB. The driver takes his foot off the gas at B and then decelerates over the interval BC. At C the car is stopped. The driver then puts on the gas and accelerates over the interval CD. The driver then releases the gas at D and decelerates over the interval DE. At E the driver stops and turns the car around. Then he accelerates from E to F. At F the driver stops accelerating and decelerates over the interval FG. At G the car is back at the original location and the driver stops.

Chapter 3 Section 3 Question 12 Page 175

a) Find the critical numbers.

$$T(t) = -0.0003t^3 + 0.012t^2 - 0.112t + 36$$

$$T'(t) = -0.0009t^2 + 0.024t - 0.112$$

$$T''(t) = -0.0018t + 0.024$$

For critical numbers,

$$-0.0009t^2 + 0.024t - 0.112 = 0$$

$$9t^2 - 240t + 1120 = 0$$

$$t = \frac{240 \pm \sqrt{(-240)^2 - 4(9)(1120)}}{2(9)}$$

$$t = \frac{240 \pm \sqrt{17\ 280}}{18}$$

$$t = \frac{40 \pm 4\sqrt{30}}{3}$$

$$t \approx 20.6,\ t \approx 6.0$$

The critical numbers are 6.0 and 20.6 days.

b) The rate of change of temperature is a maximum when $T''(t) = 0$.

$$-0.0018t + 0.024 = 0$$

$$t = \frac{40}{3}$$

$$t \approx 13.3$$

The female is most likely to conceive on the 13th day of the cycle.

c) This is a point of inflection.

Rate of change of temperature is $T'(t)$. For this quantity to be a maximum, we need its derivative $T''(t) = 0$. This is a necessary condition for an inflection point.

Chapter 3 Section 3 Question 13 Page 175

a) $f''(x) = x^2(x-2)$

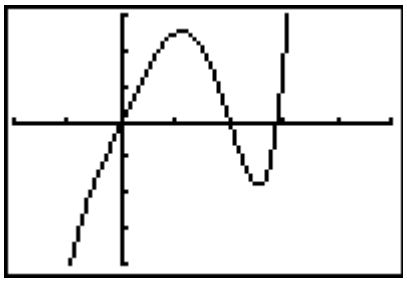
$$x^2(x-2) = 0$$

$$x = 0,\ x = 2$$

b) These values divide the domain into three intervals. Test the value of $f''(x)$ in each interval and summarize in a table.

	$x<0$	$0<x<2$	$x>2$
Test value	–1	1	3
$f''(x)$	$f''(-1)=-3$ Negative	$f''(1)=-1$ Negative	$f''(3)=9$ Positive
$f(x)$	Concave down	Concave down	Concave up

c) The graph start concave down, become straight at $x=0$, then continue concave down to $x=2$. This point (2, 1) will be a point of inflection, after which the curve will continue concave up. One possible graph is shown. (Its equation is $y=3x^5-10x^4+32x$).



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Chapter 3 Section 3 Question 14 Page 175

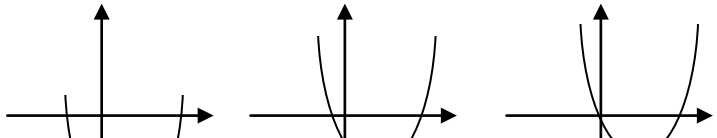
There is one point of inflection on the graph of $f(x)$.
For a point of inflection for $f(x)$, we need the slope of $f'(x)$ to change from negative to positive (or vice versa). This can only occur if the slope of $f'(x)$ is zero, which occurs exactly once on this graph of $f'(x)$.

Chapter 3 Section 3 Question 15 Page 175

A quartic function $f(x)$ will have a cubic first derivative $f'(x)$ and a quadratic second derivative $f''(x)$.

For a point of inflection, the second derivative must change sign (i.e., positive to negative or negative to positive).
Since $f''(x)$ is quadratic, it can have 0, 1, or 2 zeros.
If $f''(x)$ has no zeros, it is always positive or always negative and does not change sign. There are no points of inflection in this case.

If $f''(x)$ has one zero, it must be a 'double' zero (two identical real roots). In this case, $f''(x)$ is always positive or zero, or negative or zero. In either case there is no change of sign and there are no points of inflection.
If $f''(x)$ has two zeros, it must go from positive to negative to positive (or the reverse) to create the two distinct zeros. Clearly there are two changes of sign in this case and consequently, two points of inflection.



The diagrams below show the three possibilities for $f''(x)$

Chapter 3 Section 3 Question 16 Page 175

$$f(x) = ax^3 + bx^2 + cx + d$$
$$f'(x) = 3ax^2 + 2bx + c$$
$$f''(x) = 6ax + 2b$$

Since (0, 2) is on the curve,
$$a(0)^3 + b(0)^2 + c(0) + d = 2$$
$$d = 2$$

Since (0, 2) is a point of inflection,
$$6a(0) + 2b = 0$$
$$b = 0$$

Since $x = 2$ is a critical point,
$$3a(2)^2 + 2(0)(2) + c = 0$$
$$c = -12a$$

Since (2, 6) is on the curve,
$$a(2)^3 + (0)(2)^2 + c(2) + (2) = 6$$
$$8a + 2(-12a) = 4$$
$$-16a = 4$$
$$a = -\frac{1}{4}$$

$$c = -12\left(-\frac{1}{4}\right)$$
$$= 3$$

The required function is $f(x) = -\frac{1}{4}x^3 + 3x + 2$.

b) A cubic function has range $\{y \mid y \in \mathbb{R}\}$. The curve cannot just have a local maximum. On one side of the maximum point it must descend to a minimum point and then increase again to satisfy the required range.

Alternative explanation.

In question 19 of section 3.2 it was proved that a cubic function has either two critical points or none. Since given one critical point is given, there must be a second one as well. Since intervals of increase/decrease alternate between critical points, the second critical point must be a local minimum.

Chapter 3 Section 3 Question 17 Page 175

a) Since $f''(x) = 2$ for $x \in \mathbb{R}$, $f'(x)$ must be a linear function and $f(x)$ must be a quadratic function.

b) If $f(x) = x^n$, $n \in \mathbb{N}$, then the degree of the function must be odd of degree at least 3 since there are two intervals of opposite concavity. Some examples: $f(x) = x^5$, $f(x) = x^3$, $f(x) = x^{11}$. Note that not all cubic functions work since $f(x) = x^3 - 2x$ does not.

c) Since there are two intervals of opposite concavity, the degree of the function must be odd of degree at least 3.

d) Since there are three intervals of alternating concavity, the degree of the function must be even of degree at least 4.

e) Since there are two intervals of opposite concavity, the degree of the function must be odd of degree at least 3.

f) Since there are three intervals of alternating concavity, the degree of the function must be even of degree at least 4.

Chapter 3 Section 3 Question 18 Page 175

E is the correct answer.

$f'(a)$ and $f''(a)$ do not necessarily exist.
Therefore, A and B are not necessarily true.

C is not true because f is not necessarily strictly increasing when $x < a$ and strictly decreasing when $x > a$.

D is not true if f is a function which is not differentiable everywhere.
Such a function exists. For example,

$$f(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a \text{ is rational} \\ -1 & \text{if } x \neq a \text{ is irrational} \end{cases}$$

E is the definition of the local maximum.

Chapter 3 Section 3 Question 19 Page 175

E is the correct answer.

Let $f(x) = x^4$.

Then $f'(0) = 0 = f''(0)$ and f has a local minimum at $x = 0$.

Let $f(x) = -x^4$.
Then $f'(0) = 0 = f''(0)$ and f has a local maximum at $x = 0$.

Let $f(x) = x^3$.
Then $f'(0) = 0 = f''(0)$ and f has a point of inflection at $x = 0$.

So $f'(a) = 0 = f''(a)$ can happen when there is a local maximum, a local minimum, or a point of inflection at $x = a$.

Chapter 3 Section 4 Simple Rational Functions

Chapter 3 Section 4 Question 1 Page 183

Vertical asymptotes only occur where the denominator in a rational function equals zero.

a) $x = 5$

b) $x^2 - 4 = 0$

$x^2 = 4$

$x = 2$ or $x = -2$

c) $x^2 + 5 = 0$

$x^2 = -5$

There are no vertical asymptotes since this equation has no real solutions.

d) $x^2 - 3x + 2 = 0$

$(x - 2)(x - 1) = 0$

$x = 2$ or $x = 1$

e) $x^2 + 2x - 4 = 0$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-4)}}{2}$$

$$x = \frac{-2 \pm \sqrt{20}}{2}$$

$x = -1 + \sqrt{5}$ or $x = -1 - \sqrt{5}$

f) $x = 0$

g) $x^4 + 8 = 0$

$x^4 = -8$

There are no vertical asymptotes since this equation has no real solutions.

h) $x^2 - 6x + 9 = 0$

$(x - 3)(x - 3) = 0$

$x = 3$

Chapter 3 Section 4 Question 2 Page 183

a) Since $\frac{4.5}{4.5-5} = -9$, $\lim\limits_{x \to 5^-} f(x) = -\infty$.

Since $\frac{5.5}{5.5-5} = +11$, $\lim\limits_{x \to 5^+} f(x) = \infty$.

b) Since $\frac{1.9+3}{(1.9)^2-4} = -13$, $\lim\limits_{x \to 2^-} f(x) = -\infty$.

Since $\frac{2.1+3}{(2.1)^2-4} = 12$, $\lim\limits_{x \to 2^+} f(x) = \infty$.

Since $\frac{-2.01+3}{(-2.01)^2-4} = 25$, $\lim\limits_{x \to -2^-} f(x) = \infty$.

Since $\frac{-1.99+3}{(-1.99)^2-4} = -25$, $\lim\limits_{x \to -2^+} f(x) = -\infty$.

c) No asymptotes.

d) Since $\frac{(1.9)^2}{(1.9)^2-3(1.9)+2} = -40$, $\lim\limits_{x \to 2^-} f(x) = -\infty$.

Since $\frac{(2.1)^2}{(2.1)^2-3(2.1)+2} = 40$, $\lim\limits_{x \to 2^+} f(x) = \infty$.

Since $\frac{(0.99)^2}{(0.99)^2-3(0.99)+2} = 97$, $\lim\limits_{x \to 1^-} f(x) = \infty$.

Since $\frac{(1.01)^2}{(1.01)^2-3(1.01)+2} = -103$, $\lim\limits_{x \to 1^+} f(x) = -\infty$.

e) Since $\frac{1.23-5}{(1.23)^2+2(1.23)-4} = 139$, $\lim\limits_{x \to \left(-1+\sqrt{5}\right)^-} f(x) = \infty$.

Since $\frac{1.24-5}{(1.24)^2+2(1.24)-4} = -213$, $\lim\limits_{x \to \left(-1+\sqrt{5}\right)^+} f(x) = -\infty$.

Since $\frac{(-3.24)-5}{(-3.24)^2+2(-3.24)-4} = -468$, $\lim\limits_{x \to \left(-1-\sqrt{5}\right)^-} f(x) = -\infty$.

Since $\frac{(-3.23)-5}{(-3.23)^2+2(-3.23)-4} = 304$, $\lim\limits_{x \to \left(-1-\sqrt{5}\right)^+} f(x) = \infty$.

f) Since $2(-0.01)+\dfrac{1}{(-0.01)}=-100$, $\lim\limits_{x\to 0^-} f(x)=-\infty$.

Since $2(0.01)+\dfrac{1}{0.01}=100$, $\lim\limits_{x\to 0^+} f(x)=\infty$.

g) No asymptotes.

h) Since $\dfrac{2(2.99)-3}{(2.99)^2-6(2.99)+9}=29\ 800$, $\lim\limits_{x\to 3^-} f(x)=\infty$.

Since $\dfrac{2(3.01)-3}{(3.01)^2-6(3.01)+9}=30\ 200$, $\lim\limits_{x\to 3^+} f(x)=\infty$.

Chapter 3 Section 4 Question 3 Page 183

a) $y=\dfrac{1}{x^2}$

$=x^{-2}$

$y'=-2x^{-3}$

$=-\dfrac{2}{x^3}$

Critical points occur if the derivative is zero or undefined. Here, $x=0$ is the only possibility, but the function is not defined at $x=0$. Therefore, there are no local extrema.

b) $f(x)=\dfrac{2}{x+3}$

$=2(x+3)^{-1}$

$f'(x)=-2(x+3)^{-2}$

$=-\dfrac{2}{(x+3)^2}$

Critical points occur if the derivative is zero or undefined. Here, $x=-3$ is the only possibility, but the function is not defined at $x=-3$. Therefore, there are no local extrema.

c) $g(x)=\dfrac{x}{x-4}$

$=x(x-4)^{-1}$

$g'(x)=1(x-4)^{-1}+x(-1)(x-4)^{-2}(1)$

$=(x-4)^{-2}((x-4)-x)$

$=\dfrac{-4}{(x-4)^2}$

Critical points occur if the derivative is zero or undefined. Here, $x=4$ is the only possibility, but the function is not defined at $x=4$. Therefore, there are no local extrema.

d) $h(x) = \dfrac{-3}{(x-2)^2}$

$= -3(x-2)^{-2}$

$h'(x) = 6(x-2)^{-3}$

$= \dfrac{6}{(x-2)^3}$

Critical points occur if the derivative is zero or undefined. Here, $x = 2$ is the only possibility, but the function is not defined at $x = 2$. Therefore, there are no local extrema.

e) $y = \dfrac{x}{x^2-1}$

$= x(x^2-1)^{-1}$

$y' = 1(x^2-1)^{-1} + x(-1)(x^2-1)^{-2}(2x)$

$= (x^2-1)^{-2}(x^2-1-2x^2)$

$= -\dfrac{x^2+1}{(x^2-1)^2}$

Critical points occur if the derivative is zero or undefined. Here, $x = 1$ is the only possibility, but the function is not defined at $x = 1$. Therefore, there are no local extrema.

f) $t(x) = \dfrac{2x}{3x^2+12x}$

$= 2x(3x^2+12x)^{-1}$

$t'(x) = 2(3x^2+12x)^{-1} + 2x(-1)(3x^2+12x)^{-2}(6x+12)$

$= (3x^2+12x)^{-2}(6x^2+24x-12x^2-24x)$

$= \dfrac{-6x^2}{(3x^2+12x)^2}$

Critical points occur if the derivative is zero or undefined. Here, $x = 0, -4$ is the only possibility, but the function is not defined at $x = 0, -4$. Therefore there are no local extrema.

Chapter 3 Section 4 Question 4 Page 183

a) The two functions are both rational functions having only one vertical asymptote. The graph of *f* is the graph of *g* translated one unit left and reflected in the *x*-axis.

b) $f(x) = \dfrac{-2}{(x+1)^2}$

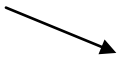
$= -2(x+1)^{-2}$

$f'(x) = 4(x+1)^{-3}$

$= \dfrac{4}{(x+1)^3}$

There are no critical points. The intervals of increase/decrease are separated by the asymptote value, $x = -1$.

Use a table to show increasing and decreasing intervals for the function.

	$x < -1$	$x = -1$	$x > -1$
Test Value	-2		0
$f'(x)$	$f'(-2) = -4$ Negative	Undefined	$f'(0) = 4$ Positive
$f(t)$	Decreasing 	Vertical Asymptote	Increasing

c) $f(x) = \dfrac{-2}{(x+1)^2}$

$f'(x) = 4(x+1)^{-3}$

$f''(x) = -12(x+1)^{-4}$

$= \dfrac{-12}{(x+1)^4}$

Concavity changes only at the asymptotes.

	$x < -1$	$x = -1$	$x > -1$
Test Value	-2		0
$f''(x)$	$f''(-2) = -12$ Negative	Undefined	$f''(0) = -12$ Negative
$f(t)$	Concave down	Vertical Asymptote	Concave down

Chapter 3 Section 4 Question 5 Page 183

a) $x^2 - 4 = 0$

$x = -2,\ x = 2$

The equations of the vertical asymptotes are $x = -2$ and $x = 2$.

b) Find the critical points.

$h(x) = \dfrac{1}{x^2 - 4}$

$= (x^2 - 4)^{-1}$

$h'(x) = -(x^2 - 4)^{-2}(2x)$

$= -\dfrac{2x}{(x^2 - 4)^2}$

There is a critical point at $x = 0$.

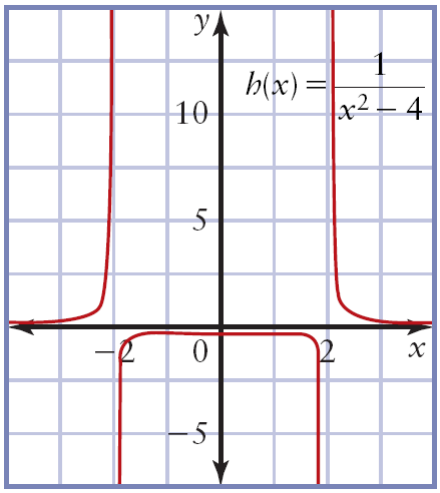
Consider the intervals defined by this point and the asymptotes.

	$x < -2$	$-2 < x < 0$	$0 < x < 2$	$x > 2$
Test Value	-3	-1	1	3
$h'(x)$	$h'(-3) = 0.24$ Positive	$h'(-1) = 0.22$ Positive	$h'(1) = -0.22$ Negative	$h'(3) = -0.24$ Negative
$h(t)$	Increasing	Increasing	Decreasing	Decreasing

c) Since the curve is increasing to the left of $x = -2$, the left hand limit must be ∞.
Since the curve is increasing to the right of $x = -2$, the right hand limit must be $-\infty$.

Since the curve is decreasing to the left of $x = 2$, the left hand limit must be $-\infty$.
Since the curve is decreasing to the right of $x = 2$, the right hand limit must be ∞.

d)



Chapter 3 Section 4 **Question 6 Page 183**

a) $C(p) = \dfrac{75\,000}{100 - p}$

$C(50) = \dfrac{75\,000}{50}$

$= 1500$

The removal cost is \$1500.

b) Since $C(99.9) = 750\ 000$, $\lim\limits_{p \to 100^-} C(p) = \infty$.

c) Since the cost increases without bound as the percentage of contaminants removed approaches 100, it is clear that there is not enough money available to achieve 100% removal.

Chapter 3 Section 4 Question 7 Page 183

If the curve is concave down for $x > 2$ and must approach the asymptote asymptotically, it will look like a parabola opening down in this interval. Therefore, on the right side of the asymptote it must approach $-\infty$. In symbols, $\lim\limits_{x \to 2^+} f(x) = -\infty$.

Chapter 3 Section 4 Question 8 Page 184

a) $N(t) = 0.5t + \dfrac{5}{10t + 1}$

$$\begin{aligned} N(0) &= 0.5(0) + \frac{5}{10(0) + 1} \\ &= 5 \end{aligned}$$

There were 5 units of pollutant in the river when the clean-up began.

b) To find the minimum, look for critical points.

$$\begin{aligned} N(t) &= 0.5t + \frac{5}{10t + 1} \\ &= 0.5t + 5(10t + 1)^{-1} \end{aligned}$$

$$\begin{aligned} N'(t) &= 0.5 - 510t + 1^{-2}(10) \\ &= 0.5 - \frac{50}{10t + 1^2} \end{aligned}$$

$$\begin{aligned} 0.5 - \frac{50}{10t + 1^2} &= 0 \\ 10t + 1^2 &= 100 \\ 10t + 1 &= \pm 10 \end{aligned}$$

$t = -1.1,\ t = 0.9$

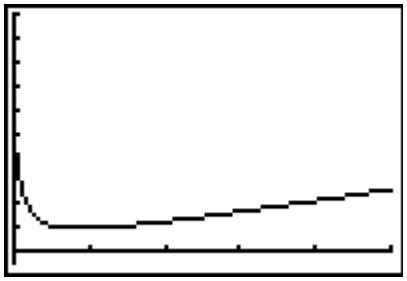
Since time must be positive, there is only one critical point at $t = 0.9$.

Since $N'(0.5) = -0.9$, N is decreasing before the critical point.
Since $N'(1) = 0.1$, N is increasing after the critical point.

The quantity of pollutant is a minimum after about 0.9 years (about 11 months).

Since $\frac{N(0.9)}{N(0)} = \frac{0.95}{5} = \frac{19}{100}$, the fraction of pollutant remaining is about $\frac{1}{5}$.

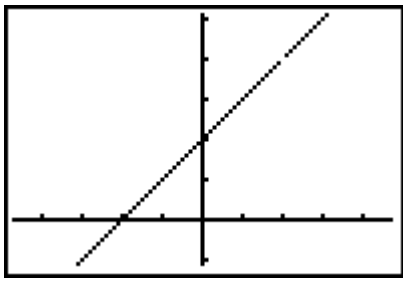
c) Since the pollutants started to increase again after this time, it may be possible that the environmental group stopped working on the project at this time. It is also possible that the source of pollution had begun to reintroduce additional pollutants into the river at this time.



Chapter 3 Section 4 Question 9 Page 184

Solutions for Achievement Checks are shown in the Teacher's Resource.

Chapter 3 Section 4 Question 10 Page 184



X	Y1
1.7	3.7
1.8	3.8
1.9	3.9
2	ERROR
2.1	4.1
2.2	4.2
2.3	4.3

X=2

a) The limit of the function values close to $x = 2$ is not ∞ or $-\infty$, which is necessary for an asymptote. Examining the function closely, we can see that the numerator and denominator of the fraction share a common factor of $x - 2$. If this divided out, the function becomes $f(x) = x + 2,\ x \neq 2$

b) No. the function is not defined at $x = 2$. In fact, $f(2) = \frac{0}{0}$ which is an undefined quantity.

Chapter 3 Section 4 Question 11 Page 184

$$f(x) = \frac{ax}{bx+c} = ax(bx+c)^{-1}$$

$$f'(x) = a(bx+c)^{-1} + ax(-1)(bx+c)^{-2}(b) \quad \text{Product rule}$$

$$= (bx+c)^{-2}(a(bx+c) - abx)$$

$$= \frac{c}{(bx+c)^2}$$

Turning points (critical points) occur only if the derivative is zero or undefined.
The derivative cannot be zero since the numerator is a non-zero constant.
The denominator cannot be zero, which would lead to an undefined derivative, since a quantity squared is always greater than zero.
Therefore there are no turning points possible.

Chapter 3 Section 4 Question 12 Page 184

The function $f(x) = \frac{1}{(x-2)(x+1)}$ will have asymptotes at $x = 2$ and $x = -1$.
Since this function is below the x-axis when $x = 1$, add a constant to the function to guarantee the x-intercept of 1.

$$\frac{1}{(1-2)(1+1)} + k = 0$$

$$-\frac{1}{2} + k = 0$$

$$k = \frac{1}{2}$$

One possible function is $f(x) = \frac{1}{(x-2)(x+1)} + \frac{1}{2}$.

Alternative solution.
Begin as above but introduce a factor in the numerator that will have a value of zero when $x = 1$.

Another possible function is $f(x) = \frac{x-1}{(x-2)(x+1)}$.

Chapter 3 Section 4 Question 13 Page 184

B is the correct answer.

$$y = \frac{(x+1)^2}{2x^2+5x+3}$$
$$= \frac{(x+1)^2}{(x+1)(2x+3)}, \text{ where } x \neq -1$$
$$= \frac{x+1}{2x+3}$$

Therefore $x = -1$ cannot be a vertical asymptote as -1 is not in the domain of the function. Similarly, -1 cannot be an x-intercept.

$\lim\limits_{x\to\infty} \frac{x+1}{2x+3} = \frac{1}{2}$ implies that $y = \frac{1}{2}$ is a horizontal asymptote.

Chapter 3 Section 4 Question 14 Page 184

C is the correct answer.

A. Since $x^n + 1 \neq 0$,

$$y = \frac{x^{2n}-1}{x^{2n}+x^n}$$
$$= \frac{(x^n-1)(x^n+1)}{x^n(x^n+1)}$$
$$= \frac{x^n-1}{x^n}$$

There is a vertical asymptote $x = 0$ and a horizontal asymptote $y = 1$.

B. $y = \frac{x^{2n}+1}{x^n+1}$ has a vertical asymptote $x = \sqrt[n]{-1}$ when n is odd.

C. Since $x^n + 1 \neq 0$,

$$y = \frac{x^{2n}-1}{x^n+1}$$
$$= \frac{(x^n-1)(x^n+1)}{x^n+1}$$
$$= x^n - 1$$

This is a polynomial function and clearly does not have an asymptote.

D. For $x^n - 1 \neq 0$,

$$y = \frac{x^{2n} - x^n}{x^{2n} + x^n - 2}$$
$$= \frac{x^n(x^n - 1)}{(x^n + 2)(x^n - 1)}$$
$$= \frac{x^n}{x^n + 2} \quad .$$
$$= \frac{x^n + 2 - 2}{x^n + 2}$$
$$= 1 - \frac{2}{x^n + 2}$$

There is a vertical asymptote $x = \sqrt[n]{-2}$ when n is odd and a horizontal asymptote $y = 1$.

E. $y = \dfrac{x^{2n+1} + x + 1}{x^{2n} + 1}$

$$= \frac{x(x^{2n} + 1) + 1}{x^{2n} + 1}$$
$$= x + \frac{1}{x^{2n} + 1}$$

This function has a slant asymptote $y = x$.

Chapter 3 Section 5 **Putting It All Together**

Chapter 3 Section 5 **Question 1 Page 192**

a) $f(x) = x^3 - 6x$

$f'(x) = 3x^2 - 6$

$f''(x) = 6x$

Find the critical points.

$3x^2 - 6 = 0$

$x^2 = 2$

$x = \pm\sqrt{2}$

Use the second derivative test for extrema.

$f''\left(\sqrt{2}\right) = 6\left(\sqrt{2}\right)$

$= 8$

$f''\left(-\sqrt{2}\right) = 6\left(-\sqrt{2}\right)$

$= -8$

Therefore, $\left(-\sqrt{2}, 4\sqrt{2}\right)$ is a local maximum and $\left(\sqrt{2}, -4\sqrt{2}\right)$ is a local minimum.

b) $g(x) = -x^4 + 2x^2$

$g'(x) = -4x^3 + 4x$

$g''(x) = -12x^2 + 4$

Find the critical points.

$-4x^3 + 4x = 0$

$-4x(x^2 - 1) = 0$

$-4x(x + 1)(x - 1) = 0$

$x = 0,\ x = 1,\ x = -1$

Use the second derivative test for extrema.

$g''(-1) = -12(-1)^2 + 4$

$= -8$

$g''(1) = -12(1)^2 + 4$

$= -8$

$g''(0) = -12(0)^2 + 4$

$= 4$

Therefore, (–1, 1) and (1, 1) are local maximum points and (0, 0) is a local minimum.

c) $f(x) = -x^3 + 3x - 2$

$f'(x) = -3x^2 + 3$

$f''(x) = -6x$

Find the critical points.

$-3x^2 + 3 = 0$

$x^2 = 1$

$x = 1, x = -1$

Use the second derivative test for extrema.

$f''(-1) = -6(-1)$

$= 6$

$f''(1) = -6(-1)$

$= -6$

Therefore, (1, 0) is a local maximum and (–1, –4) is a local minimum.

d) $h(x) = 2x^2 + 4x + 5$

$h'(x) = 4x + 4$

$h''(x) = 4$

Find the critical points.

$4x + 4 = 0$

$x = -1$

Use the second derivative test for extrema.

$h''(-1) = 4$

Therefore, (–1, 3) is a local minimum.

Chapter 3 Section 5 Question 2 Page 192

a) $f(x) = 2x^3 - 4x^2$

$f'(x) = 6x^2 - 8x$

$f''(x) = 12x - 8$

For points of inflection, let $f''(x) = 0$.

$12x - 8 = 0$

$x = \frac{2}{3}$

Test if $f''(x)$ changes sign.

$f''(0) = -8$

$f''(1) = 4$

Since there is a sign change, $\left(\frac{2}{3}, -\frac{32}{27}\right)$ is a point of inflection.

b) $f(x) = x^4 - 6x^2$

$f'(x) = 4x^3 - 12x$

$f''(x) = 12x^2 - 12$

For points of inflection, let $f''(x) = 0$.

$12x^2 - 12 = 0$

$x^2 = 1$

$x = 1, x = -1$

Test if $f''(x)$ changes sign.

For $x = -1$,

$f''(-2) = 36$

$f''(0) = -12$

For $x = 1$,

$f''(0) = -12$

$f''(2) = 36$

Since there is a sign change in each case, (–1, –5) and (1, –5) are points of inflection.

c) $f(x) = x^5 - 30x^3$

$f'(x) = 5x^4 - 90x^2$

$f''(x) = 20x^3 - 180x$

For points of inflection, let $f''(x) = 0$.

$20x^3 - 180x = 0$

$20x(x^2 - 9) = 1$

$x = 0, x = 3, x = -3$

Test if $f''(x)$ changes sign.

For $x = -3$,

$f''(-4) = -560$

$f''(-1) = 160$

For $x = 3$,
$f''(1) = -160$
$f''(4) = 560$

For $x = 0$,
$f''(-1) = 160$
$f''(1) = -160$

Since there is a sign change in each case, (–3, 567), (3, –567), and (0, 0) are points of inflection.

d) $f(x) = 3x^5 - 5x^4 - 40x^3 + 120x^2$
$f'(x) = 15x^4 - 20x^3 - 120x^2 + 240x$
$f''(x) = 60x^3 - 60x^2 - 240x + 240$

For points of inflection, let $f''(x) = 0$.
$60x^3 - 60x^2 - 240x + 240 = 0$
$60x^2(x-1) - 240(x-1) = 0$
$(x-1)(60)(x^2 - 4) = 0$

$x = 1,\ x = 2,\ x = $ Ğ2

Test if $f''(x)$ changes sign.
For $x = -2$,
$f''(-3) = -1200$
$f''(-1) = 360$

For $x = 2$,
$f''(1.5) = -52.5$
$f''(3) = 600$

For $x = 1$,
$f''(0) = 240$
$f''(1.5) = -52.5$

Since there is a sign change in each case, (–2, 624), (2, 176), and (0, 0) are points of inflection.

Chapter 3 Section 5 Question 3 Page 192

a) $f(x) = x^4 - 8x^3$

$-f(x) = -x^4 + 8x^3$

$f(-x) = (-x)^4 - 8(-x)^3$

$= x^4 + 8x^3$

The function is neither even nor odd.

b) The function is a polynomial. The domain is $\{x \in \mathbb{R}\}$.

c) For the *y*-intercept, let $x = 0$.

$f(0) = 0$

The *y*-intercept is 0.

For the *x*-intercept, let $y = 0$.

$x^4 - 8x^3 = 0$

$x^3(x - 8) = 0$

$x = 0,\ x = 8$

The *x*-intercepts are 0 and 8.

d) Find the critical points.

$f(x) = x^4 - 8x^3$

$f'(x) = 4x^3 - 24x^2$

$f''(x) = 12x^2 - 48x$

$4x^3 - 24x^2 = 0$

$4x^2(x - 6) = 0$

$x = 0,\ x = 6$

Use the second derivative test for extrema.

$f''(0) = 12(0)^2 - 48(0)$

$= 0$

$f''(-1) = 12(-1)^2 - 48(-1)$

$= 60$

$f''(1) = 12(1)^2 - 48(1)$

$= -36$

Note that for $x = 0$, check the sign of $f''(x)$ on either side of $x = 0$. Since the function is changing from concave up to concave down there, (0, 0) is a point of inflection.

$f''(6) = 12(6)^2 - 48(6)$

$= 144$

Therefore, (6, –432) is a local minimum.

The critical points create three intervals. Test the derivative in each interval.

	$x < 0$	$0 < x < 6$	$x > 6$
Test Value	–1	1	7
$f'(x)$	$f'(-1) = -28$ Negative	$f'(1) = -20$ Negative	$f'(7) = 196$ Positive
$f(x)$	Decreasing	Decreasing	Increasing

The function is decreasing for $x < 0$ and $0 < x < 6$ and is increasing for $x > 6$.

For concavity, we need to find all points of inflection. Let $f''(x) = 0$.

$12x^2 - 48x = 0$

$12x(x - 4) = 0$

$x = 0,\ x = 4$

Test for concavity in the intervals between the possible points of inflection.

	$x < 0$	$x = 0$	$0 < x < 4$	$x = 4$	$x > 4$
Test value	–1		1		5
$f''(x)$	$f''(-1) = 60$ Positive	0	$f''(1) = -36$ Negative	0	$f''(5) = 60$ Positive
$f(x)$	Concave up ⌣	Point of Inflection (0, 0)	Concave down ⌢	Point of Inflection (4, –256)	Concave up ⌣

Chapter 3 Section 5 Question 4 Page 192

a) Follow the six step plan.

$f(x) = x^3$

$f'(x) = 3x^2$

$f''(x) = 6x$

Step 1. The domain is $\{x \mid x \in \mathbb{R}\}$.

Step 2. $f(0) = 0$

The y-intercept is 0.

$x^3 = 0$

$x = 0$

The x-intercept is 0.

Step 3. Find the critical numbers.

$3x^2 = 0$

$x = 0$

Use the second derivative test to classify the critical point.

$f''(0) = 6(0)$

$= 0$ Check further.

$f''(-1) = -6$

$f''(1) = 6$

Step 3. ④ Therefore (0, 1) is a point of inflection and not an extremum.

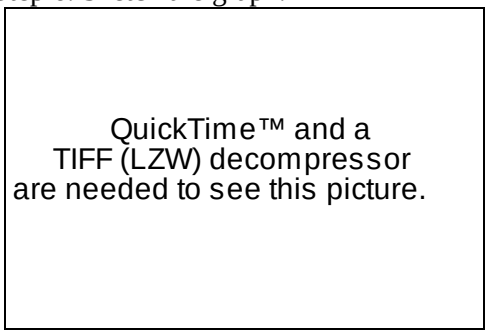
Step 5. $f'(-1) = 3$

$f'(1) = 3$

Therefore, f is increasing for $x < 0$ and $x > 0$.

From Step 3, f is concave down for $x < 0$ and concave up for $x > 0$.

Step 6. Sketch the graph.



($x \in [-4, 4]$, $y \in [-4, 4]$)

b) Follow the six step plan.

$h(x) = x^5 + 20x^2 + 5$

$h'(x) = 5x^4 + 40x$

$h''(x) = 20x^3 + 40$

Step 1. The domain is $\{x \in \mathbb{R}\}$.

Step 2. $h(0) = 5$.

The y-intercept is 5.

$x^5 + 20x^2 + 5 = 0$

This equation is not factorable. Trial and error suggests there is an x-intercept of around $x = -3$.

Step 3. Find the critical numbers.

$$5x^4 + 40x = 0$$
$$5x(x^3 + 8) = 0$$
$$5x(x+2)(x^2 - 2x + 4) = 0 \qquad \text{The third factor has no real roots.}$$

$$x = 0,\ x = -2$$

Use the second derivative test to classify the critical points.

$$h''(0) = 20(0)^3 + 40$$
$$= 40$$
$$h''(-2) = 20(-2)^3 + 40$$
$$= -120$$

Therefore, (–2, 53) is a local maximum and (0, 5) is a local minimum.

Step 4. Find the possible points of inflection.

$$20x^3 + 40 = 0$$
$$x^3 + 2 = 0 \qquad \text{Factor the sum of cubes.}$$
$$\left(x + \sqrt[3]{2}\right)\left(x^2 - \sqrt[3]{2}x + \left(\sqrt[3]{2}\right)^2\right) = 0 \qquad \text{The second factor has no real roots.}$$

$$x = -\sqrt[3]{2}$$

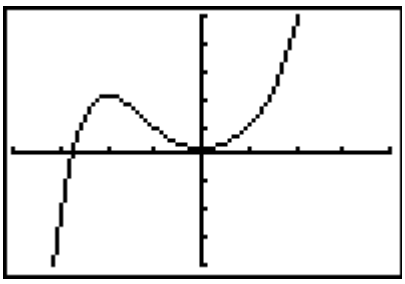
Test the intervals.

$$h''(-2) = 20(-2)^3 + 40$$
$$= -120$$
$$h''(0) = 20(0)^3 + 40$$
$$= 40$$

Therefore $\left(-\sqrt[3]{2}, 18\left(\sqrt[3]{2}\right)^2 + 5\right)$ or (–1.26, 33.57) is a point of inflection.

Step 5. From Step 3, h is increasing for $x < -2$ and $x > 0$ and decreasing for $-2 < x < 0$.
From Step 4, h is concave down for $x < -\sqrt[3]{2}$ and concave up for $x > -\sqrt[3]{2}$.

Step 6. Sketch the graph.



($x \in [-4, 4],\ y \in [-100, 125];\ \text{Yscl:}25$)

c) Follow the six step plan.

$$k(x) = \frac{1}{2}x^4 - 2x^3$$
$$k'(x) = 2x^3 - 6x^2$$
$$k''(x) = 6x^2 - 12x$$

Step 1. The domain is $\{x \in \mathbb{R}\}$.

Step 2. $k(0) = 0$

The y-intercept is 0.

$$\frac{1}{2}x^4 - 2x^3 = 0$$
$$x^3(x-4) = 0$$

$x = 0,\ x = 4$

The x-intercepts are 0 and 4.

Step 3. Find the critical numbers.

$$2x^3 - 6x^2 = 0$$
$$2x^2(x-3) = 0$$

$x = 0,\ x = 3$

Use the second derivative test to classify the critical points.

$$k''(0) = 6(0)^2 - 12(0)$$
$$= 0$$ Check further.

$$k''(-1) = 6(-1)^2 - 12(-1)$$
$$= 18$$
$$k''(1) = 6(1)^2 - 12(1)$$
$$= -6$$
$$k''(3) = 6(3)^2 - 12(3)$$
$$= 18$$

Therefore (3, –13.5) is a local minimum.

Step 4. Find the possible points of inflection.

$$6x^2 - 12x = 0$$
$$6x(x-2) = 0$$

$x = 0,\ x = 2$

Test the intervals.

$x = 0$ was tested already.

$k''(1) = -6$

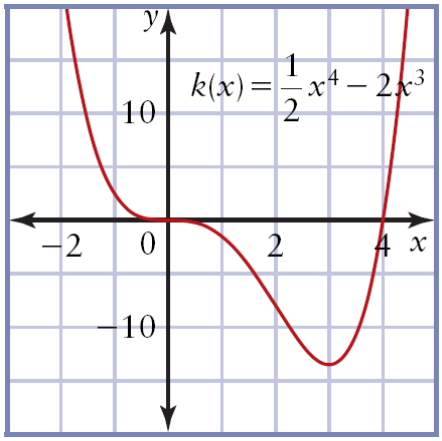
$k''(3) = 6(3)^2 - 12(3)$

$= 18$

Therefore (0, 0) and (2, –8) are points of inflection.

Step 5. From Step 3, k is increasing for $x > 3$ and decreasing for $x < 0$ and $0 < x < 3$.
From Step 4, k is concave down for $0 < x < 2$ and concave up for $x < 0$ and $x > 2$.

Step 6. Sketch the graph.



d) Follow the six step plan.

$b(x) = -(2x-1)(x^2 - x - 2)$

$= -2x^3 + 3x^2 + 3x - 2$

$b'(x) = -6x^2 + 6x + 3$

$b''(x) = -12x + 6$

Step 1. The domain is $\{x \in \mathbb{R}\}$.

Step 2. $b(0) = -2$

The y-intercept is –2.

$-(2x-1)(x^2 - x - 2) = 0$

$-(2x-1)(x-2)(x+1) = 0$

$x = \frac{1}{2}, x = 2, x = -1$

The x-intercepts are –1, 0.5, and 2.

Step 3. Find the critical numbers.

$$-6x^2 + 6x + 3 = 0$$
$$-3(2x^2 - 2x - 1) = 0$$
$$x = \frac{2 \pm \sqrt{12}}{4}$$
$$= \frac{1 \pm \sqrt{3}}{2}$$

$x \approx -0.4,\ x \approx 1.4$

Use the second derivative test to classify the critical points.

$$b''\left(\frac{1-\sqrt{3}}{2}\right) = -12\left(\frac{1-\sqrt{3}}{2}\right) + 6$$
$$= 6\sqrt{3}$$
$$b''\left(\frac{1+\sqrt{3}}{2}\right) = -12\left(\frac{1+\sqrt{3}}{2}\right) + 6$$
$$= -6\sqrt{3}$$

Therefore, (–0.4, –2.6) is a local minimum and (1.4, 2.6) is a local maximum.

Step 4. Find the possible points of inflection.

$$-12x + 6 = 0$$
$$x = \frac{1}{2}$$

Test the intervals.

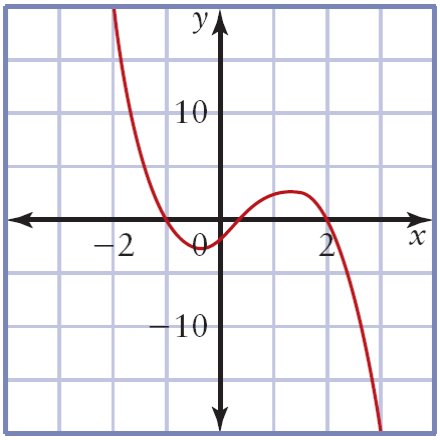
$$b''(0) = -12(0) + 6$$
$$= 6$$
$$b''(1) = -12(1) + 6$$
$$= -6$$

Therefore, (0.5, 0) is a point of inflection.

Step 5. From Step 3, b is increasing for $\frac{1-\sqrt{3}}{2} < x < \frac{1+\sqrt{3}}{2}$ and decreasing for $x < \frac{1-\sqrt{3}}{2}$ and $x > \frac{1+\sqrt{3}}{2}$.

From Step 4, b is concave up for $x < \frac{1}{2}$ and concave down for $x > \frac{1}{2}$.

Step 6. Sketch the graph.



Chapter 3 Section 5 **Question 5 Page 192**

$f(x) = 2x^3 - 3x^2 - 72x + 7$

a) The first derivative of f will be of degree 2. Setting the derivative to zero can yield at most two critical values. Therefore, the function has at most two local extrema.

b) The second derivative of f will be of degree 1. Setting this derivative equal to zero will yield one possible value for a point of inflection. Therefore, f can have at most one point of inflection.

c) $f'(x) = 6x^2 - 6x - 72$

$f''(x) = 12x - 6$

For critical values,

$6x^2 - 6x - 72 = 0$

$x^2 - x - 12 = 0$

$(x - 4)(x + 3) = 0$

$x = 4,\ x = -3$

Use the second derivative test to classify the critical points.

$f''(-3) = 12(-3) - 6$

$= -42$

$f''(4) = 12(4) - 6$

$= 42$

Therefore, (–3, 142) is a local maximum and (4, –201) is a local minimum.

Find the possible points of inflection.

$12x - 6 = 0$

$x = \frac{1}{2}$

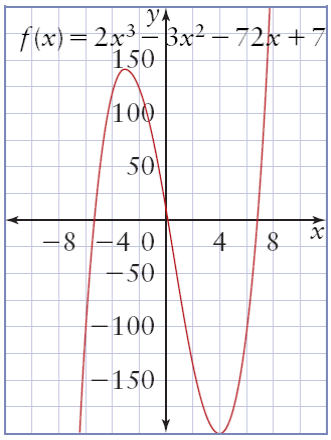
Test the intervals.

$$f''(0) = 12(0) - 6$$
$$= -6$$
$$f''(1) = 12(1) - 6$$
$$= 6$$

Therefore, (0.5, –29.5) is a point of inflection.

Therefore, f is increasing for $x < -3$ and $x > 4$ and decreasing for $-3 < x < 4$.
Also f is concave down for $x < 0.5$ and concave up for $x > 0.5$.

d)



e) This function has the maximum number of critical points and points of inflection predicted in parts a) and b).

Chapter 3 Section 5 Question 6 Page 192

i) $h(x) = 3x^2 - 27$

a) The first derivative of h will be of degree 1. Setting the derivative to zero will yield one critical value. Therefore, the function has one local extremum.

b) The second derivative of h will be of degree 0. There will be no points of inflection.

c) $h'(x) = 6x$
$h''(x) = 6$

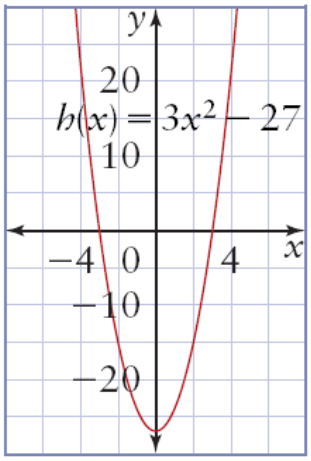
For critical values,
$6x = 0$
$x = 0$

Use the second derivative test to classify the critical points.
$h''(0) = 6$

Therefore, (0, –27) is a local minimum.
There are no points of inflection and the second derivative is always positive.

Therefore h is decreasing for $x < 0$ and increasing for $x > 0$.
Also h is always concave up.

d)



e) This function has the predicted number of critical points and points of inflection.

ii) $t(x) = x^5 - 2x^4 + 3$

a) The first derivative of t will be of degree 4. Setting the derivative to zero can yield at most four critical values. Therefore, the function has at most three local extrema.

b) The second derivative of t will be of degree 3. Setting this derivative equal to zero will yield at most three possible values for points of inflection. Therefore t can have at most three points of inflection.

c) $t'(x) = 5x^4 - 8x^3$

$t''(x) = 20x^3 - 24x^2$

For critical values,

$5x^4 - 8x^3 = 0$

$x^3(5x - 8) = 0$

$x = 0,\ x = 1.6$

Use the second derivative test to classify the critical points.

$$t''(0) = 20(0)^3 - 24(0)^2$$
$$= 0$$

Check further.

$$t''(-1) = 20(-1)^3 - 24(-1)^2$$
$$= -4$$
$$t''(1) = 20(1)^3 - 24(1)^2$$
$$= -4$$
$$t''(1.6) = 20(1.6)^3 - 24(1.6)^2$$
$$= -20.48$$

Therefore (0, 3) is a local maximum and (1.6, 0.37856) is a local minimum.

Find the possible points of inflection.

$$20x^3 - 24x^2 = 0$$
$$4x^2(5x - 6) = 0$$
$$x = 0,\ x = 1.2$$

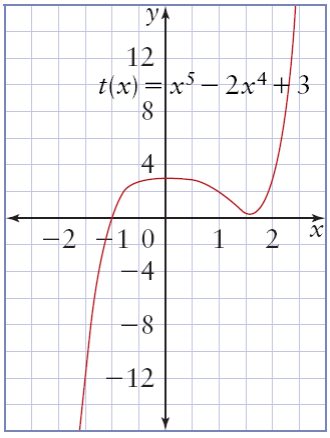
Test the intervals.

$$t''(-1) = 20(-1)^3 - 24(-1)^2$$
$$= -44$$
$$t''(1) = 20(1)^3 - 24(1)^2$$
$$= -4$$
$$t''(2) = 20(2)^3 - 24(2)^2$$
$$= 64$$

Therefore, (1.2, 1.3411) is the only point of inflection.

Therefore t is increasing for $x < 0$ and $x > 1.6$ and decreasing for $0 < x < 1.6$.
Also t is concave down for $x < 1.2$ and concave up for $x > 1.2$

d)



e) This function has less than the maximum number of predicted number of critical points and points of inflection.

iii) $g(x) = x^4 - 8x^2 + 16$

a) The first derivative of g will be of degree 3. Setting the derivative to zero can yield at most three critical values. Therefore, the function has at most 3 local extrema.

b) The second derivative of g will be of degree 2. Setting this derivative equal to zero will yield at most two possible values for points of inflection. Therefore g can have at most two points of inflection.

c) $g'(x) = 4x^3 - 16x$

$g''(x) = 12x^2 - 16$

For critical values,

$4x^3 - 16x = 0$

$4x(x^2 - 4) = 0$

$x = 0,\ x = 2,\ x - 2$

Use the second derivative test to classify the critical points.

$g''(-2) = 12(-2)^2 - 16$

$= 32$

$g''(0) = 12(0)^2 - 16$

$= -16$

$g''(2) = 12(2)^2 - 16$

$= 32$

Therefore, (–2, 0) and (2, 0) are local minimums and (0, 16) is a local maximum.

Find the possible points of inflection.

$12x^2 - 16 = 0$

$x^2 = \frac{4}{3}$

$x = \pm\frac{2}{\sqrt{3}}$

$x \approx -1.15,\ x \approx 1.15$

Test the intervals.

$g''(-2) = 32$

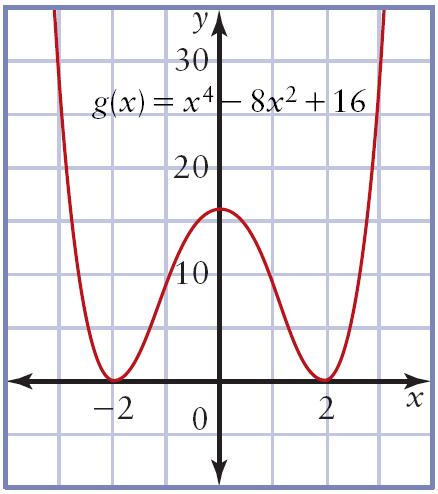
$g''(0) = -16$

$g''(2) = 32$

Therefore, (–1.15, 7.11) and (1.15, 7.11) are points of inflection.

Therefore, g is decreasing for $x < -2$ and $0 < x < 2$ and increasing for $-2 < x < 0$ and $x > 2$.
Also g is concave up for $x < -1.15$ and $x > 1.15$ and concave down for $-1.15 < x < 1.15$.

d)



e) This function has the predicted maximum number of critical points and points of inflection.

iv) $k(x) = -2x^4 + 16x^2 - 12$

a) The first derivative of k will be of degree 3. Setting the derivative to zero can yield at most three critical values. Therefore, the function has at most three local extrema.

b) The second derivative of k will be of degree 2. Setting this derivative equal to zero will yield at most two possible values for points of inflection. Therefore, k can have at most two points of inflection.

c) $k'(x) = -8x^3 + 32x$

$k''(x) = -24x^2 + 32$

For critical values,

$-8x^3 + 32x = 0$

$-8x(x^2 - 4) = 0$

$x = 0,\ x = 2,\ x = -2$

Use the second derivative test to classify the critical point.

$$k''(0) = -24(0)^2 + 32$$
$$= +32$$
$$k''(-2) = -24(-2)^2 + 32$$
$$= -64$$
$$k''(2) = -24(2)^2 + 32$$
$$= -64$$

Therefore (0, –12) is a local minimum and (–2, 20) and (2, 20) are local maximum points.

Find the possible points of inflection.

$$-24x^2 + 32 = 0$$
$$-8(3x^2 - 4) = 0$$

$$x = \sqrt{\frac{4}{3}}, x = -\sqrt{\frac{4}{3}}$$
$$x \approx 1.15, x \approx -1.15$$

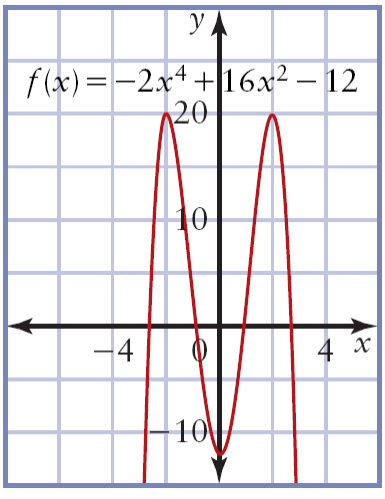
Test the intervals. (Note this was performed earlier.)

$$k''(-2) = -64$$
$$k''(0) = 32$$
$$k''(2) = -64$$

Therefore, (–1.15, 5.67) and (1.15, 5.67) are points of inflection.

Therefore, k is increasing for $x < -2$ and $0 < x < 2$ and decreasing for $-2 < x < 0$ and $x > 2$.
Also k is concave down for $x < -1.15$ and $x > 1.15$ and concave up for $-1.15 < x < 1.15$.

d)



e) This function has the predicted maximum number of critical points and points of inflection.

Chapter 3 Section 5 Question 7 Page 192

a) The maximum possible number of local extrema is one less than the degree of the polynomial function.

b) The number may be less that the maximum if one of the roots of the first derivative function is a 'double' root. In this case, the function is increasing (or decreasing) on both sides of the root and so there is no local extremum present.

Chapter 3 Section 5 Question 8 Page 192

Such a polynomial can have a maximum of 4 points of inflection. It is possible for it to have zero points of inflection. A simple example is $f(x) = x^6$.

Chapter 3 Section 5 Question 9 Page 192

a) Follow the six step plan.

$k(x) = 3x^3 + 7x^2 + 3x - 1$

$k'(x) = 9x^2 + 14x + 3$

$k''(x) = 18x + 14$

Step 1. Since this is a polynomial function, the domain is $\{x \in \mathbb{R}\}$.

Step 2. $k(0) = -1$

The y-intercept is -1.

For x-intercepts, let $y = 0$.

$3x^3 + 7x^2 + 3x - 1 = 0$ Use the factor theorem.

$(x+1)(3x^2 + 4x - 1) = 0$

$x = -1,\ x = \dfrac{-4 \pm \sqrt{28}}{6}$

$x = -1,\ x \approx 0.22,\ x \approx -1.55$

The x-intercepts are -1 and $\dfrac{-2 \pm \sqrt{7}}{3}$.

Step 3. Find the critical numbers.

$9x^2 + 14x + 3 = 0$

$x = \dfrac{-14 \pm \sqrt{88}}{18}$

$= \dfrac{-7 \pm \sqrt{22}}{9}$

$x \approx -1.3, x \approx -0.26$

Use the second derivative test to classify the critical points.

$k''(-1.3) = -9.4$

$k''(-0.26) = 9.32$

Therefore (–1.3, –0.34) is a local maximum and (–0.26, –1.36) is a local minimum.

Step 4. Find the possible points of inflection.

$18x + 14 = 0$

$$x = -\frac{7}{9}$$

x B –0.8

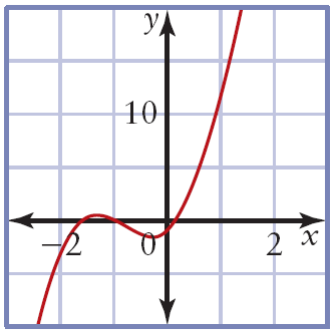
Test the intervals.

The intervals for $x = -\frac{7}{9}$ have already been tested.

Therefore (–0.8, –0.5) is a point of inflection.

Step 5. From Step 3, k is increasing for $x < -1.3$ and $x > -0.3$ and decreasing for $-1.3 < x < -0.3$.
From Step 4, k is concave down for $x < -0.8$ and concave up for $x > -0.8$.

Step 6. Sketch the graph.



b) Follow the six step plan.

$t(x) = 2x^3 - 12x^2 + 18x - 4$

$t'(x) = 6x^2 - 24x + 18$

$t''(x) = 12x - 24$

Step 1. Since t is a polynomial function, the domain is $\{x \mid x \in \mathbb{R}\}$.

Step 2. $t(0) = -4$

The y-intercept is –4.

For x-intercepts, let $y = 0$.

$2x^3 - 12x^2 + 18x - 4 = 0$ Use factor theorem.

$t(1) = 4$

$t(-1) = -36$

$t(2) = 0$ $(x - 2)$ is a factor.

$(x - 2)(2x^2 - 8x + 2) = 0$

$x = 2,\ x = \dfrac{8 \pm \sqrt{48}}{4}$

$x = 2,\ x = 2 \pm \sqrt{3}$

The x-intercepts are 2 and $2 \pm \sqrt{3}$.

Step 3. Find the critical numbers.

$6x^2 - 24x + 18 = 0$

$6(x^2 - 4x + 3) = 0$

$(x - 3)(x - 1) = 0$

$x = 1,\ x = 3$

Use the second derivative test to classify the critical points.

$t''(1) = -12$

$t''(3) = 12$

Therefore, (1, 4) is a local maximum and (3, –4) is a local minimum.

Step 4. Find the possible points of inflection.

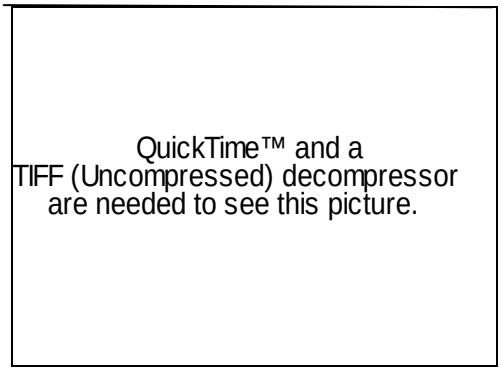
$12x - 24 = 0$

$x = 2$

Test the intervals. The intervals for $x = 2$ have been tested already.
Therefore, (2, 0) is a point of inflection.

Step 5. From Step 3, t is increasing for $x < 1$ and $x > 3$ and decreasing for $1 < x < 3$.
From Step 4, t is concave down for $x < 2$ and concave up for $x > 2$.

Step 6. Sketch the graph.



($x \in [-4, 10], y \in [-10, 10]$)

c) Follow the six step plan.

$f(x) = 2x^4 - 26x^2 + 72$

$f'(x) = 8x^3 - 52x$

$f''(x) = 24x^2 - 52$

Step 1. Since this is a polynomial function, the domain is $\{x \mid x \in \mathbb{R}\}$.

Step 2. $f(0) = 72$

The y-intercept is 72.

For x-intercepts, let $y = 0$.

$$2x^4 - 26x^2 + 72 = 0$$
$$2(x^4 - 13x^2 + 36) = 0$$
$$2(x^2 - 9)(x^2 - 4) = 0$$
$$2(x+3)(x-3)(x+2)(x-2) = 0$$

$x = 2, x = -2, x = 3, x = -3$

The x-intercepts are ± 2 and ± 3.

Step 3. Find the critical numbers.

$$8x^3 - 52x = 0$$
$$4x(2x^2 - 13) = 0$$

$x = 0,\ x = \sqrt{\frac{13}{2}}, x = -\sqrt{\frac{13}{2}}$

$x = 0, x \approx 2.55, x \approx -2.55$

Use the second derivative test to classify the critical points.

$f''(-2.55) = 104$

$f''(2.55) = 104$

$f''(0) = -52$

Therefore, (0, 72) is a local maximum and (–2.55, –12.5) and (2.55, –12.5) are local minima.

Step 4. Find the possible points of inflection.

$24x^2 - 52 = 0$

$x^2 = \frac{13}{6}$

$x = \sqrt{\frac{13}{6}}, x = -\sqrt{\frac{13}{6}}$

$x \approx 1.5, x \approx -1.5$

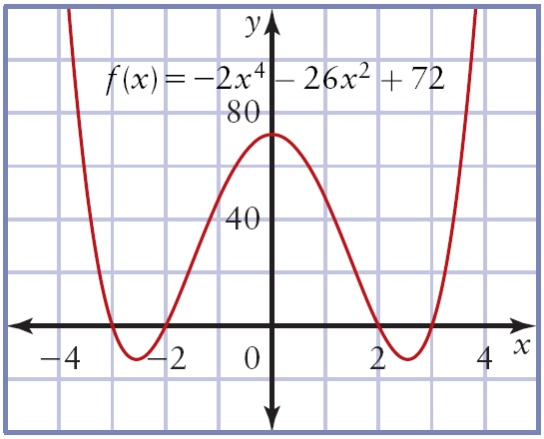
Test the intervals. The intervals for these values have already been tested.

Therefore, (–1.5, 25.2) and (1.5, 25.2) are points of inflection.

Step 5. From Step 3, f is increasing for $-2.55 < x < 0$ and $x > 2.55$ and decreasing for $x < -2.55$ and $0 < x < 2.55$.

From ④, we know f is concave down for $-1.5 < x < 1.5$ and concave up for $x < 1.5$ and $x > 1.5$.

Step 6. Sketch the graph.



d) Follow the six step plan.

$h(x) = 5x^3 - 3x^5$

$h'(x) = 15x^2 - 15x^4$

$h''(x) = 30x - 60x^3$

Step 1. Since this is a polynomial function, the domain is $\{x \in \mathbb{R}\}$.

Step 2. $h(0) = 0$

The y-intercept is 0.

For x-intercepts, let $y = 0$.

$$5x^3 - 3x^5 = 0$$
$$x^3(5 - 3x^2) = 0$$

$$x = 0, x = \sqrt{\frac{5}{3}}, x = -\sqrt{\frac{5}{3}}$$
$$x = 0, x \approx 1.3, x \approx -1.3$$

The x-intercepts are $\pm\sqrt{\frac{5}{3}}$ and 0.

Step 3. Find the critical numbers.

$$15x^2 - 15x^4 = 0$$
$$15x^2(1 - x^2) = 0$$

$$x = 0,\ x = 1,\ x = -1$$

Use the second derivative test to classify the critical points.

$h''(-1) = 30$

$h''(1) = -30$

$h''(0) = 0$ Check on either side of $x = 0$.

$h''(-0.5) = -7.5$

$h''(0.5) = 7.5$

Therefore, (–1, –2) is a local minimum and (1, 2) is a local maximum. (0, 0) is an inflection point.

Step 4. Find the possible points of inflection.

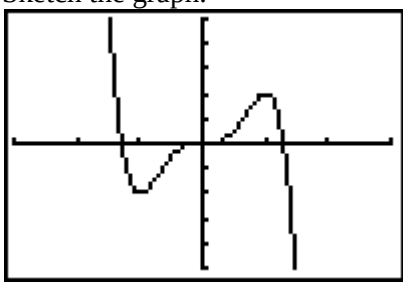
$$30x - 60x^3 = 0$$
$$30x(1 - 2x^2) = 0$$

$$x = 0,\ x = \sqrt{0.5}, x = -\sqrt{0.5},$$
$$x = 0,\ x \approx 0.7, x \approx -0.7$$

Test the intervals. The intervals for these values have already been tested.
Therefore, (0, 0), (–0.7, –1.2) and (0.7, 1.2) are points of inflection.

Step 5. From Step 3, h is increasing for $-1 < x < 0$ and $0 < x < 1$ and decreasing for $x < -1$ and $x > 1$.
From Step 4, h is concave down for $-0.7 < x < 0$ and $x > 0.7$ and concave up for $x < -0.7$ and $0 < x < 1.7$.

Step 6. Sketch the graph.



e) Follow the six step plan.

$g(x) = 3x^4 + 2x^3 - 15x^2 + 12x - 2$

$g'(x) = 12x^3 + 6x^2 - 30x + 12$

$g''(x) = 36x^2 + 12x - 30$

Step 1. Since this is a polynomial function, the domain is $\{x \in \mathbb{R}\}$.

Step 2. $g(0) = -2$

The y-intercept is –2.

For x-intercepts, let $y = 0$.

$3x^4 + 2x^3 - 15x^2 + 12x - 2 = 0$

$f(1) = 0$ $(x - 1)$ is a factor.

$f(-1) = -28$

$f(0.5) = 0.6875$

$f(0.25) \approx 0.1055$

$f(0.23) \approx -0.0007$

$3x^4 + 2x^3 - 15x^2 + 12x - 2$

$= (x + 1)(3x^3 - x^2 - 14x + 26) - 28$

The second factor is not easily factorable.

The x-intercept is 1.

Step 3. Find the critical numbers.

$12x^3 + 6x^2 - 30x + 12 = 0$

$6(2x^3 + x^2 - 5x + 2) = 0$ Use the factor theorem.

$f(1) = 0$ $(x - 1)$ is a factor.

$6(x-1)(2x^2 + 3x - 2) = 0$

$6(x-1)(2x-1)(x+2) = 0$

$x = 1,\ x = -2,\ x = \frac{1}{2}$

Use the second derivative test to classify the critical points.

$g''(1) = 18$

$g''(-2) = 90$

$g''(0.5) = -15$

Therefore, (–2, –54) and (1, 0) are local minimum points. (0.5, 0.7) is a local maximum.

Step 4. Find the possible points of inflection.

$36x^2 + 12x - 30 = 0$

$6(6x^2 + 2x - 5) = 0$

$x = \frac{-2 \pm \sqrt{124}}{12}$

$= \frac{-1 \pm \sqrt{31}}{6}$

$x \approx -1.1,\ x \approx 0.8$

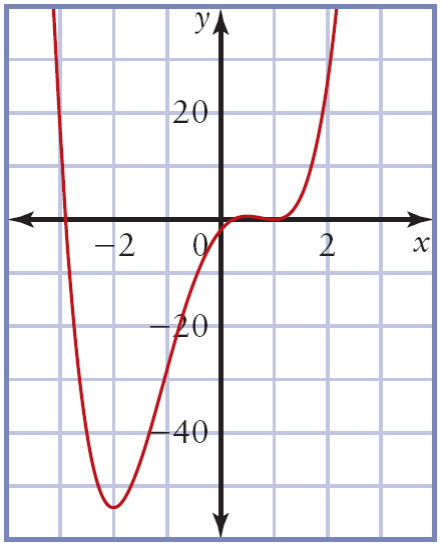
Test the intervals. The intervals for these values have already been tested.

Therefore, (–1.1, –32) and (0.8, 0.3) are points of inflection.

Step 5. From Step 3, g is increasing for $-2 < x < 0.5$ and $x > 1$ and decreasing for $x < -2$ and $0.5 < x < 1$.

From Step 4, g is concave down for $-1.1 < x < 0.8$ and concave up for $x < -1.1$ and $x > 0.8$.

Step 6. Sketch the graph.



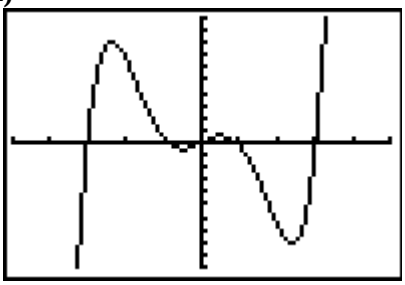
Chapter 3 Section 5 Question 10 Page 193

This is false. For example, constant functions have no turning points. Even polynomial functions of higher degrees may not have turning points. The function $f(x) = x^3$ has a point of inflection at the origin, but no turning points.

Chapter 3 Section 5 Question 11 Page 193

a) $f(x) = 2x^5 - 20x^3 + 15x$

i)



($x \in [-5, 5]$, $y \in [-100, 100]$, $\text{Yscl} = 10$ **)**

ii) Points of inflection appear to be at the origin and also when $x = -1.5$ or $x = 1.5$
Thus the function appears to be concave down for $x < -1.5$ and $0 < x < 1.5$. The curve appears to be concave up for $-1.5 < x < 0$ and $x > 1.5$.

iii) $f'(x) = 10x^4 - 60x^2 + 15$

$f''(x) = 40x^3 - 120x$

For inflection points, let $f''(x) = 0$

$40x^3 - 120x = 0$

$40x(x^2 - 3) = 0$

$x = 0,\ x = \sqrt{3}, x = -\sqrt{3}$

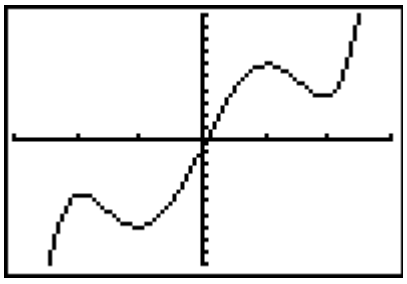
Therefore, the actual inflection points are (0, 0), $\left(-\sqrt{3},\ 46.77\right)$ and $\left(\sqrt{3},\ -46.77\right)$.

Also, the function is concave down for $x < -\sqrt{3}$ and $0 < x < \sqrt{3}$ and concave up for $-\sqrt{3} < x < 0$ and $x > \sqrt{3}$.

Thus the function is concave down for $x < -1.5$ and $0 < x < 1.5$. The curve is concave up for $1.5 < x < 0$ and $x > 1.5$.

b) $g(x) = x^5 - 8x^3 + 20x - 1$

i)



$(x \in [-3, 3],\ y \in [-29, 20],\ \text{Yscl} = 2)$

ii) Points of inflection appear to be at the origin and also when $x = -1.5$ or $x = 1.5$.
Thus the function appears to be concave down for $x < -1.5$ and $0 < x < 1.5$. The curve appears to be concave up for $-1.5 < x < 0$ and $x > 1.5$.

iii) $f'(x) = 5x^4 - 24x^2 + 20$

$f''(x) = 20x^3 - 48x$

For inflection points, let $f''(x) = 0$.

$20x^3 - 48x = 0$

$4x(5x^2 - 12) = 0$

$x = 0,\ x = \sqrt{2.4}, x = -\sqrt{2.4}$

Therefore, the actual inflection points are (0, 0), $\left(-\sqrt{2.4},\ 7.66\right)$ and $\left(\sqrt{2.4},\ 2.94\right)$.

Also, the function is concave down for $x < -\sqrt{2.4}$ and $0 < x < \sqrt{2.4}$ and concave up for $-\sqrt{2.4} < x < 0$ and $x > \sqrt{2.4}$.

Chapter 3 Section 5 Question 12 Page 193

Many different functions are possible.
Since $x = 2$ is a critical value, $(x - 2)$ must be a factor of the first derivative.
Since there is a point of inflection when $x = 0$, x must be a factor of $f''(x)$.

Try $f'(x) = (x - 2)$. This does not work since $f''(x) = 1$ (which does not have x as a factor).

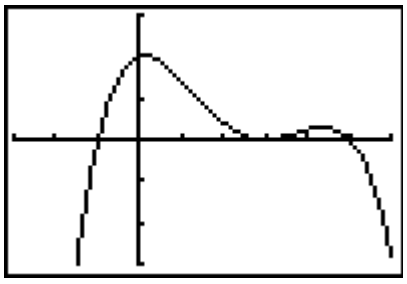
Try $f'(x) = (x - 2)(x + 2)$
$= x^2 - 4$
This works since $f''(x) = 2x$ (which has x as a factor).

A possible function is $f(x) = \frac{1}{3}x^3 - 4x + k$ which has the necessary first and second derivatives.
(Note that k can be any constant value.)

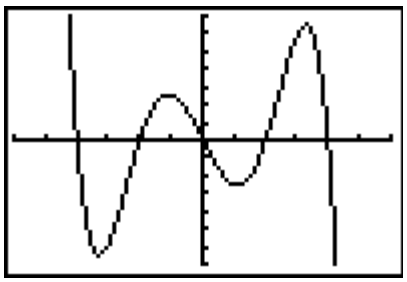
Chapter 3 Section 5 Question 13 Page 193

a) The function must be increasing to x-intercept –1, continuing to increase through the y-intercept of 2 until it reaches a local maximum at $x = 2$. It then decreases to the point (3, 0) which is a local minimum. Then it increases to a local maximum at $x = 4$. Finally it descends through the x-intercept 5 and continues descending thereafter.

A possible equation is $f(x) = -\frac{2}{45}(x + 1)(x - 3)^2(x - 5)$.



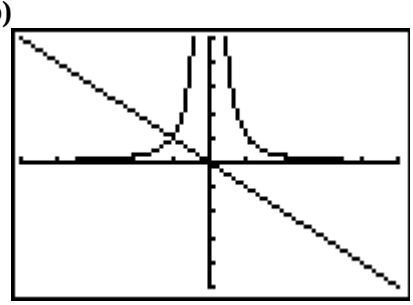
b) The 5 intercepts are –4, –2, 2, 4, and 0. The missing ones are added because the function is odd and must be symmetric about the origin. The function starts in the second quadrant, descends through the x-intercept –4, increases through the (–1, 5) and then descends to the origin. The right half of the graph is a reflection in the origin of this path.
The function is of the form $g(x) = kx(x^2 - 4)(x^2 - 16)$.
The local point (–1, 5) determines the scale factor k.
A possible equation is $g(x) = -\frac{1}{9}x(x^2 - 4)(x^2 - 16)$.


$(x \in [-6, 6],\ y \in [-14, 14],\ \text{Yscl} = 2)$

Chapter 3 Section 5 Question 14 Page 193

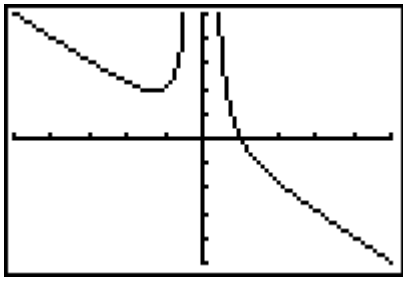
a) Two possible functions are $g(x) = -x$ and $h(x) = \frac{1}{x^2}$.

b)



c) The graph will look more like $h(x)$ in the centre of the graph but more like $g(x)$ farther from the origin. The sum will have a vertical asymptote since $h(x)$ has one. The horizontal asymptote will be 'bent' by the influence of $g(x)$. In fact, the sum curve will get closer and closer to $g(x)$ as one gets farther from the origin ($g(x)$ will become an asymptote for the sum curve).

d)



e) $f(x) = -x + x^{-2}$

$f'(x) = -1 - 2x^{-3}$

$= -1 - \frac{2}{x^3}$

For a critical point,

$$-1-\frac{2}{x^3}=0$$
$$x^3-2=0$$
$$x=\sqrt[3]{2}$$

The turning point is $\left(-\sqrt[3]{2}, -\sqrt[3]{2}-\frac{1}{\sqrt[3]{4}}\right)$

f) $f''(x)=6x^{-4}$

$$=\frac{6}{x^4}$$

There are no inflection points ($x=0$ is not in the domain of f).

The vertical asymptote may divide areas of different concavity.
Test the intervals.

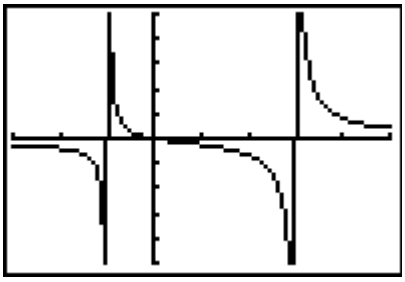
$f''(-1)=6$

$f''(1)=6$

The curve is concave up for $x<0$ and $x>0$.

Chapter 3 Section 5 Question 15 Page 193

a)



b) Yes. The function can be stretched vertically by any positive factor.

One possible equation is $y=\frac{x}{(x+1)(x-3)}$.

c) It is reasonable to assume that there could be a minimum in the concave up interval or a maximum in the concave down interval. However, finding such an equation is not simple. It likely involves function types not studied in this course.

Chapter 3 Section 5 Question 16 Page 193

a) $\lim\limits_{x\to\infty} g(x)=0$ and $\lim\limits_{x\to-\infty} g(x)=0$

This means that the y-axis is a horizontal asymptote for this function. This result can be verified by testing large (positive and negative) values of x.

$$g(100) = \frac{1}{(100)^2 - 1}$$

$$\approx 0.0001$$

$$g(-100) = \frac{1}{(-100)^2 - 1}$$

$$\approx 0.0001$$

b) The vertical asymptotes are $x = -1$ and $x = 1$.

c) Check $x = 1$.

$$g(1.01) = \frac{1}{(1.01)^2 - 1}$$

$$\approx 50$$

$$g(1.001) = \frac{1}{(1.001)^2 - 1}$$

$$\approx 500$$

$$g(0.99) = \frac{1}{(0.99)^2 - 1}$$

$$\approx -50$$

$$g(0.999) = \frac{1}{(0.999)^2 - 1}$$

$$\approx -500$$

Therefore, $\lim\limits_{x \to 1^-} g(x) = -\infty$ and $\lim\limits_{x \to 1^+} g(x) = +\infty$.

d) The function is an even function. It is symmetric about the *y*-axis. This allows us to determine the limits for the other asymptote directly.

$\lim\limits_{x \to -1^-} g(x) = -\infty$ and $\lim\limits_{x \to -1^+} g(x) = +\infty$.

e) $g(x) = (x^2 - 1)^{-1}$

$$g'(x) = -2x(x^2 - 1)^{-2}$$

$$= \frac{-2x}{(x^2 - 1)}$$

$$g''(x) = -2(x^2 - 1)^{-2} + (-2x)(-2)(x^2 - 1)^{-3}(2x)$$

$$= (x^2 - 1)^{-3}(-2x^2 + 2 + 8x^2)$$

$$= \frac{6x^2 + 2}{(x^2 - 1)^3}$$

For critical points,

$$\frac{-2x}{(x^2-1)} = 0$$
$$x = 0$$

Check concavity with the second derivative.

$$g''(0) = \frac{6(0)^2 + 2}{((0)^2-1)^3}$$
$$= -2$$

The point (0, –1) is a local maximum.

Chapter 3 Section 5 Question 17 Page 194

a) Let $x = 0$.

$f(0) = -1$.

The y-intercept is –1.

Let $y = 0$.

$$\frac{x-1}{x+1} = 0$$
$$x - 1 = 0$$
$$x = 1$$

The x-intercept is 1.

b) The vertical asymptote is found by letting the denominator equal zero. The asymptote is $x = -1$.

A horizontal asymptote can be found by examining limits at infinity.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x-1}{x+1}$$ Limit is indeterminate: $\frac{\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{1-\frac{1}{x}}{1+\frac{1}{x}}$$ Divide by x in the numerator and denominator.

$$= 1$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x-1}{x+1}$$ Limit is indeterminate: $\frac{-\infty}{-\infty}$

$$= \lim_{x\to-\infty} \frac{1-\frac{1}{x}}{1+\frac{1}{x}}$$ Divide by x in the numerator and denominator.

$$= 1$$

There is a horizontal asymptote at $y = 1$.

c) $f(x) = (x-1)(x+1)^{-1}$

$f'(x) = 1(x+1)^{-1} + (x-1)(-1)(x+1)^{-2}(1)$

$= (x+1)^{-2}(x+1-x+1)$

$= \dfrac{2}{(x+1)^2}$

There are no critical values.

However, you need to test the intervals on either side of the asymptote.

$f'(-2) = 2$

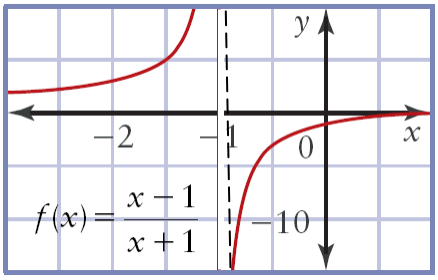
$f'(1) = \dfrac{1}{2}$

The function is increasing for $x < -1$ and $x > -1$.

d) Since the function is increasing as it approaches the asymptote from the left, the $\lim\limits_{x \to -1^-} f(x) = \infty$.

Since the function is decreasing as it approaches the asymptote from the right, the $\lim\limits_{x \to -1^+} f(x) = -\infty$.

e)



Chapter 3 Section 5 **Question 18 Page 194**

a) $g(x) = x^3 - 27x$

$g'(x) = 3x^2 - 27$

$g''(x) = 6x$

i) This is a polynomial function whose 'end' behaviour is determined by the sign of the coefficient of the term of highest degree. In this case, the coefficient is +1.

Therefore, $\lim\limits_{x \to \infty} g(x) = \infty$ and $\lim\limits_{x \to -\infty} g(x) = -\infty$.

ii) Examine the following.

$g(-x) = (-x)^3 - 27(-x)$

$= -x^3 + 27x$

$= -g(x)$

The function is an odd function and is symmetric about the origin.

iii) Find critical values.

$3x^2 - 27 = 0$

$x^2 = 9$

$x = 3, x = \check{G}B$

Test using the second derivative test.

$g''(-3) = 6(-3)$

$= -18$

$g''(3) = 6(3)$

$= 18$

Point (–3, 54) is a local maximum and (3, –54) is a local minimum.

iv) Find possible points of inflection.

$6x = 0$

$x = 0$

	$x<0$	$x=0$	$x>0$
$g''(x)$	Negative	0	Positive
$g(x)$	Concave down	Point of inflection	Concave up

There is a point of inflection at (0, 0).

b) $y = f(x)$

$= x^4 - 8x^2 + 16$

$f'(x) = 4x^3 - 16x$

$f''(x) = 12x^2 - 16$

i) This is a polynomial function whose 'end' behaviour is determined by the sign of the coefficient of the term of highest degree. In this case, the coefficient is +1.

Therefore, $\lim_{x \to \infty} f(x) = \infty$ and $\lim_{x \to -\infty} f(x) = -\infty$.

ii) Examine the following.

$f(-x) = (-x)^4 - 8(-x)^2 + 16$

$= x^4 - 8x^2 + 16$

$= f(x)$

The function is an even function as so it is symmetric about the *y*-axis.

iii) Find critical values.

$4x^3 - 16x = 0$

$4x(x^2 - 4) = 0$

$x = 0, x = 2, x = \check{G}2$

Test using the second derivative test.

$$f''(-2) = 12(-2)^2 - 16$$
$$= 32$$
$$f''(2) = 12(2)^2 - 16$$
$$= 32$$
$$f''(0) = 12(0)^2 - 16$$
$$= -16$$

Point (0, 16) is a local maximum and points (–2, 0) and (2, 0) are local minimums.

iv) Find possible points of inflection.

$$12x^2 - 16 = 0$$
$$x^2 = \frac{4}{3}$$
$$x = \frac{2}{\sqrt{3}}, x = -\frac{2}{\sqrt{3}}$$
$$x \approx 1.15, x \approx -1.15$$

	$x < \sqrt{\frac{4}{3}}$	$x < -\sqrt{\frac{4}{3}}$	$-\sqrt{\frac{4}{3}} < x < +\sqrt{\frac{4}{3}}$	$x = +\sqrt{\frac{4}{3}}$	$x > +\sqrt{\frac{4}{3}}$
y''	Positive	0	Negative	0	Positive
y	Concave up	Point of inflection	Concave down	Point of inflection	Concave up

There are points of inflection at $\left(-\frac{2}{\sqrt{3}}, \frac{64}{9}\right)$ and $\left(\frac{2}{\sqrt{3}}, \frac{64}{9}\right)$.

c) $k(x) = \frac{1}{1-x^2}$
$$= (1-x^2)^{-1}$$
$$k'(x) = -(1-x^2)^{-2}(-2x)$$
$$= \frac{2x}{(1-x^2)^2}$$
$$k''(x) = 2(1-x^2)^{-2} + (2x)(-2)(1-x^2)^{-3}(-2x)$$
$$= (1-x^2)^{-3}(2 - 2x^2 + 8x^2)$$
$$= \frac{6x^2 + 2}{(1-x^2)^3}$$

i) Take limits.

$\lim_{x\to\infty} k(x) = \lim_{x\to\infty} \frac{1}{1-x^2} = 0$ 1 divided by a large positive number.

$\lim_{x\to-\infty} k(x) = \lim_{x\to-\infty} \frac{1}{1-x^2} = 0$ 1 divided by a large positive number.

ii) Examine $k(-x) = \frac{1}{1-(-x)^2} = \frac{1}{1-x^2} = k(x)$.

The function is an even function as so it is symmetric about the *y*-axis.

iii) Find critical values.

$$\frac{2x}{(1-x^2)^2} = 0$$

$$x = 0$$

Test using the second derivative test.

$$k''(0) = \frac{6(0)^2 + 2}{(1-(0)^2)^3}$$

$$= 2$$

Point (0, 1) is a local minimum.

iv) Find possible points of inflection.

$$\frac{6x^2 + 2}{(1-x^2)^3} = 0$$

$$6x^2 + 2 = 0$$

There are no real roots to this equation. There are no points of inflection.
Concavity can only change at the vertical asymptotes ($x = \neq 1$) that are present in this case.

	$x<-1$	$x=-1$	$-1<x<1$	$x=1$	$x>1$
$k''(x)$	Negative	Undefined	Positive	Undefined	Negative
$k(x)$	Concave down	Vertical asymptote	Concave up	Vertical asymptote	Concave down

d) $f(x) = \dfrac{x}{x^2+1}$

$= x(x^2+1)^{-1}$

$f'(x) = 1(x^2+1)^{-1} + x(-1)(x^2+1)^{-2}(2x)$

$= (x^2+1)^{-2}(x^2+1-2x^2)$

$= \dfrac{-x^2+1}{(x^2+1)^2}$

$f''(x) = -2x(x^2+1)^{-2} + (-x^2+1)(-2)(x^2+1)^{-3}(2x)$

$= (x^2+1)^{-3}(-2x^3-2x+4x^3-4x)$

$= \dfrac{-6x+2x^3}{(x^2+1)^3}$

i) Take limits.

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x}{x^2+1}$

$= \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}}{x+\frac{1}{x}}$ Divide by x^2/x^2.

$= 0$

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{x}{x^2+1}$

$= \lim\limits_{x\to-\infty} \dfrac{\frac{1}{x}}{x+\frac{1}{x}}$ Divide by x^2/x^2.

$= 0$

ii) Examine the following.

$f(-x) = \dfrac{(-x)}{(-x)^2+1}$

$= \dfrac{-x}{x^2+1}$.

$= -f(x)$

The function is an odd function so it is symmetric about the origin.

iii) Find critical values.

$\dfrac{-x^2+1}{(x^2+1)^2} = 0$

$x^2 = 1$

$x = 1, x = -1$

Test using the second derivative test.

$$f''(-1) = \frac{-6(-1) + 2(-1)^3}{((-1)^2 + 1)^3}$$

$$= \frac{4}{8}$$

$$f''(1)^2 = \frac{-6(1) + 2(1)^3}{((1)^2 + 1)^3}$$

$$= \frac{-4}{8}$$

Point $\left(-1, -\frac{1}{2}\right)$ is a local minimum and $\left(1, \frac{1}{2}\right)$ is a local maximum.

iv) Find possible points of inflection.

$$\frac{-6x + 2x^3}{(x^2 + 1)^3} = 0$$

$$-6x + 2x^3 = 0$$

$$2x(x^2 - 3) = 0$$

$x = 0,\ x = \sqrt{3}, x = \check{G}\sqrt{3}$

There are three possible points of inflection.

<<There are 6 instances of a square root sign where the plus/minus sign is misplaced in front of it. $\pm\sqrt{3}$ is correct placement.>><<pick up corrected artwork from answers>>

	$x < -\sqrt{3}$	$x = -\sqrt{3}$	$-\sqrt{3} < x < 0$	$x = 0$
$f(x)''$	Negative	0	Positive	0
$f(x)$	Concave down	Point of inflection	Concave up	Point of inflection

	$0 < x < +\sqrt{3}$	$x = +\sqrt{3}$	$x > +\sqrt{3}$
$f(x)''$	Negative	0	Positive
$f(x)$	Concave down	Point of inflection	Concave up

The points of inflection are $\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right), \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$, and $(0, 0)$.

e) $h(x) = \dfrac{x-4}{x^2} = x^{-2}(x-4)$

$h'(x) = x^{-2}(1) + (-2)x^{-3}(x-4)$

$= x^{-3}(x - 2x + 8)$

$= \dfrac{8-x}{x^3}$

$h'(x) = x^{-3}(-1) - 3x^{-4}(8-x)$

$= x^{-4}(-x + 3x - 24)$

$= \dfrac{2x-24}{x^4}$

i) Take limits.

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{x-4}{x^2}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{4}{x^2}}{1}$$ Divide by x^2/x^2.

$$= 0$$

$$\lim_{x\to-\infty} h(x) = \lim_{x\to-\infty} \frac{x-4}{x^2}$$

$$= \lim_{x\to-\infty} \frac{\frac{1}{x} - \frac{4}{x^2}}{1}$$ Divide by x^2/x^2.

$$= 0$$

ii) Examine the following.

$$h(-x) = \frac{(-x)-4}{(-x)^2}$$

$$= \frac{-x-4}{x^2}$$

The function is neither odd nor even. There is no resulting symmetry.

iii) Find critical values.

$$\frac{8-x}{x^3} = 0$$

$$8 - x = 0$$

$$x = 8$$

Test using the second derivative test.

$$h''(8) = \frac{2(8)-24}{(8)^4}$$

$$= -0.002$$

Point $\left(8, \frac{1}{16}\right)$ is a maximum

iv) Find possible points of inflection.

$$\frac{2x-24}{x^4}=0$$
$$2x-24=0$$
$$x=12$$

There is 1 possible point of inflection.

	$x<0$	$x=0$	$0<x<12$	$x=12$	$x>12$
$h''(x)$	Negative	Undefined	Negative	0	Positive
$h(x)$	Concave Down	Vertical Asymptote	Concave Down	Point of Inflection	Concave Up

The point of inflection is $\left(12, \frac{1}{18}\right)$.

Chapter 3 Section 5 Question 19 Page 194

a) There is no x-intercept since the fraction can never equal zero. The y-intercept is 1 since

$$g(0)=\frac{1}{0^2+1}$$
$$=1$$

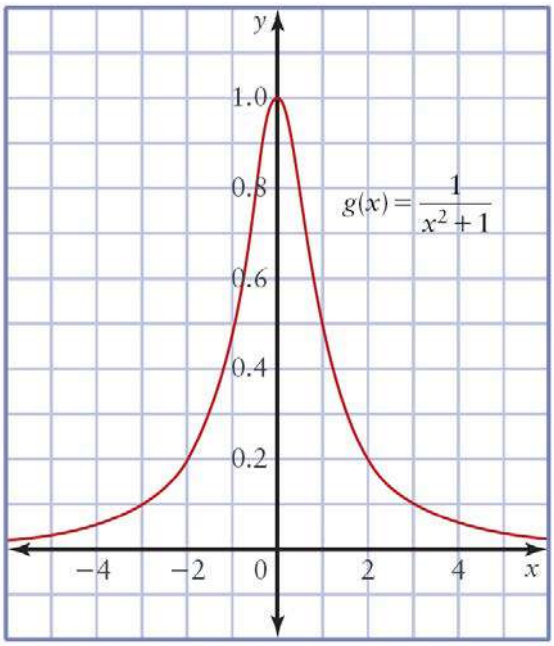
b) The maximum value for $g(x)$ is 1. The function $g(x)$ is a fraction whose numerator is fixed as 1. For a maximum value, the denominator must be as small as possible. Since x^2 is always greater than or equal to zero, the smallest the denominator can be is 0 + 1 = 1. Therefore, the maximum value for the fraction is 1.

c) For large (positive or negative) values of x, $g(x)$ becomes 1 over a very large number, which gets closer and closer to the value zero. The horizontal asymptote is the y-axis or $y=0$.

d) No. The function has a positive y-intercept that is its maximum. As x gets large, the function decreases and approaches the x-axis from above. It is very unlikely that there would be any extrema in this situation. (However, it might be possible to imagine a damped sinusoidal curve that could satisfy the conditions of parts a) to c) of this question, but still have multiple local maxima and minima as the function approaches the x-axis.)
Clearly, using the tools of calculus (e.g., derivatives) would verify this conjecture.

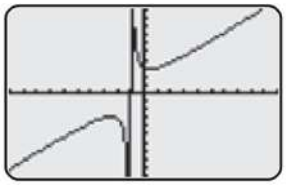
e) Two. The function has one maximum where the function must be concave down. At the limit, the function must be concave up (or it will intersect the axis). There must be one point of inflection between the maximum and the end of the x-axis in either direction.

f)



Chapter 3 Section 5 Question 20 Page 194

a)



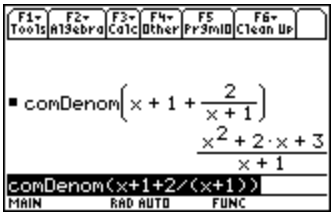
b) $\frac{x^2+2x+3}{x+1} = \frac{x^2+2x+1+2}{x+1}$

$$= \frac{x^2+2x+1}{x+1} + \frac{2}{x+1}$$

$$= \frac{(x+1)^2}{x+1} + \frac{2}{x+1}$$

$$= x+1+\frac{2}{x+1}$$

Another method is to use a CAS.



c) The two functions are $s(x) = x + 1$ and $t(x) = \frac{2}{x+1}$.

The function *t* has a vertical asymptote; therefore *f* will also have the same vertical asymptote.
Function *t* approaches the value zero as *x* increases; therefore *f* will approach the curve *s* as *x* increases.
That is, *s* will be a slant asymptote for *f*.

Chapter 3 Section 5 Question 21 Page 194

C is the correct answer.

$y = (x+2)^5(x^2-1)^4$

$\frac{dy}{dx} = (x+2)^4(x^2-1)^3(13x^2+16x-5)$

When $\frac{dy}{dx} = 0$, $x = -2$ or ± 1, or two other real roots from the quadratic equation $13x^2 + 16x - 5 = 0$.

With the exception of -2, $\frac{dy}{dx}$ will have a sign change when passing through each of these roots. Hence, there are four local maximum or minimum points.

Chapter 3 Section 5 Question 22 Page 194

D is the correct answer.

The functions $y = \frac{1}{x^2}$, $y = \frac{1}{x}$, $y = \frac{x^2}{x}$, and $y = \frac{1}{\sqrt{x}}$ are discontinuous at $x = 0$.
Hence, their derivatives are also discontinuous at $x = 0$.
However, $x = 0$ is not in the domains of these functions.
The function $y = \sqrt{x}$ has derivative $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$, which is discontinuous at $x = 0$.
In this case, $x = 0$ is in the domain of the function.

Chapter 3 Section 6 **Optimization Problems**

Chapter 3 Section 6 **Question 1 Page 201**

For a maximum, find the first derivative and let it equal zero to find the critical points.

$$h(t) = -4.9t^2 + 19.6t + 2$$
$$h'(t) = -9.8t + 19.6$$
$$0 = -9.8t + 19.6$$
$$t = 2$$

Since this function is a parabola opening down, this critical point is a maximum.

$$h(2) = -4.9(2)^2 + 19.6(2) + 2$$
$$= 21.6$$

The maximum height of the ball is 21.6 m.

Chapter 3 Section 6 **Question 2 Page 201**

Let the integers be represented by x and $20 - x$ and P be their product (the quantity to be maximized).

$$P(x) = x(20 - x)$$
$$= 20x - x^2$$
$$P'(x) = 20 - 2x$$

Find the critical points.

$$20 - 2x = 0$$
$$x = 10$$

Since this function is a parabola opening down, this critical point is a maximum.
The two integers are 10 and (20 – 10) = 10.

Chapter 3 Section 6 **Question 3 Page 201**

Find the critical points.

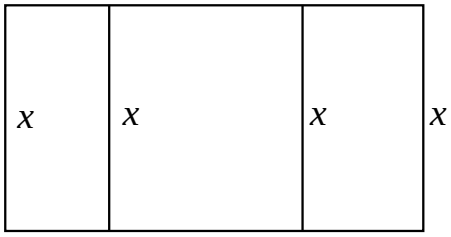
$$N(t) = -0.05t^2 + 3t + 5$$
$$N'(t) = -0.1t + 3$$
$$0 = -0.1t + 3$$
$$t = 30$$

Since this function is a parabola opening down, this critical point is a maximum.
It takes 30 years of experience to reach maximum productivity.

Chapter 3 Section 6 **Question 4 Page 201**

a) Let the length of each small pen be x metres. This uses $4x$ of fencing.
$1200 - 4x$ of fencing remains for the two long sides.
Each long side has length:
$\frac{1200 - 4x}{2} = 600 - 2x$.

Let A be the area enclosed.



$$A(x) = x(600 - 2x)$$
$$= 600x - 2x^2$$

Find the critical points.
$$A'(x) = 600 - 4x$$
$$0 = 600 - 4x$$
$$x = 150$$

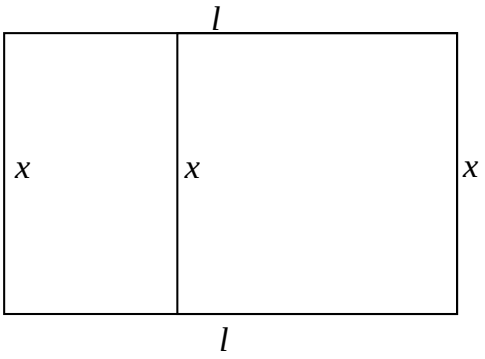
Since this function is a parabola opening down, this critical point is a maximum.
$$A(150) = 600(150) - 2(150)^2$$
$$= 45\ 000$$

The maximum area that can be enclosed is 45 000 m^2.

b) This condition would limit the function to the interval: $180 < x < 210$. (The right hand limit occurs because the long sides of the big pen must also be at least 180 m long and that would leave only 840 m for the four shorter sides.)
Consequently, there would be no critical point in the interval. A maximum value would have to occur at an endpoint. Since $A(180) = 43\ 200$ and $A(210) = 37\ 800$, the maximum possible area would be 43 200 m^2 in this case.

Chapter 3 Section 6 **Question 5 Page 201**

Let the width of the square be x metres.
Let l be the length of the combined pen.



$$3x + 2l = 60$$
$$l = \frac{60 - 3x}{2}$$

Let A be the overall area.

$$A(x) = x\left(\frac{60 - 3x}{2}\right)$$
$$= 30x - \frac{3}{2}x^2$$

Find the critical points.

$$A'(x) = 30 - 3x$$
$$0 = 30 - 3x$$
$$x = 10$$

Since this function is a parabola opening down, this critical point is a maximum.

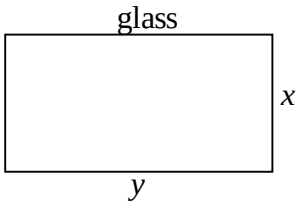
$$A(10) = 30(10) - \frac{3}{2}(10)^2$$
$$= 150$$

$$l = \frac{60 - 3(10)}{2}$$
$$= 15$$

The maximum area possible is 150 m^2 and the dimensions that give this area are 10 m by 15 m.

Chapter 3 Section 6 **Question 6 Page 201**

a) Let the dimensions of the showroom be x and y.



$$xy = 500$$
$$y = \frac{500}{x}$$

Let the total cost be represented by C.

$$C(x) = 600\left(\frac{500}{x}\right) + 1200\left(\frac{500}{x} + 2x\right)$$
$$= 2400x + \frac{900\ 000}{x}$$
$$= 2400x + 900\ 000x^{-1}$$

Find the critical points.

$$C'(x) = 2400 - 900\ 000x^{-2}$$
$$0 = 2400 - 900\ 000x^{-2}$$
$$0 = 2400x^2 - 900\ 000$$
$$x^2 = 375$$
$$x = \pm\sqrt{375}$$
$$x = \pm 5\sqrt{15}$$

Considering the context, $0 < x < 500$. Check the total cost at $x = 5\sqrt{15}$ and the endpoints.

$$C\left(5\sqrt{15}\right) = 2400\left(5\sqrt{15}\right) + \frac{900\ 000}{\left(5\sqrt{15}\right)}$$
$$= 92\ 952$$
$$C(1) = 2400(1) + \frac{900\ 000}{(1)}$$
$$= 902\ 400$$
$$C(500) = 2400(500) + \frac{900\ 000}{(500)}$$
$$= 1\ 200\ 000$$

The minimum cost occurs when the dimensions are $5\sqrt{15}$ and $\frac{500}{5\sqrt{15}} = \frac{20\sqrt{15}}{3}$.

The showroom should be about 19.4 m by 25.8 m.

b) Roofing will have a fixed cost since its area (500 m^2) does not change as different showroom rectangles are considered.

Chapter 3 Section 6 Question 7 Page 201

a) Let the perimeter be P. Then $P = 2h + 2r + \pi r$ and so $h = \frac{1}{2}(P - 2r - \pi r)$.

$$A = \frac{\pi r^2}{2} + 2rh$$
$$= \frac{\pi r^2}{2} + 2r\left(\frac{P - 2r - \pi r}{2}\right)$$
$$= \frac{\pi r^2}{2} + Pr - 2r^2 - \pi r^2$$
$$A' = \pi r + P - 4r - 2\pi r$$
$$A' = P - 4r - \pi r$$
$$P - 4r - \pi r = 0$$
$$r = \frac{P}{4 + \pi}$$

$$h = \frac{1}{2}(P - 2r - \pi r)$$
$$= \frac{1}{2}\left[P - 2\left(\frac{P}{4+\pi}\right) - \pi\left(\frac{P}{4+\pi}\right)\right]$$
$$= \frac{P}{2}\left(1 - \frac{2+\pi}{4+\pi}\right)$$
$$= \frac{P}{2}\left(\frac{4+\pi}{4+\pi} - \frac{2+\pi}{4+\pi}\right)$$
$$= \frac{P}{2}\left(\frac{2}{4+\pi}\right)$$
$$= \frac{P}{4+\pi}$$

So the ratio is:

$$\frac{h}{r} = \frac{\frac{P}{4+\pi}}{\frac{P}{4+\pi}}$$
$$= 1$$

b) Let A be the area and assume the cost for the straight edges is \$$x$/m.

$$A = \frac{\pi r^2}{2} + 2rh$$

$$h = \frac{1}{2r}\left(A - \frac{\pi r^2}{2}\right)$$

$$= \frac{A}{2r} - \frac{\pi r}{4}$$

$$C(x) = x(2h + 2r) + 3x(\pi r)$$

$$= 2x\left[\frac{A}{2r} - \frac{\pi r}{4}\right] + 2xr + 3x\pi r$$

$$= x\left(\frac{A}{r} - \frac{\pi r}{2} + 2r + 3\pi r\right)$$

$$C'(x) = x\left(\frac{-A}{r^2} - \frac{\pi}{2} + 2 + 3\pi\right)$$

$$0 = x\left(\frac{-A}{r^2} + \frac{5\pi}{2} + 2\right)$$

$$\frac{A}{r^2} = \frac{5\pi}{2} + 2$$

$$A = \left(\frac{5\pi}{2} + 2\right)r^2$$

$$h = \frac{1}{2r}\left[\left(\frac{5\pi}{2} + 2\right)r^2\right]$$

$$= \frac{r}{2}\left(\frac{5\pi}{2} + 2\right) - \frac{\pi r}{4}$$

$$= \frac{5\pi r}{4} + r - \frac{\pi r}{4}$$

$$= \pi r + r$$

$$= r(\pi + 1)$$

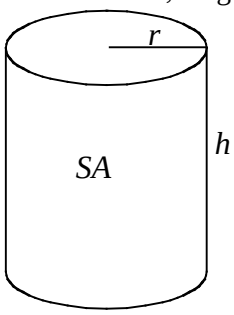
So the ratio is:

$$\frac{h}{r} = \frac{r(\pi + 1)}{r}$$

$$= \pi + 1$$

Chapter 3 Section 6 **Question 8 Page 201**

a) Let the radius, height, and surface area be represented by r, h, and S.A. respectively.



S.A. is the variable to be maximized. Need to eliminate one of the other variables.
The volume is 1.

$1 = \pi r^2 h$

$h = \dfrac{1}{\pi r^2}$

The surface area is:

$\text{S.A.} = 2\pi r^2 + 2\pi rh$

$= 2\pi r^2 + 2\pi r\left(\dfrac{1}{\pi r^2}\right)$

$= 2\pi r^2 + \dfrac{2}{r}$

$= 2\pi r^2 + 2r^{-1}$

For a minimum, find the critical points.

$\text{S.A.}' = 4\pi r - 2r^{-2}$

$0 = 4\pi r - 2r^{-2}$

$0 = 4\pi r^3 - 2$

$r = \sqrt[3]{\dfrac{1}{2\pi}}$

$r \approx 0.54$

To verify this is a minimum, test values that are close.

$S.A.(0.54) = 5.54$

$S.A.(0.5) = 5.57$

$S.A.(0.6) = 5.60$

$$h = \frac{1}{\pi r^2}$$
$$= \frac{\sqrt[3]{4\pi^2}}{\pi}$$
$$= \sqrt[3]{\frac{4}{\pi}}$$

The surface area is a minimum when $r = \sqrt[3]{\frac{1}{2\pi}}$ and $h = \sqrt[3]{\frac{4}{\pi}}$ or, in decimals, when $r \approx 0.54$ and $h \approx 1.08$.

b) $$\frac{h}{d} = \frac{h}{2r}$$
$$= \frac{1.08}{2(0.54)}$$
$$= \frac{1}{1}$$

The ratio of height to diameter is 1:1.

c) No. The diameter of a pop can is determined by the size of the hand that will hold it, irrespective of volume and cost of materials. Regular pop cans have a 2:1 ratio. Other types of drink cans may have different ratios depending on the volume of their contents. Some mini pop cans have a ratio close to 1:1.

Chapter 3 Section 6 Question 9 Page 201

Let the radius, height, and total cost be represented by r, h, and C respectively.
C is the variable to be minimized. Need to eliminate one of the other variables.

The volume is 500 cm^2.

$$V = \pi r^2 h$$
$$500 = \pi r^2 h$$
$$h = \frac{500}{\pi r^2}$$

The cost function is:

$$C(r) = (0.4)\pi r^2 + (0.2)\left(\pi r^2 + 2\pi rh\right)$$

$$= (0.4)\pi r^2 + (0.2)\left(\pi r^2 + 2\pi r\left(\frac{500}{\pi r^2}\right)\right)$$

$$= 0.6\pi r^2 + \frac{200}{r}$$

$$= 0.6\pi r^2 + 200r^{-1}$$

For a minimum cost, find the critical points.

$$C'(r) = 1.2\pi r - 200r^{-2}$$

$$0 = 1.2\pi r - 200r^{-2}$$

$$0 = 1.2\pi r^3 - 200$$

$$r = \sqrt[3]{\frac{500}{3\pi}}$$

$r \approx 3.76$

To verify this is a minimum, test values that are close.

$$C(3.76) = 79.84$$

$$C(3.7) = 79.86$$

$$C(3.8) = 79.85$$

The minimum occurs when $r = \sqrt[3]{\frac{500}{3\pi}}$ cm and $h = \frac{5\sqrt[3]{36\pi^2}}{\pi}$ cm or, in decimals, when $r \approx 3.76$ cm and $h = 11.27$ cm.

Chapter 3 Section 6 **Question 10 Page 201**

a) Let the radius, height, and volume be represented by r, h, and V respectively.

$$V = \pi r^2 h$$

The surface area is 1 m^2.

$$\text{S.A.} = \pi r^2 + 2\pi rh$$

$$1 = \pi r^2 + 2\pi rh$$

$$h = \frac{1 - \pi r^2}{2\pi r}$$

$$V = \pi r^2\left(\frac{1 - \pi r^2}{2\pi r}\right)$$

$$= \frac{r - \pi r^3}{2}$$

b) For maximum volume, find the critical points.

$$V'(r) = \frac{1}{2}\left(1 - 3\pi r^2\right)$$

$$0 = \frac{1}{2}\left(1 - 3\pi r^2\right)$$

$$r^2 = \frac{1}{3\pi}$$

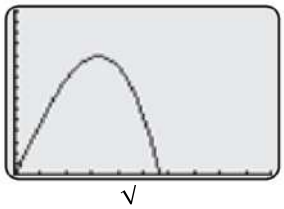
$$r = \pm\sqrt{\frac{1}{3\pi}}$$

$$r = \pm\frac{\sqrt{3\pi}}{3\pi}$$

$r \approx \pm 0.33$

Radius must be positive so discard –33 m. The maximum volume drum will occur when the radius is approximately 0.33 m.

c)



$(x \in [0, 1],\ \text{Xscl} = 0.1,\ y \in [0, 0.2],\ \text{Yscl} = 0.01)$

d) If the domain of r is restricted to $0 < r < 0.2$, then there are no critical values to consider. The maximum volume can only occur at one of the endpoints. In fact, since this function is increasing on this interval (as is evident from the graph), the maximum volume will occur at the end of the interval, i.e., when $r = 0.2$.

$$V(0.2) = \frac{(0.2) - \pi(0.2)^3}{2}$$

$$= 0.1 - 0.004\pi$$

The maximum volume in this case will be 0.087 m^3.

Chapter 3 Section 6 **Question 11 Page 202**

Let the radius, height, and volume of the paper cylinder be represented by r, h, and V respectively.
V is the variable to be maximized.

$V = \pi r^2 h$

Need this equation to only involve two variables.
The perimeter is to be 100 cm.

$100 = 2h + 2(2\pi r)$

$$r = \frac{50 - h}{2\pi}$$

Therefore, the formula becomes:

$$V(h) = \pi\left(\frac{50-h}{2\pi}\right)^2 h$$
$$= \frac{1}{4\pi}h^3 - \frac{100}{4\pi}h^2 + \frac{2500}{4\pi}h$$

For maximum volume, find the critical points.

$$V'(h) = \frac{3}{4\pi}h^2 - \frac{200}{4\pi}h + \frac{2500}{4\pi}$$
$$0 = \frac{3}{4\pi}h^2 - \frac{200}{4\pi}h + \frac{2500}{4\pi}$$
$$0 = 3h^2 - 200h + 2500$$
$$0 = (3h - 50)(h - 50)$$
$$h = \frac{50}{3},\ h = 50$$

Check the volume at the critical values.

$$V(50) = \frac{1}{4\pi}(50)^3 - \frac{100}{4\pi}(50)^2 + \frac{2500}{4\pi}(50)$$
$$= 0$$
$$V\left(\frac{50}{3}\right) = \frac{1}{4\pi}\left(\frac{50}{3}\right)^3 - \frac{100}{4\pi}\left(\frac{50}{3}\right)^2 + \frac{2500}{4\pi}\left(\frac{50}{3}\right)$$
$$= 1473.7$$

If one dimension of the paper is $\frac{50}{3}$, then the other dimension is $\frac{100 - 2\left(\frac{50}{3}\right)}{2} = \frac{100}{3}$

The maximum volume occurs when the paper is $\frac{50}{3}$ cm by $\frac{100}{3}$ cm.

Chapter 3 Section 6 **Question 12 Page 202**

Let n be the number of additional trees planted and P be the total crop.
The number of trees is now $50 + n$ and the production per tree becomes $200 - 5n$.
$P = (50 + n)(200 - 5n)$

To maximize the crop, find the critical points.
$P'(n) = (50 + n)(-5) + (1)(200 - 5n)$ Product rule.
$0 = (50 + n)(-5) + (1)(200 - 5n)$
$0 = -250 - 5n + 200 - 5n$
$n = -5$

The optimal number of trees to be planted is 50 – 5 = 45.

Note: Increasing the number of trees above 50 actually reduces the total crop.

$P(45) = 10\ 125$

$P(50) = 10\ 000$

$P(55) = 9\ 625$

Chapter 3 Section 6 **Question 13 Page 202**

a) $R(x) = (30 + x)(5000 - 100x)$

$= -100x^2 + 2000x + 150\ 000$

b) $-30 < x < 50$

Answers may vary. For example:
Assume that lowering the price increases the number of attendees. When $x = -30$, the ticket price is \$0, which is clearly impractical. If $x = 50$, then the number of attendees becomes zero which is also impractical.

c) Find the critical points.

$R'(x) = -200x + 2000$

$0 = -200x + 2000$

$x = 10$

To verify that this is a maximum, check values of $R(x)$..

$R(10) = 160\ 000$

$R(9) = 159\ 900$

$R(11) = 159\ 900$

A ticket price of \$40 will maximize the revenue from the concert.

d) The \$40 price results in 4000 people attending the concert. If only 1200 people can attend, the price should be set at \$68 (a \$38 price increase leads to a reduction of 3800 people). Any further increases in price will lead to a lower total revenue.

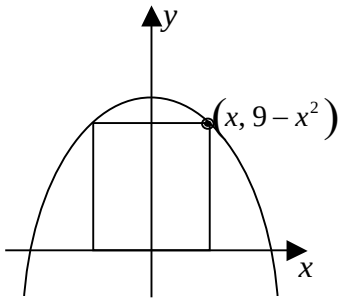
$R(38) = 81,\ 600$

$R(39) = 75\ 900$

$R(40) = 70\ 000$

Chapter 3 Section 6 **Question 14 Page 202**

Let $(x, 9, -x^2)$ be any point on the parabola.
The resulting inscribed rectangle has width $2x$ and height $9 - x^2$.



Let A represent the area.

$$A(x) = 2x(9 - x^2)$$
$$= 18x - 2x^3$$

For a maximum, find the critical points

$$A'(x) = 18 - 6x^2$$
$$0 = 18 - 6x^2$$
$$x = \sqrt{3}$$

Check that this is a maximum.

$$A\left(\sqrt{3}\right) = 12\sqrt{3}$$
$$A(2) = 20$$
$$A(1) = 16$$

The rectangle with the largest area has area 20.8 units^2.

Chapter 3 Section 6 **Question 15 Page 202**

a) To minimize fuel cost per kilometre, examine the critical points.

$$C(v) = \frac{v}{100} + \frac{25}{v}$$
$$= \frac{1}{100}v + 25v^{-1}$$
$$C'(v) = \frac{1}{100} - 25v^{-2}$$
$$0 = \frac{1}{100} - 25v^{-2}$$
$$0 = v^2 - 100(25)$$
$$v = \pm\sqrt{2500}$$
$$v = \pm 50$$

The negative value makes no sense for this problem. Check that $v = 50$ is a minimum.

$$C(50) = 1.00$$
$$C(40) = 1.025$$
$$C(60) = 1.017$$

The speed that results in the lowest cost per kilometre is 50 km/h.

b) The total cost function, T, is made up from fuel cost and driver cost.

$$T(v) = \left(\frac{v}{100} + \frac{25}{v}\right)(1000) + 40\left(\frac{1000}{v}\right)$$
$$= 10v + 25\,000v^{-1} + 40\,000v^{-1}$$
$$= 10v + 65\,000v^{-1}$$
$$T'(v) = 10 - 65\,000v^{-2}$$
$$0 = 10 - 65\,000v^{-2}$$
$$0 = 10v^2 - 65\,000$$
$$v = \pm\sqrt{6500}$$
$$v \approx \pm 80.6$$

The negative value makes no sense for this problem. Check that $v = 80.6$ is a minimum.

$$C(80) = 1612.50$$
$$C(80.6) = 1612.45$$
$$C(81) = 1612.47$$

The speed that results in the lowest total cost for the 1000-km trip is about 80.6 km/h

Chapter 3 Section 6 **Question 16 Page 202**

Let the number be x and the sum to be minimized be S.

$$S(x) = x^2 + \frac{1}{x}$$
$$= x^2 + x^{-1}$$

Examine the critical values.

$$S'(x) = 2x - x^{-2}$$
$$0 = 2x - x^{-2}$$
$$0 = 2x^3 - 1$$
$$x = \sqrt[3]{0.5}$$
$$x \approx 0.794$$

Check that $x = 0.794$ is a minimum.

$$S(0.794) = 1.8899$$
$$S(0.85) = 1.899$$
$$S(0.75) = 1.8958$$

The number is $\sqrt[3]{0.5} \approx 0.794$.

Chapter 3 Section 6 **Question 17 Page 202**

Brenda's costs for the trip include the running costs for every 100 km (15 since she is travelling 1500 km) of the vehicle and the costs of paying herself.

$$C_{tot}(v) = 15\left(0.9 + 0.0016v^2\right) + \frac{30(1500)}{v}$$
$$= 13.5 + 0.024v^2 + \frac{45\,000}{v}$$

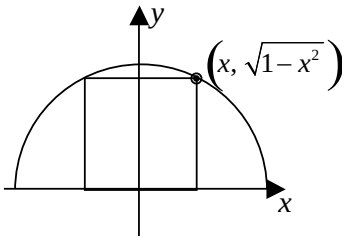
Find the derivative of the total cost and set it to zero to find the speed that will minimize cost.

$$C'_{tot}(v) = 0.048v - \frac{45\,000}{v^2}$$
$$0 = 0.048v - \frac{45\,000}{v^2}$$
$$0 = 0.048v^3 - 45\,000$$
$$0.048v^3 = 45\,000$$
$$v^3 = 937\,500$$
$$v \approx 97.9$$

The speed 97.9 km/h (approximately) will minimize Brenda's costs.

Chapter 3 Section 6 **Question 18 Page 202**

Choose coordinate axes so that the plexiglass is a semicircle with equation $y = \sqrt{1 - x^2}$ as shown.
Each point on the semicircle determines a possible rectangle.
The resulting inscribed rectangle has width $2x$ and height $\sqrt{1 - x^2}$.



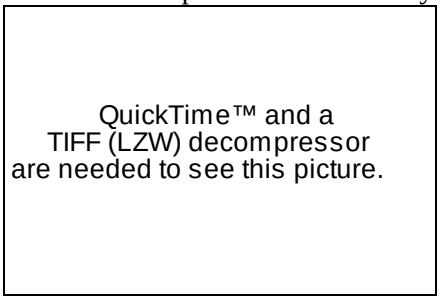
Let A represent the area.

$$A(x) = 2x\sqrt{1 - x^2}$$
$$= 2x(1 - x^2)^{\frac{1}{2}}$$

For a maximum, find the critical points:

$$A'(x) = 2x\left(\frac{1}{2}\right)(1 - x^2)^{-\frac{1}{2}}(-2x) + 2(1 - x^2)^{\frac{1}{2}}$$
$$= (1 - x^2)^{-\frac{1}{2}}(-2x^2 + 2 - 2x^2)$$
$$= (2 - 4x^2)(1 - x^2)^{-\frac{1}{2}}$$
$$= \frac{2 - 4x^2}{(1 - x^2)^{\frac{1}{2}}}$$
$$0 = \frac{2 - 4x^2}{(1 - x^2)^{\frac{1}{2}}}$$
$$x^2 = \frac{1}{2}$$
$$x = \pm\sqrt{0.5}$$
$$x \approx \pm 0.71$$

The maximum point is confirmed by the graph of $A(x)$:.



The largest rectangle has dimensions 1.42 m by 0.71 m.

Chapter 3 Section 6 **Question 19 Page 202**

a) If x m are used for the quarter circle, then there are $(20 - x)$ m for the two sides of the square.

Each side will be $\left(\frac{20-x}{2}\right)$ m.

b) $4x = 2\pi r$

$$r = \frac{4x}{2\pi}$$

$$r = \frac{2x}{\pi}$$

c) $A(x)$ = area of square + area of quarter circle

$$= \left(\frac{20-x}{2}\right)^2 + \frac{1}{4}\pi\left(\frac{2x}{\pi}\right)^2$$

d) Simplifying,

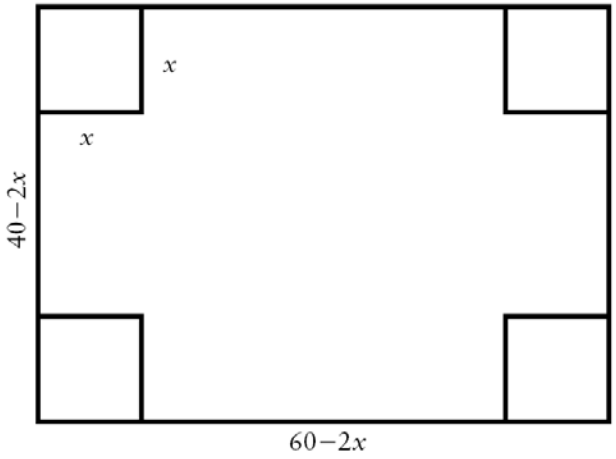
$$A(x) = \frac{400 - 40x + x^2}{4} + \frac{x^2}{\pi}$$

$$= 100 - 10x + \left(\frac{1}{4} + \frac{1}{\pi}\right)x^2$$

$$= \left(\frac{4+\pi}{4\pi}\right)x^2 - 10x + 100$$

Chapter 3 Section 6 **Question 20 Page 202**

a) Draw a diagram.



b) $V(x) = x(40 - 2x)(60 - 2x)$

$$= 4x^3 - 200x^2 + 2400x$$

c) $0 < x < 20$

If $x < 0$, no tin would be cut from the corners and the box would have no volume.

If $x > 20$, all of the tin is cut away from the width and no box can be formed.

d) Find the critical values.

$V'(x) = 12x^2 - 400x + 2400$

$0 = 12x^2 - 400x + 2400$

$0 = 3x^2 - 100x + 600$

$x = \dfrac{100 \pm \sqrt{10000 - 7200}}{6}$

$x = \dfrac{100 \pm 20\sqrt{7}}{6}$

$x = \dfrac{50 \pm 10\sqrt{7}}{3}$

$x \approx 25.5$, $x \approx 7.85$

Check values for the one critical value in the domain for x.

$V(7.85) = 8450.4$

$V(7) = 8372$

$V(9) = 8316$

Therefore:

$x = \dfrac{50 - 10\sqrt{7}}{3}$ $\quad 40 - 2x = \dfrac{20 + 20\sqrt{7}}{3}$ $\quad 60 - 2x = \dfrac{80 + 20\sqrt{7}}{3}$

≈ 7.85 $\quad \approx 24.3$ $\quad \approx 44.3$

The box dimensions that will maximize the volume are 7.85 cm by 24.3 cm by 44.3 cm.

Chapter 3 Section 6 **Question 21 Page 203**

a) Find the volume and surface area.

$V = \pi r^2(2r)$ $\quad$ S.A. $= 2\pi r^2 + 2\pi r(2r)$

$= 2\pi r^3$ $\quad = 6\pi r^2$

The profit function is

$P(r) = 7(6\pi r^2) - 10(2\pi r^3)$.

$= 42\pi r^2 - 20\pi r^3$

To maximize the profit, find the critical values.

$P'(r) = 84\pi r - 60\pi r^2$

$0 = 84\pi r - 60\pi r^2$

$0 = 12\pi r(7 - 5r)$

$r = 0$, $r = 1.4$

Check values for this critical point.

$P(1.4) = 86.21$

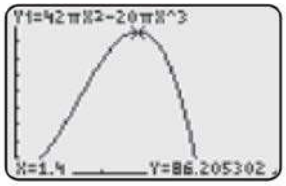
$P(1.5) = 84.82$

$P(1.3) = 84.95$

The container radius that maximizes the profit is 1.4 m.

b) The profit per container is \$86.21

c)



Chapter 3 Section 6 **Question 22 Page 203**

a) A cube is the most efficient shape. Non-cubical shapes require more material to enclose the same volume.

b) No. The volume involved does not affect the efficient shape considerations.

c) The shape would be expected to change.
Let the volume be 1 m^3. Also let the base be x by x and the height be h,

$V = (x)(x)(h)$

$1 = (x)(x)(h)$

$h = \frac{1}{x^2}$

The surface area is

$\text{S.A.} = x^2 + 4x\left(\frac{1}{x^2}\right)$

$= x^2 + 4x^{-1}$

For a minimum surface area, find the critical values.

$\text{S.A.}' = 2x - 4x^{-2}$

$0 = 2x - 4x^{-2}$

$0 = 2x^3 - 4$

$x^3 = 2$

$x = \sqrt[3]{2}$

$x \doteq 1.26$

The corresponding height is:

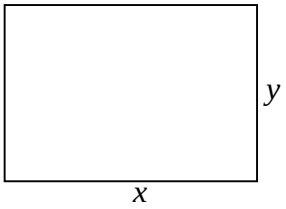
$$h = \frac{1}{\left(\sqrt[3]{2}\right)^2}$$

$$\approx 0.63$$

If the box has no top, the most efficient shape is a square base with a height that is one-half the base dimensions.

Chapter 3 Section 6 Question 23 Page 203

a) i) Let the dimensions be x and y as shown.



There is 1000 m of fencing.

$$2x + 2y = 1000$$
$$y = 500 - x$$

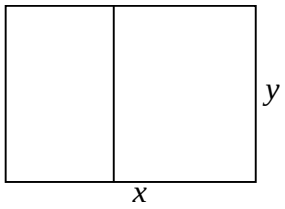
$$A(x) = x(500 - x)$$
$$= 500x - x^2$$

For a maximum:

$$A'(x) = 500 - 2x$$
$$0 = 500 - 2x$$
$$x = 250$$

The dimensions of the pen with maximum area are 250 m by 250 m.

ii)



There is 1000 m of fencing,

$$2x + 3y = 1000$$
$$y = \frac{1000 - 2x}{3}$$

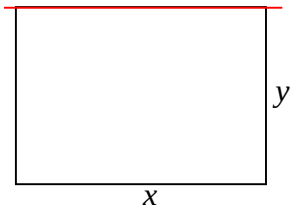
$$A(x) = x\left(\frac{1000 - 2x}{3}\right)$$
$$= \frac{1000}{3}x - \frac{2}{3}x^2$$

For a maximum:

$$A'(x) = \frac{1000}{3} - \frac{4}{3}x$$
$$0 = \frac{1000}{3} - \frac{4}{3}x$$
$$x = 250$$

The dimensions of the pen with maximum area are 250 m by 166.7 m.

iii)



There is 1000 m of fencing.

$$x + 2y = 1000$$
$$y = \frac{1000 - x}{2}$$

$$A(x) = x\left(\frac{1000 - x}{2}\right)$$
$$= 500x - \frac{1}{2}x^2$$

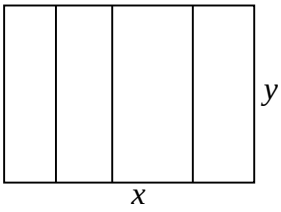
For a maximum,

$$A'(x) = 500 - x$$
$$0 = 500 - x$$
$$x = 500$$

The dimensions of the pen with maximum area are 500 m by 250 m.

b) The dimension with two sections of fence is always 250 m. The other dimension is 500 divided by the number of fence sections in that dimension.

c) The dimensions, if there were three dividers and four equal parts, would be 250 m by 100 m.
This result could be found by modifying the calculus technique used in part a).
It could also be found by using the pattern identified in part b).



Chapter 3 Section 6 **Question 24 Page 203**

a) The cube has the least surface area. This can be verified using calculus, assuming the base is a square.
Let the volume be 1 m^3. Also let the base be x by x and the height be h,

$$V = (x)(x)(h)$$
$$1 = (x)(x)(h)$$
$$h = \frac{1}{x^2}$$

The surface area is

$$\text{S.A.} = 2x^2 + 4x\left(\frac{1}{x^2}\right)$$
$$= 2x^2 + 4x^{-1}$$

For a minimum surface area, find the critical values.

$$\text{S.A.}' = 4x - 4x^{-2}$$
$$0 = 4x - 4x^{-2}$$
$$0 = 4x^3 - 4$$
$$x^3 = 1$$
$$x = \sqrt[3]{1}$$
$$x = 1$$

The corresponding height is 1.
The dimensions are 1 m by 1 m by 1 m, a cube.

b) It appears that the best shape is one that has all dimensions equal. In three-space, that shape is a sphere. Compare a cube and a sphere. Let the volume be 1 m^3.

For the cube, the dimensions are 1 m by 1 m by 1 m and the total surface area is 6 m^2.

For the sphere:

$$1 = \frac{4}{3}\pi r^3$$

$$r = \sqrt[3]{\frac{3}{4\pi}}$$

$$r \approx 0.62$$

$$\text{S.A.} = 4\pi(0.62)^2$$

$$\approx 4.84$$

Clearly, the sphere has a smaller surface area than a cube of the same volume.

c) Answers may vary. For example:

:

- Packaging may not be a significant cost related to the product.
- Packaging is advertising. A larger surface pointed at the consumer provides the company with a l space in which to promote their product.
- Spheres are very hard to stack or store on rectangular shelves.
- Packages need to be handled. The size of the human hand may determine the thickness of the box.
- Aesthetically, a rectangle in the shape of the golden ratio is more pleasing than a square.

Chapter 3 Section 6 Question 25 Page 203

The correct answer is E.

Let r be the radius of the circle, where $0 \le r \le \frac{L}{2\pi}$.

The side length of the square is $\frac{L - 2\pi r}{4}$.

The total area is:

$$A = \pi r^2 + \left(\frac{L - 2\pi r}{4}\right)^2$$

$$\frac{dA}{dr} = \frac{\pi(\pi + 4)r}{2} - \frac{\pi L}{4}$$

$\frac{dA}{dr} < 0$ when $0 \le r < \frac{L}{2(\pi + 4)}$

$\frac{dA}{dr} > 0$ when $\frac{L}{2(\pi + 4)} < r \le \frac{L}{2\pi}$.

Therefore, A has a minimum when $r = \frac{L}{2(\pi + 4)}$.

Hence $2\pi r = \frac{\pi L}{\pi + 4}$ is used for the circle.

Chapter 3 Section 6 **Question 26 Page 203**

The correct answer is E.

$$f(x) = \frac{x^n}{x-1}$$
$$f'(x) = \frac{x^{n-1}\left[(n-1)x - n\right]}{(x-1)^2}$$

If n is even, then f will have a local maximum at $x = 0$ and a local minimum at $x = \frac{n}{n-1}$.

If $n \neq 1$ is odd, then f will have only a local minimum at $x = \frac{n}{n-1}$.

If $n = 1$, then f will have no local maximum or minimum.

Chapter 3 Review

Chapter 3 Review Question 1 Page 204

a) $f(x) = 7 + 6x - x^2$

$f'(x) = 6 - 2x$

The function is increasing when $f'(x) > 0$. First solve the related equation.

$0 = 6 - 2x$

$x = 3$

Test the inequality for arbitrary values in the intervals $(-\infty, 3)$ and $(3, \infty)$.
For the first interval, use $x = 0$.

$\text{L.S.} = 6 - 2(0)$ $\quad 4 > 0$

$= 4$

For the second interval, use $x = 4$.

$\text{L.S.} = 6 - 2(4)$ $\quad -2 < 0$

$= -2$

$f(x)$ is increasing on the interval $(-\infty, 3)$.
$f(x)$ is decreasing on the interval $(3, \infty)$.

b) Let $y = f(x)$.

$f(x) = x^3 - 48x + 5$

$f'(x) = 3x^2 - 48$

The function is increasing when $f'(x) > 0$. First solve the related equation.

$3x^2 - 48 = 0$

$x^2 = 16$

$x = \pm 4$

Test the inequality for arbitrary values in the intervals $(-\infty, -4)$, $(-4, 4)$, and $(4, \infty)$.
For the first interval, use $x = -5$.

$\text{L.S.} = 3(-5)^2 - 48$ $\quad 27 > 0$

$= 27$

For the second interval, use $x = 0$.

$\text{L.S.} = 3(0)^2 - 48$ $\quad -48 < 0$

$= -48$

For the third interval, use $x = 5$.

$\text{L.S.} = 3(5)^2 - 48 \qquad 27 > 0$

$= 27$

$f(x)$ is increasing on the intervals $(-\infty, -4)$ and $(4, \infty)$.
$f(x)$ is decreasing on the interval $(-4, 4)$.

c) $g(x) = x^4 - 18x^2$

$g'(x) = 4x^3 - 36x$

The function is increasing when $g'(x) > 0$. First solve the related equation.

$0 = 4x^3 - 36x$

$0 = 4x(x^2 - 9)$

$x = 0,\ x = \pm 3$

Test the inequality for arbitrary values in the intervals $(-\infty, -3)$, $(-3, 0)$, $(0, 3)$, and $(3, \infty)$.
For the first interval, use $x = -5$.

$\text{L.S.} = 4(-5)^3 - 36(-5) \qquad -320 < 0$

$= -320$

For the second interval, use $x = -1$,

$\text{L.S.} = 4(-1)^3 - 36(-1) \qquad 27 > 0$

$= 32$

For the third interval, use $x = 1$.

$\text{L.S.} = 4(1)^3 - 36(1) \qquad -32 < 0$

$= -32$

For the fourth interval, use $x = 5$.

$\text{L.S.} = 4(5)^3 - 36(5) \qquad 320 > 0$

$= 320$

$g(x)$ is increasing on the intervals $(-3, 0)$ and $(3, \infty)$.
$g(x)$ is decreasing on the intervals $(-\infty, -3)$, $(0, 3)$.

d) $f(x) = x^3 + 10x - 9$

$f'(x) = 3x^2 + 10$

The function is increasing when $f'(x) > 0$. First solve the related equation.

$0 = 3x^2 + 10$

$x^2 = -\frac{10}{3}$

There are no solutions to this equation. Test the inequality for one arbitrary value, such as $x = 0$.

$\text{L.S.} = 3(0)^2 + 10 \qquad 10 > 0$
$= 10$

$f(x)$ is always increasing.

Chapter 3 Review **Question 2 Page 204**

$f'(x) = x(x-3)^2$
$0 = x(x-3)^2$

$x = 0, x = 3$

Test points in the intervals.
$f'(-1) = -16$
$f'(1) = 4$
$f'(4) = 4$

	$x < 0$	$x = 0$	$0 < x < 3$	$x = 3$	$x > 3$
$f'(x)$	Negative	0	Positive	0	Positive
$f(x)$	Decreasing	Minimum	Increasing	Point of inflection	Increasing

Chapter 3 Review **Question 3 Page 204**

a) Let $y = f(x)$.
$f(x) = 3x^2 + 24x - 8$
$f'(x) = 6x + 24$
$0 = 6x + 24$
$x = -4$

Use a table to show increasing and decreasing intervals for the function.

	$x < -4$	$x = -4$	$x > -4$
Test value	–5	–4	0
$f'(x)$	$f'(-5) = -6$ Negative	0	$f'(0) = 24$ Positive
y	Decreasing	(–4, –56)	Increasing

The critical point (–4, –56) is a local minimum.

b) $f(x) = 16 - x^4$
$f'(x) = -4x^3$
$0 = -4x^3$
$x = 0$

Use a table to show increasing and decreasing intervals for the function.

	$x < 0$	$x = 0$	$x > 0$
Test value	-1	0	1
$f'(x)$	$f'(-1) = 4$ Positive	0	$f'(1) = -4$ Negative
$f(x)$	Increasing	$(0, 16)$	Decreasing

The critical point (0, 16) is a local maximum.

c) $g(x) = x^3 + 9x^2 - 21x - 12$

$g'(x) = 3x^2 + 18x - 21$

$0 = 3x^2 + 18x - 21$

$0 = 3(x + 7)(x - 1)$

$x = -7,\ x = 1$

Use a table to show increasing and decreasing intervals for the function.

	$x < -7$	$x = -7$	$-7 < x < 1$	$x = 1$	$x > 1$
Test value	-10	-7	0	1	2
$g'(x)$	$g'(-10) = 99$ Positive	0	$g'(0) = -21$ Negative	0	$g'(2) = 27$ Positive
$g(x)$	Increasing	$(-2, 233)$	Decreasing	$(1, -23)$	Increasing

The critical point (–2, 233) is a local maximum and the point (1, –23) is a local minimum.

Chapter 3 Review **Question 4 Page 204**

a) Find the critical points for this function.

$v(t) = 3t^2 - 24t + 88$

$v'(t) = 6t - 24$

$6t - 24 = 0$

$t = 4$

Check that it is a minimum.

	$x < 4$	$x = 4$	$x > 4$
Test value	3	4	5
$v'(t)$	$v'(3) = -6$ Negative	0	$v'(5) = 6$ Positive
$v(t)$	Decreasing	(4, 40)	Increasing

The critical point (4, 40) is a local minimum.
The minimum sped of the car is 40 km/h.

b) There are no critical points that can be maxima in this interval. The maximum must occur at an endpoint.

$v(2) = 52$

$v(5) = 43$

The maximum speed of the car in the interval is 52 km/h.

Chapter 3 Review **Question 5 Page 204**

Find the critical numbers.

$$f(x) = x^3 - 8x^2 + 5x + 2$$
$$f'(x) = 3x^2 - 16x + 5$$
$$0 = 3x^2 - 16x + 5$$
$$0 = (3x - 1)(x - 5)$$
$$x = \frac{1}{3}, x = 5$$

The critical numbers are $\frac{1}{3}$ and 5.

To find local extrema, use a table to show increasing and decreasing intervals for the function.

	$x < \frac{1}{3}$	$x = \frac{1}{3}$	$\frac{1}{3} < x < 5$	$x = 5$	$x > 5$
Test value	0	$\frac{1}{3}$	1	5	6
$f'(x)$	$f'(0) = 5$ Positive	0	$f'(1) = -8$ Negative	0	$f'(6) = 17$ Positive
$f(x)$	Increasing	$\left(\frac{1}{3}, \frac{76}{27}\right)$	Decreasing	(5, –48)	Increasing

The critical point $\left(\frac{1}{3}, \frac{76}{27}\right)$ is a local maximum and the critical point (2, –48) is local minimum.

Check the function values at the endpoints of the interval.

$f(0) = 2$

$f(6) = -40$

The absolute minimum is (5, –48) and the absolute maximum is $\left(\frac{1}{3}, \frac{76}{27}\right)$ for the interval $[0, 6]$.

Chapter 3 Review **Question 6 Page 204**

B is the correct answer.
A cubic function will have a quadratic (second degree) first derivative and a linear (first degree) second derivative. A linear function always has exactly one zero and thus a cubic function must have exactly one point of inflection.

Chapter 3 Review **Question 7 Page 204**

True.
The second derivative of a quadratic function is of second degree. This derivative will have either zero, one, or two roots. Zero or two roots lead to zero or two points of inflection. If the quadratic has only one root, it means that it is always positive or always negative, which means that it does not change sign. In this case there can be no point of inflection because there is no sign change.

Chapter 3 Review **Question 8 Page 204**

$f(x) = x^4 - 2x^3 - 12x^2 + 3$

$f'(x) = 4x^3 - 6x^2 - 24x$

$f''(x) = 12x^2 - 12x - 24$

Find the possible points of inflection.

$0 = 12x^2 - 12x - 24$

$0 = x^2 - x - 2$

$0 = (x - 2)(x + 1)$

$x = 2,\ x = -1$

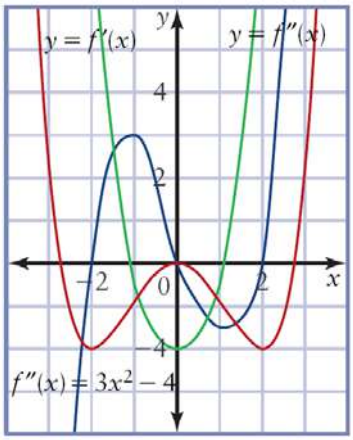
These values divide the domain into three intervals. Test the value of $f''(x)$ in each interval and summarize in a table.

	$x < -1$	$x = -1$	$-1 < x < 2$	$x = 2$	$x > 2$
Test value	–2	–1	0	2	3
$f''(x)$	$f''(-2) = 48$ Positive	0	$f''(0) = -24$ Negative	0	$f''(3) = 48$ Positive
$f(x)$	Concave up	Point of inflection (–1, –6)	Concave down	Point of Inflection (2, –45)	Concave up

Chapter 3 Review **Question 9 Page 204**

a) From the graph of $f'(x)$, f is increasing when $-2 < x < 0$ and $x > 2$ and deceasing otherwise. It will have local minima at $x = \pm 2$ and a local maximum at $x = 0$.

b) From the graph of $f'(x)$, $f''(x)$ will have zeros at –1.25 and +1.25.
$f''(x)$ will have negative values between –1.25 and +1.25 and positive values elsewhere. Since $f'(x)$ has an inflection point when $x = 0$, $f''(x)$ will have a local minimum at $x = 0$.



Chapter 3 Review **Question 10 Page 204**

Find the critical values.

$f(x) = 2x^3 - x^4$

$f'(x) = 6x^2 - 4x^3$

$0 = 6x^2 - 4x^3$

$0 = 2x^2(3 - 2x)$

$x = 0, x = \frac{3}{2}$

Use the second derivative test to determine maxima and minima.

$f''(x) = 12x - 12x^2$

Test the critical points.

$f''\left(\frac{3}{2}\right) = -9$

$f''(0) = 0$

$f''(-0.5) = -9$

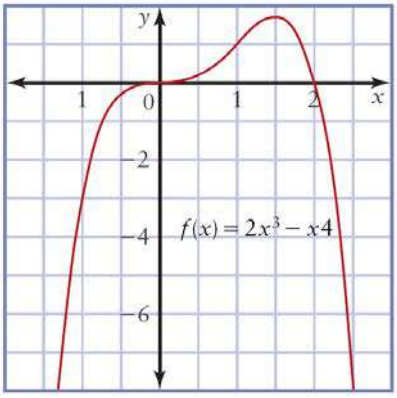
$f''(0.5) = 3$

The second derivative test indicates that there is a local maximum at $x = 1.5$.
Since the sign of the second derivative changes at $x = 0$, there is a point of inflection there.

To sketch the curve, more information is needed.
Use a table to show increasing and decreasing intervals for the function.

	$x < 0$	$x = 0$	$0 < x < 1.5$	$x = 1.5$	$x > 1.5$
Test value	-1	0	1	0	2
$f'(x)$	$f'(-1) = 10$ Positive	0	$f'(1) = 2$ Positive	0	$f'(2) = -8$ Negative
$f(x)$	Increasing	(0,0)	Increasing	(1.5, 1.7)	Decreasing

Then sketch the graph.



Chapter 3 Review **Question 11 Page 204**

Vertical asymptotes only occur where the denominator in a rational function equals zero.

a) $x = 0$

b) $0 = 2x - 4$
$x = 2$

c) $0 = x^2 - 3x - 10$
$0 = (x - 5)(x + 2)$
$x = 5,\ x = -2$

d) $0 = x^2 + 2x + 1$
$0 = (x + 1)(x + 1)$
$x = -1$

Chapter 3 Review **Question 12 Page 205**

a) Substituting a small, positive number for x means the numerator is close to 4 and the denominator is a very small, but positive number. The quotient of these two numbers will make a very large, positive number. Thus, $\lim\limits_{x \to 0^+} f(x) = \infty$.

b) The limit has the same value, $\lim_{x \to 0^-} f(x) = \infty$, since substituting a small but negative number will still lead to a quotient of 4 and a small positive number.

c) Let $y = 0$.

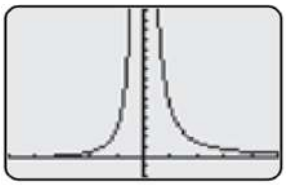
$$\frac{x+4}{x^2} = 0$$
$$x + 4 = 0$$
$$x = -4$$

d) To find the turning point, let the derivative equal zero.

$$f(x) = (x+4)x^{-2}$$
$$f'(x) = (x+4)(-2)x^{-3} + (1)x^{-2}$$
$$= x^{-3}(-2x - 8 + x)$$
$$0 = x^{-3}(-2x - 8 + x)$$
$$0 = -x - 8$$
$$x = -8$$

The turning point is $\left(-8, -\frac{1}{16}\right)$

e)



Chapter 3 Review **Question 13 Page 205**

a) $f(-x) = (-x)^3 - 3(-x)$
$= -x^3 + 3x$
$= -f(x)$

Therefore the function is an odd function.

b) This is a polynomial function. The domain is $\mathbb{R}$.

c) Let $x = 0$.

$f(0) = (0)^3 - 3(0)$
$= 0$

The y–intercept is 0.

Let $y = 0$.

$0 = x^3 - 3x$

$0 = x(x^2 - 3)$

$x = 0,\ x = \pm\sqrt{3}$

There are three x-intercepts at $-\sqrt{3}$, 0, and $\sqrt{3}$.

d) Find the critical points.

$f(x) = x^3 - 3x$

$f'(x) = 3x^2 - 3$

$f''(x) = 6x$

$0 = 3x^2 - 3$

$0 = 3(x^2 - 1)$

$x = \pm 1$

Use the second derivative test for extrema.

$f''(-1) = 6(-1)$

$= -6$

$f''(1) = 6(1)$

$= 6$

Therefore (1, –2) is a local minimum and (–1, 2) is a local maximum.

The critical points create three intervals. Test the derivative in each interval.

	$x < -1$	$-1 < x < 1$	$x > 1$
Test value	–2	0	2
$f'(x)$	$f'(-2) = 9$	$f'(0) = -3$	$f'(2) = 9$
	Positive	Negative	Positive
$f(x)$	Increasing	Decreasing	Increasing

The function is increasing for $x < -1$ and $x > 1$ and is decreasing for $-1 < x < 1$.

For concavity, need to find all points of inflection. Let $f''(x) = 0$.

$6x = 0$

$x = 0$

Test for concavity in the intervals between the possible points of inflection.

	$x < 0$	$x = 0$	$x > 0$
Test value	-1		1
$f''(x)$	$f''(-1) = -6$ Negative	0	$f''(1) = 6$ Positive
$f(x)$	Concave down	Point of Inflection $(0, 0)$	Concave up

Chapter 3 Review **Question 14 Page 205**

a) Follow the six-step plan.

$f(x) = -x^2 + 2x$

$f'(x) = -2x + 2$

$f''(x) = -2$

Step 1: Since this is a polynomial function, the domain is $\mathbb{R}$.

Step 2: $f(0) = 0$. The y-intercept is 0.

$0 = -x^2 + 2x$

$0 = x(x - 2)$

$x = 0, x = 2$

The x-intercepts are 0 and 2.

Step 3: Find the critical numbers.

$0 = -2x + 2$

$x = 1$

Use the second derivative test to classify the critical points.

$f''(1) = -2$

Therefore (1, 1) is a local maximum.

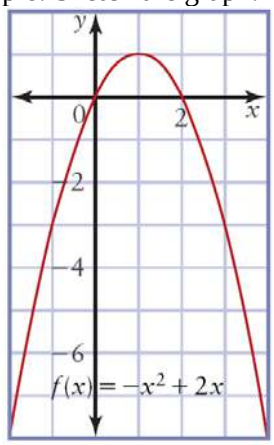
Step 4: Find the possible points of inflection.

Since $f''(x) - 2 \neq 0$ for any x, there are no points of inflection.

Step 5: From Step 3, $f(x)$ is decreasing for $x > 1$ and increasing for $x < 1$.

From Step 4, $f(x)$ is concave down for all x.

Step 6: Sketch the graph.



b) Follow the six-step plan.

$$k(x) = \frac{1}{4}x^4 - \frac{9}{2}x^2$$
$$k'(x) = x^3 - 9x$$
$$k''(x) = 3x^2 - 9$$

Step 1: Since this is a polynomial function, the domain is $\mathbb{R}$.

Step 2: $k(0) = 0$. The y-intercept is 0.

$$0 = \frac{1}{4}x^4 - \frac{9}{2}x^2$$
$$0 = x^2(x^2 - 18)$$
$$x = 0,\ x = \pm\sqrt{18}$$

The x-intercepts are 0 and $\pm 3\sqrt{2}$.

Step 3: Find the critical numbers.

$$0 = x^3 - 9x$$
$$0 = x(x^2 - 9)$$
$$x = 0,\ x = \pm 3$$

Use the second derivative test to classify the critical points.

$$k''(-3) = 18$$
$$k''(0) = -9$$
$$k''(3) = 18$$

Therefore (–3, –20.25) and (3, –20.25) are local minimum points and (0, 0) is a local maximum.

Step 4: Find the possible points of inflection.

$0 = 3x^2 - 9$

$x^2 = 3$

$x = \pm\sqrt{3}$

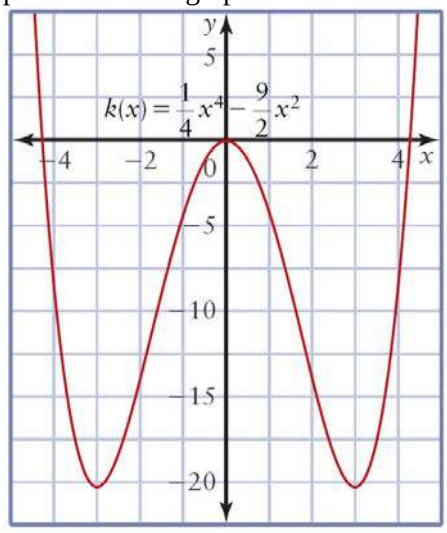
Test the intervals. Have already tested $x = 0, -3,$ and 3.

Therefore $\left(-\sqrt{6}, -18\right)$ and $\left(\sqrt{6}, -18\right)$ are points of inflection.

Step 5: From Step 3, $k(x)$ is decreasing for $x < -3$ and $0 < x < 3$ and increasing for $-3 < x < 0$ and $x > 3$.

From Step 4, $k(x)$ is concave down for $-\sqrt{6} < x < \sqrt{6}$ and concave up elsewhere.

Step 6: Sketch the graph.



c) Follow the six-step plan.

$h(x) = 2x^3 - 3x^2 - 3x + 2$

$h'(x) = 6x^2 - 6x - 3$

$h''(x) = 12x - 6$

Step 1: Since this is a polynomial function, the domain is $\mathbb{R}$.

Step 2: $h(0) = 2$. The y-intercept is 2.

$0 = 2x^3 - 3x^2 - 3x + 2$

$0 = 2(x^3 + 1) - 3x(x + 1)$

$0 = 2(x + 1)(x^2 - x + 1) - 3x(x + 1)$

$0 = (x + 1)(2x^2 - 2x + 2 - 3x)$

$0 = (x + 1)(2x^2 - 5x + 2)$

$0 = (x + 1)(2x - 1)(x - 2)$

$x = -1,\ x = \frac{1}{2},\ x = 2$

The x-intercepts are -1, $\frac{1}{2}$ and 2.

Step 3: Find the critical numbers.

$0 = 6x^2 - 6x - 3$

$0 = 3(2x^2 - 2x - 1)$

$x = \frac{2 \pm \sqrt{12}}{4}$

$x = \frac{1 \pm \sqrt{3}}{2}$

$x \approx -0.37,\ x \approx 1.37$

Use the second derivative test to classify the critical points.

$h''(-0.37) = -10.44$

$h''(1.37) = 10.44$

Therefore (–0.37, 2.60) is a local maximum point and (1.37, –2.60) is a local minimum.

Step 4: Find the possible points of inflection.

$0 = 12x - 6$

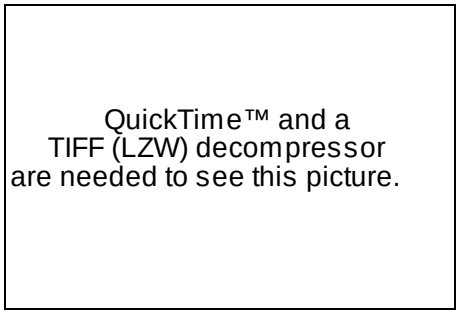
$x = \frac{1}{2}$

Have already tested the intervals around 0.5.
Therefore (0.5, 0) is a point of inflection.

Step 5: From ③, $h(x)$ is decreasing for $\frac{1-\sqrt{3}}{2} < x < \frac{1+\sqrt{3}}{2}$ and increasing for $x < \frac{1-\sqrt{3}}{2}$ and $x > \frac{1+\sqrt{3}}{2}$.

From ④, $h(x)$ is concave down for $x < 0$ and concave up for $x > 0$.

Step 6: Sketch the graph.



Chapter 3 Review **Question 15 Page 205**

a) Find the critical points.

$$C(t) = \frac{0.12t}{t^2 + 2t + 2}$$
$$= 0.12t(t^2 + 2t + 2)^{-1}$$
$$C'(t) = 0.12t(-1)(t^2 + 2t + 2)^{-2}(2t + 2) + (0.12)(t^2 + 2t + 2)^{-1}$$
$$= (t^2 + 2t + 2)^{-2}(-0.24t^2 - 0.24t + 0.12t^2 + 0.24t + 0.24)$$
$$= \frac{-0.12t^2 + 0.24}{(t^2 + 2t + 2)^2}$$
$$0 = -0.12t^2 + 0.24$$
$$0 = -0.12(t^2 - 2)$$
$$t = \pm\sqrt{2}$$

Since there is only one critical value in the interval $0 \le x \le 4$, test the function at the critical point and the endpoints.

$$C(0) = 0$$
$$C\left(\sqrt{2}\right) = 0.025$$
$$C(4) = 0.018$$

The maximum concentration of the drug during the interval $0 \le x \le 4$ is 0.025 mg/cm^3.

b) The maximum concentration occurs approximately 1.414 hours after administration of the drug.

Chapter 3 Review **Question 16 Page 205**

Use the variables suggested in the question.
Let S.A. be the surface area, the variable to be minimized.

$$\begin{aligned}\text{S.A.} &= 1(2w^2) + 2(2wh) + 2(wh) \\ &= 4w^2 + 6wh\end{aligned}$$

Volume is to be 2400 cm^2.

$$\begin{aligned}V &= 2w^2h \\ 2400 &= 2w^2h \\ h &= \frac{1200}{w^2}\end{aligned}$$

Simplify the surface area function.

$$\begin{aligned}\text{S.A.} &= 2w^2 + 6w\left(\frac{1200}{w^2}\right) \\ &= 2w^2 + 7200w^{-1}\end{aligned}$$

For a minimum, find the critical points.

$$\begin{aligned}\text{S.A.}' &= 4w - 7200w^{-2} \\ 0 &= 4w - 7200w^{-2} \\ 0 &= 4w^3 - 7200 \\ w^3 &= 1800 \\ w &= \sqrt[3]{1800}\end{aligned}$$

$w \approx 12.16$

Test values on either side of this point.

$$\text{S.A.}'(11) = -15.5$$

$$\text{S.A.}'(13) = 9.40$$

Since the slope changes sign, the point must be a minimum.

$$2w = 24.32$$

$$w = 12.16$$

$$\begin{aligned}h &= \frac{1200}{(12.16)^2} \\ &= 8.11\end{aligned}$$

The dimensions of the box will be 12.16 cm by 24.32 cm by 8.11 cm.

Chapter 3 Practice Test

Chapter 3 Practice Test Question 1 Page 206

B is the correct answer.

$$f'(x) = 2x - 8$$
$$0 = 2x - 8$$
$$x = 4$$
$$f'(0) = -8$$
$$f'(3) = -2$$

The only critical point is not in the given interval. The function is decreasing at both endpoints of the interval. Therefore it is always decreasing on the interval.

Chapter 3 Practice Test Question 2 Page 206

D is the correct answer.

A, B, and C are incorrect. The function is decreasing for all values of *x*.

Chapter 3 Practice Test Question 3 Page 206

C is the correct answer.

A: $(2, f(2))$ is a critical point because $f'(2) = 0$.
B: $(2, f(2))$ $(2, f(2))$ might be a turning point.
D: $(2, f(2))$ $(2, f(2))$ could be a local maximum (or a point of inflection) since f is increasing on the interval preceding $x = 2$. It certainly cannot be a local minimum.

Chapter 3 Practice Test Question 4 Page 206

C is the correct answer.

For an odd function, $f(-x) = -f(x)$.
Therefore:
$$f(-a) = -f(a)$$
$$= -5$$

Chapter 3 Practice Test **Question 5 Page 206**

C is the correct answer.

A: The graph has no x-intercepts since the numerator of the fraction cannot be zero.

B: $f(x) = -3(x-2)^{-2}$

$f'(x) = 6(x-2)^{-3}$

$f''(x) = -18(x-2)^{-4}$

$= \dfrac{-18}{(x-2)^4}$

The second derivative is always negative (the denominator is always positive). The curve is always concave down.

C: $f'(0) = \dfrac{6}{-8}$

$= -0.75$

This contradicts the statement $f'(x) > 0$ when $x < 2$.

D: If a number close to 2 is substituted for x, the fraction becomes –3 divided by a very small positive number, the result of which is a very large negative number.

Chapter 3 Practice Test **Question 6 Page 206**

Given $f'(x) = x(x-1)^2$, the graph of $f(x)$ has **two** critical points and **one** turning point.

$f'(x) = x(x-1)^2$

$0 = x(x-1)^2$

$x = 0,\ x = 1$

Check the intervals.

$f'(-1) = -4$

$f'(0.5) = 0.125$

$f'(2) = 2$

There is a minimum when $x = 0$ and a point of inflection when $x = 1$.

Chapter 3 Practice Test **Question 7 Page 206**

A corresponds to a local maximum since slopes for $f(x)$ will change from positive to negative at that point.

B corresponds to a point of inflection since slopes for $f(x)$ will not change at that point but will be negative before and after the point.

C corresponds to a local minimum since slopes for $f(x)$ will change from negative to positive at that point.

Chapter 3 Practice Test **Question 8 Page 206**

$$f(x) = x^3 - 5x^2 + 6x + 2$$
$$f'(x) = 3x^2 - 10x + 6$$
$$0 = 3x^2 - 10x + 6$$
$$x = \frac{10 \pm \sqrt{28}}{6}$$
$$x = \frac{5 \pm \sqrt{7}}{3}$$

$x \approx 2.5,\ x \approx 0.8$

Check values at the critical points and the endpoints of the interval.

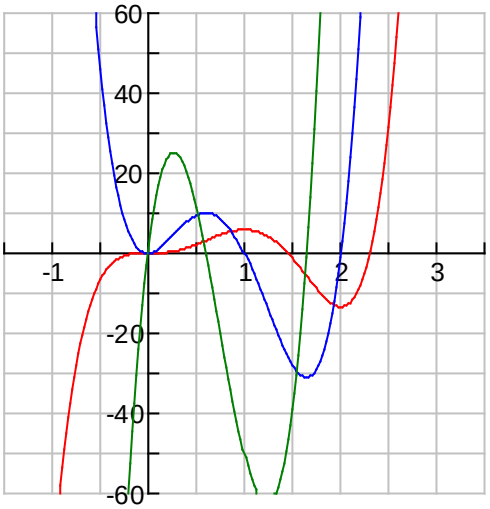
$$f(0) = 2$$
$$f\left(\frac{5-\sqrt{7}}{3}\right) \approx 4.1$$
$$f\left(\frac{5+\sqrt{7}}{3}\right) \approx 1.4$$
$$f(4) = 10$$

The absolute maximum is (4, 10) and the absolute minimum is (2.5, 1.4)

Chapter 3 Practice Test **Question 9 Page 206**

Answers may vary. For example:

$f(x)$ in red, $f'(x)$ in blue, and $f''(x)$ in green.
Note that zeros of $f'(x)$ correspond to local extrema on $f(x)$ and zeros of $f''(x)$ correspond to points of inflection on $f(x)$.



Chapter 3 Practice Test **Question 10 Page 207**

a) This is a quadratic polynomial function with leading coefficient +3. Would expect the limit to be $+\infty$ in both cases.

Substitute large numbers for x to check.

$f(100) = 284\,180\,000$

$f(-100) = 316\,018\,000$

Clearly $\lim\limits_{x\to-\infty} f(x) = \infty$ and $\lim\limits_{x\to\infty} f(x) = \infty$.

b) $f(x) = 3x^4 - 16x^3 + 18x^2$

$f'(x) = 12x^3 - 48x^2 + 36x$

$f''(x) = 36x^2 - 96x + 36$

Find the critical points.

$0 = 12x^3 - 48x^2 + 36x$

$0 = 12x(x^2 - 4x + 3)$

$0 = 12x(x-3)(x-1)$

$x = 0,\ x = 1,\ x = 3$

Test the critical points using the second derivative test.

$f''(0) = 36(0)^2 - 96(0) + 36$

$= 36$

$f''(1) = 36(1)^2 - 96(1) + 36$

$= -24$

$f''(3) = 36(3)^2 - 96(3) + 36$

$= 72$

There are local minima at (0, 0) and (3, –27) and a local maximum at (1, 5).

c) For possible points of inflection, set the second derivative to zero.

$0 = 36x^2 - 96x + 36$

$0 = 12\left(3x^2 - 8x + 3\right)$

$x = \dfrac{8 \pm \sqrt{64 - 36}}{6}$

$x = \dfrac{4 \pm \sqrt{7}}{3}$

$x \doteq 0.45,\ x \doteq 2.22$

Test values for the intervals created by these points have already been checked above.

There are two points of inflection at (0.45, 2.31) and (2.22, –13.48).

Chapter 3 Practice Test **Question 11 Page 207**

a) $C(x) = 0.1x^2 + 1.2x + 3.6$

$$U(x) = \frac{0.1x^2 + 1.2x + 3.6}{x}$$
$$= 0.1x + 1.2 + 3.6x^{-1}$$

b) To minimize unit cost, find the critical points for $U(x)$.

$$U'(x) = 0.1 - 3.6x^{-2}$$
$$0 = 0.1 - 3.6x^{-2}$$
$$0 = x^2 - 36$$
$$x = \pm 6$$

The negative value has no meaning in this situation. Check the values around $x = 6$.

$$U'(5) = 0.1 - 3.6(5)^{-2}$$
$$= -0.044$$
$$U'(7) = 0.1 - 3.6(7)^{-2}$$
$$= 0.027$$

There is a minimum when $x = 6$. The company should produce 6 ATVs per day to minimize the cost per unit.

Chapter 3 Practice Test **Question 12 Page 207**

a) Vertical asymptotes occur when the denominator in a rational function is zero. In this case, the vertical asymptote is $x = 0$.

b) Let $y = f(x)$.

$$f(x) = x^2 + \frac{1}{x^2}$$
$$= x^2 + x^{-2}$$
$$f'(x) = 2x - 2x^{-3}$$
$$f''(x) = 2 + 6x^{-4}$$

Find the critical points.

$$0 = 2x - 2x^{-3}$$
$$0 = x^4 - 1$$
$$x = \pm 1$$

Test using the second derivative test.

$f''(-1) = 8$

$f''(1) = 8$

There are local minima at (1, 2) and (–1, 2).

c) To find intervals of concavity, let $f''(x) = 0$

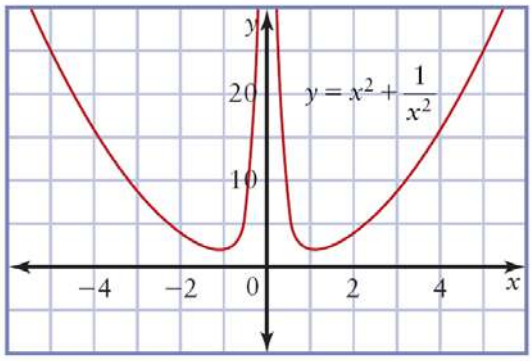
$0 = 2 + 6x^{-4}$

$0 = 2x^4 + 6$

$x^4 = -3$

There are no real roots to this equation.
The only value dividing regions of concavity is related to the asymptote.
Based on earlier tests, the function is concave up for $x < 0$ and also $x > 0$.

d)



Chapter 3 Practice Test **Question 13 Page 207**

a) i) $f'(0)$ B -3

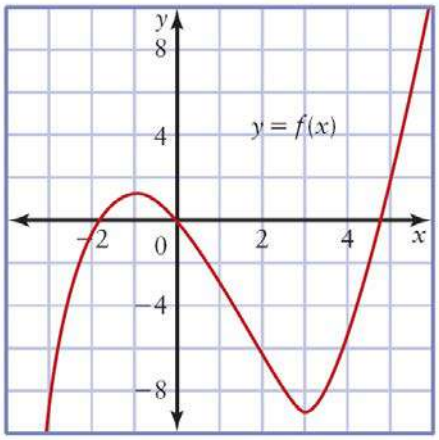
$f'(1)$ B -4

$f'(0) > f'(1)$

ii) Between $x = -1$ and $x = 3$, all the slopes for f are negative (below zero on the $f'(x)$ graph).
Therefore, the function is decreasing from $x = -1$ to $x = 3$ and $f(-1) > f(3)$.

iii) There are no critical numbers between 5 and 10 and all the slopes of f are positive in this interval.
Therefore, $f(10) > f(5)$

b) The sketch should have a maximum at $x = -1$ and, a minimum $x = 3$, and a point of inflection at x = 1.



Chapter 3 Practice Test **Question 14 Page 207**

Answers may vary. For example:

Since the factor in the denominator is squared, all of the values of the function will be positive. The curve will always be above the *x*-axis. Therefore, the limit as an asymptote is approached from either side must be $+\infty$

Chapter 3 Practice Test **Question 15 Page 207**

a) Let x be the number of \$10 reductions in price. The revenue function is the number of rooms rented times the price per room.

$$R(x) = (40 + 10x)(120 - 10x)$$
$$= -100x^2 + 800x + 4800$$

$$R'(x) = -200x + 800$$
$$0 = -200x + 800$$
$$x = 4$$

To maximize revenue, the hotel should decrease the price by four increments to \$80 per room.

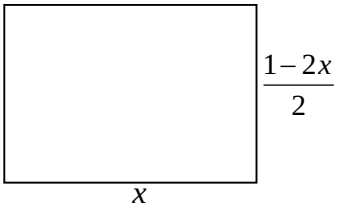
b) The revenue function is increasing until $x = 4$. If there are only 50 rooms, the hotel can only make one decrease of \$10. Compare the revenues.

$$R(0) = 4800$$
$$R(1) = 5500$$
$$R(4) = 6400$$

With 50 rooms, the maximum revenue is \$5500.

Chapter 3 Practice Test **Question 16 Page 207**

a)



Let the perimeter be 1 unit. The dimensions of the rectangle will be x and $\frac{1-2x}{2}$.

$$A(x) = x\left(\frac{1-2x}{2}\right)$$
$$= 0.5x - x^2$$

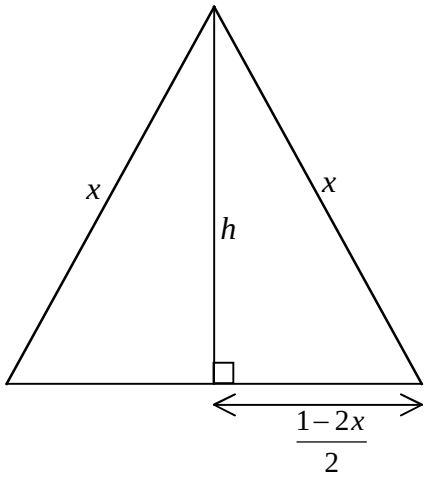
For a maximum area:

$$A'(x) = 0.5 - 2x$$
$$0 = 0.5 - 2x$$
$$x = 0.25$$

$$\frac{1-2(0.25)}{2} = 0.25$$

Therefore the dimensions are 0.25 units by 0.25 units.
The maximum area rectangle is a square.

b)



Assume that the triangle is isosceles.

Then for perimeter = 1,

$$x^2 = h^2 + \left(\frac{1-2x}{2}\right)^2$$

$$x^2 = h^2 + \frac{1}{4} - x + x^2$$

$$h = \sqrt{-\frac{1}{4} + x}$$

$$A(x) = \left(\frac{1}{2}\right)(1-2x)\sqrt{-\frac{1}{4}+x}$$

$$= (0.5-x)(x-0.25)^{\frac{1}{2}}$$

$$A'(x) = (-1)(x-0.25)^{\frac{1}{2}} + (0.5-x)\left(\frac{1}{2}\right)(x-0.25)^{-\frac{1}{2}}$$

For a maximum, let $A'(x) = 0$.

$$(-1)(x-0.25)^{\frac{1}{2}} + (0.5-x)\left(\frac{1}{2}\right)(x-0.25)^{-\frac{1}{2}} = 0$$

$-(x-0.25) + (0.5-x)0.5 = 0$ Multiply by $(x-0.25)^{\frac{1}{2}}$.

$0.25 - x + 0.25 - 0.5x = 0$

$x = \frac{1}{3}$

Therefore each side must be $\frac{1}{3}$ of the perimeter.

c) For a given perimeter, the octagon will enclose more area than the pentagon. The circle is the most efficient shape for a given perimeter in two dimensions and the octagon is closer to the circle shape than the pentagon would be. Again, the proof of these statements is beyond the scope of this course, but an investigation with specific numbers could confirm them.

d) In general, a circle or a square will enclose a maximum area for a perimeter.

Chapter 3 Practice Test Question 17 Page 207

a) For a maximum yield, let the derivative equal zero.

$$B(n) = -0.1n^2 + 10n$$

$$B'(n) = -0.2n + 10$$

$$0 = -0.2n + 10$$

$$n = 50$$

They should plant 50 000 seeds per acre for a maximum yield.

b) Create a profit function.

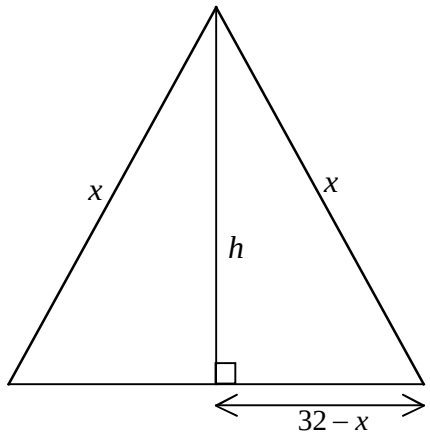
$$P(n) = 3(-0.1n^2 + 10n) - 2n$$
$$= -0.3n^2 + 28n$$

For a maximum profit, let the derivative equal zero.

$$P'(n) = -0.6n + 28$$
$$0 = -0.6n + 28$$
$$n = 46.667$$

For maximum profit, they should plant about 47 000 seeds per acre.

Chapter 3 Practice Test **Question 18 Page 207**



Let the two equal sides of the isosceles triangle have length x cm.
In the right triangle:

$$x^2 = h^2 + (32 - x)^2$$
$$h = \sqrt{-1024 + 64x}$$

The area of the isosceles triangle is:

$$A(x) = \left(\frac{1}{2}\right)(64 - 2x)\sqrt{64x - 1024}$$
$$= (32 - x)\sqrt{64x - 1024}$$

For a maximum:

$$A'(x) = (-1)\sqrt{64x - 1024} + (32 - x)\left(\frac{1}{2}\right)(64x - 1024)^{-\frac{1}{2}}(64)$$

$$A'(x) = -\sqrt{64x - 1024} + (1024 - 32x)(64x - 1024)^{-\frac{1}{2}}$$

$$0 = -\sqrt{64x - 1024} + (1024 - 32x)(64x - 1024)^{-\frac{1}{2}}$$

$$0 = -64x + 1024 + 1024 - 32x$$

$$96x = 2048$$

$$x = \frac{64}{3}$$

$$x \approx 21.33$$

For a maximum area the three sides need to have the same length of approximately 21.33 cm.

$\sqrt{}$

$\sqrt{}$

Chapter 1 to 3 Review

Chapter 1 to 3 Review **Question 1 Page 208**

a) $\frac{h(20)-h(10)}{20-10}=\frac{0.04-0.82}{10}$

$\square$ $\&-0.078$

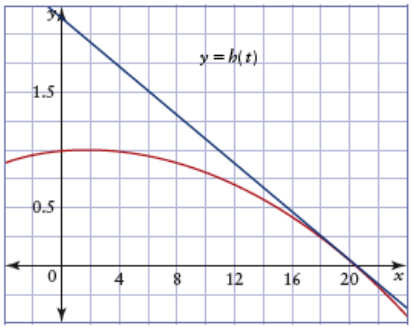
The average rate of change of the height is –0.078 m/s.
This is the average vertical velocity between 10 s and 20 s.

b) $\frac{h(20+0.1)-f(20)}{0.1}=\frac{0.02\,917-0.04}{0.1}$

$\square$ $\&-0.11$

The instantaneous rate of change at t = 20 s is approximately –0.11 m/s.
This is the instantaneous vertical velocity at t = 20 s.

c)



Chapter 1 to 3 Review **Question 2 Page 208**

a) $t_1=\frac{3}{2};\hat{I}_2=\frac{6}{5};\hat{I}_3=\frac{9}{10};\hat{I}_4=\frac{12}{17};\hat{I}_5=\frac{15}{26}$

b) Yes, the limit is zero. As x increases, the denominator increases faster than the numerator so the sequence approaches 0.

Chapter 1 to 3 Review **Question 3 Page 208**

The limits as x approaches 5 from the left and right sides are both equal to –2, so the limit as x approaches 5 exists and equals –2. The value of the function at x = 5 is not –2, so there is a removable discontinuity at this point. The function is discontinuous at x = 5.

Chapter 1 to 3 Review **Question 4 Page 208**

a) $\lim\limits_{x\to 3}(-2x^3+x^2-4x+7)=-50$

b) $\lim\limits_{x\to 4}\dfrac{x^2-16}{x-4}=\lim\limits_{x\to 4}\dfrac{(x-4)(x+4)}{x-4}$

$=\lim\limits_{x\to 4}(x+4)$

$=8$

c) $\lim\limits_{x\to 0}\dfrac{3x-5}{7+4x^3}=-\dfrac{5}{7}$

Chapter 1 to 3 Review **Question 5 Page 208**

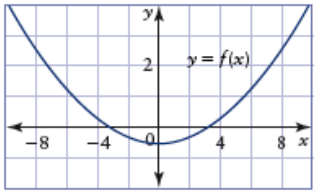
a) $\{x \mid x\neq 3, x\in \text{°}\}$

b) $\lim\limits_{x\to 3^+} f(x)=\infty$

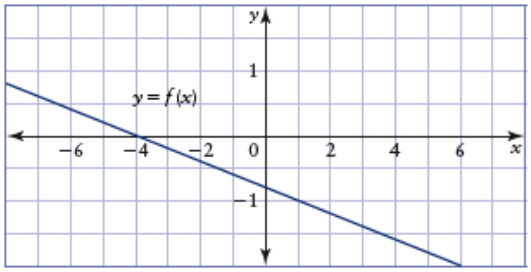
c) No. The graph is discontinuous at $x = 3$, since there is a vertical asymptote at this point.

Chapter 1 to 3 Review **Question 6 Page 208**

a)



b)



Chapter 1 to 3 Review **Question 7 Page 208**

a) $$\frac{dy}{dx} = \lim_{h\to 0} \frac{\frac{2}{3}(x+h)^3 - 8(x+h) - \left(\frac{2}{3}x^3 - 8x\right)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{2}{3}(x^3 + 3x^2h + 3xh^2 + h^3) - 8(x+h) - \frac{2}{3}x^3 + 8x}{h}$$

$$= \lim_{h\to 0} \frac{2x^2h + 2xh^2 + \frac{2}{3}h^3 - 8h}{h}$$

$$= \lim_{h\to 0} (2x^2 + 2xh + \frac{2}{3}h^2 - 8)$$

$$= 2x^2 - 8$$

b) $$\frac{dy}{dx} = \lim_{h\to 0} \frac{\sqrt{2(x+h)-1} - \sqrt{2x-1}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{2(x+h)-1} - \sqrt{2x-1}\right)\left(\sqrt{2(x+h)-1} + \sqrt{2x-1}\right)}{h\left(\sqrt{2(x+h)-1} + \sqrt{2x-1}\right)}$$

$$= \lim_{h\to 0} \frac{2(x+h)-1-(2x-1)}{h\left(\sqrt{2(x+h)-1} + \sqrt{2x-1}\right)}$$

$$= \lim_{h\to 0} \frac{2}{\sqrt{2(x+h)-1} + \sqrt{2x-1}}$$

$$= \frac{2}{2\sqrt{2x-1}}$$

$$= \frac{1}{\sqrt{2x-1}}$$

c) $$\frac{dy}{dx} = \lim_{h\to 0} \frac{3(x+h)^2 - (x+h) + 1 - (3x^2 - x + 1)}{h}$$

$$= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 - x - h + 1 - 3x^2 + x - 1}{h}$$

$$= \lim_{h\to 0} \frac{6xh + 3h^2 - h}{h}$$

$$= \lim_{h\to 0} (6x + 3h - 1)$$

$$= 6x - 1$$

d) $f'(t) = \lim\limits_{h \to 0} \dfrac{\dfrac{-2(t+h)}{(t+h)+3} - \left(\dfrac{-2t}{t+3}\right)}{h}$

$$= \lim_{h \to 0} \frac{\dfrac{(t+3)(-2t-2h)+2t(t+h+3)}{(t+h+3)(t+3)}}{h}$$

$$= \lim_{h \to 0} \frac{-6h}{h(t+h+3)(t+3)}$$

$$= \lim_{h \to 0} \frac{-6}{(t+h+3)(t+3)}$$

$$= -\frac{6}{(t+3)^2}$$

Chapter 1 to 3 Review **Question 8 Page 208**

a) $\dfrac{dy}{dx} = 24x^3 - 15x^2 + 3$

b) $\dfrac{dy}{dx} = 1(4t-7) + (3+t)(4)$

$$= 8t + 5$$

c) $g'(x) = 4(-2x^3+3)^3(-6x^2)$

$$= -24x^2(-2x^3+3)^3$$

d) $s'(t) = \dfrac{(5t-1)^{\frac{1}{2}}(1) - t\left(\dfrac{1}{2}\right)(5t-1)^{-\frac{1}{2}}(5)}{5t-1}$

$$= \frac{1}{(5t-1)^{\frac{1}{2}}} - \frac{5t}{2(5t-1)^{\frac{3}{2}}}$$

$$= \frac{2(5t-1)-5t}{2(5t-1)^{\frac{3}{2}}}$$

$$= \frac{5t-2}{2(5t-1)^{\frac{3}{2}}}$$

Chapter 1 to 3 Review **Question 9 Page 208**

a) $\frac{dy}{dx} = \frac{(x+3)^2(-4) + 4x(2)(x+3)}{(x+3)^4}$

$= \frac{-4x^2 - 24x - 36 + 8x^2 + 24x}{(x+3)^4}$

$= \frac{4x^2 - 36}{(x+3)^4}$

$= \frac{4(x+3)(x-3)}{(x+3)^4}$

$= \frac{4(x-3)}{(x+3)^3}$

The slope of the tangent at $x = -2$ is $\left.\frac{dy}{dx}\right|_{x=-2}$

$\left.\frac{dy}{dx}\right|_{x=-2} = \frac{4(-2-3)}{(-2+3)^3}$

$= -20$

When $x = -2$, $y = 8$.
Use the point $(-2, 8)$ and $m = -20$ to find b in the equation of the tangent, $y = mx + b$.
$8 = -20(-2) + b$
$b = -32$

The equation of the tangent at $x = -2$ is $y = -20x - 32$.

b) The slope of the tangent at $x = -1$ is $\left.\frac{dy}{dx}\right|_{x=-1}$

$\left.\frac{dy}{dx}\right|_{x=-1} = \frac{4(-1-3)}{(-1+3)^3}$

$= -2$

The normal is perpendicular to the tangent, so the slope of the normal is $-\left(\frac{1}{-2}\right) = \frac{1}{2}$.

When $x = -1$, $y = 1$.

Use the point $(-1, 1)$ and $m = \frac{1}{2}$ to find b in the equation of the normal, $y = mx + b$.

$1 = \frac{1}{2}(-1) + b$

$b = \frac{3}{2}$

The equation of the normal at $x = -1$ is $y = \frac{1}{2}x + \frac{3}{2}$.

Chapter 1 to 3 Review **Question 10 Page 208**

a) $y - x + 2 = 0$

$y = x - 2$

The slope of the tangent lines is perpendicular to the line above, so the tangents have a slope of -1.

Now, find $f'(x)$ and the values of x where $f'(x) = -1$.

$f'(x) = 3x^2 + 4x$

$-1 = 3x^2 + 4x$

$0 = 3x^2 + 4x + 1$

$0 = (3x + 1)(x + 1)$

$x = -\frac{1}{3}, x = -1$

Therefore, the points are $\left(-\frac{1}{3}, \frac{5}{27}\right)$ and $(-1, 1)$.

b) The slope of the tangents is -1.

Use the point $\left(-\frac{1}{3}, \frac{5}{27}\right)$ and $m = -1$ to find b in the equation of the first tangent, $y = mx + b$.

$\frac{5}{27} = -\left(-\frac{1}{3}\right) + b$

$b = -\frac{4}{27}$

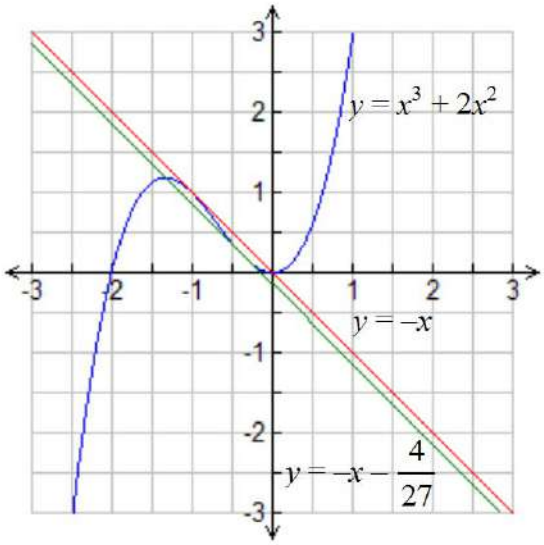
Use the point $(-1, 1)$ and $m = -1$ to find b in the equation of the second tangent, $y = mx + b$.

$1 = -1(-1) + b$

$b = 0$

Therefore, the equation of the first tangent is $y = -x - \frac{4}{27}$ and the equation of the second tangent is $y = -x$.

c) Sketch the graph of the function and the tangent lines.



Chapter 1 to 3 Review **Question 11 Page 209**

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= (5 - 4u + 3u^2)\left(\frac{1}{2}(2-x)^{-\frac{1}{2}}(-1)\right)$$

$$= \left(5 - 4\sqrt{2-x} + 3(2-x)\right)\left(\frac{-1}{2\sqrt{2-x}}\right)$$

$$= \left(11 - 4\sqrt{2-x} - 3x\right)\left(\frac{-1}{2\sqrt{2-x}}\right)$$

$$\left.\frac{dy}{dx}\right|_{x=-2} = \left(11 - 4\sqrt{2-(-2)} - 3(-2)\right)\left(\frac{-1}{2\sqrt{2-(-2)}}\right)$$

$$= 2.25$$

Chapter 1 to 3 Review **Question 12 Page 209**

a) i) Graph 1; Velocity and acceleration are derivatives of position.

ii) Graph 2; Velocity is the derivative of position, which is cubic, so the velocity function is quadratic.

iii) Graph 3; Acceleration is the derivative of velocity, which is quadratic, so the acceleration function is linear.

b) The object is slowing down when $v(t) \times a(t) < 0$, so over the intervals $x < 1$ and $2 < x < 3$.
The object is speeding up when $v(t) \times a(t) > 0$, so over the intervals $1 < x < 2$ and $x > 3$.

c) If the slope of the position function is moving towards zero then the object is slowing down. If the slope is moving away from zero then the object is speeding up.

Chapter 1 to 3 Review **Question 13 Page 209**

a) $V(t) = (3.72 - 0.02t^2)(13.00 + 0.21t)$
$= -0.0042t^3 - 0.26t^2 + 0.7812t + 48.36$

b) $V'(t) = -0.0126t^2 - 0.52t + 0.7812$

This is the change of voltage as time changes.

c) i) $V'(3) = -0.0126(3)^2 - 0.52(3) + 0.7812$
$= -0.8922$

The rate of change of voltage at $t = 3$ s is –0.8922 V/s.

ii) $I'(t) = -0.04t$
$I'(3) = -0.04(3)$
$= -0.12$

The rate of change of current at $t = 3$ s is –0.12 A/s.

iii) $R'(t) = 0.21$
$R'(3) = 0.21$

The rate of change of resistance at $t = 3$ s is 0.21 Ω/s.

Chapter 1 to 3 Review **Question 14 Page 209**

a) $C'(x) = -0.002x + 1.5$
$C'(300) = 0.90$

The marginal cost is $0.90 per gadget.

b) $C(301) - C(300) = -0.001(301)^2 + 1.5(301) + 500 - (-0.001(300)^2 + 1.5(300) + 500)$
$= 0.899$

The actual cost of producing the 301st gadget is $0.90.

c) $R(x) = xp(x)$

$= 4.5x - 0.1x^2$

$R'(x) = 4.5 - 0.2x$

$R'(300) = 4.5 - 0.2(300)$

$= -55.50$

The marginal revenue is –$55.50 per gadget.

$P(x) = R(x) - C(x)$

$= -0.099x^2 + 3x - 500$

$P'(x) = -0.198x + 3$

$P'(300) = 3 - 0.198(300)$

$= -56.40$

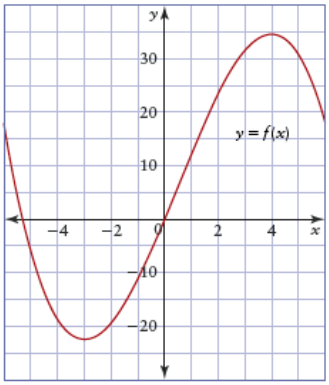
The marginal profit is –$56.40 per gadget.

Chapter 1 to 3 Review **Question 15 Page 209**

a) i) $f'(2)$ since $f'(-2) = 6$ and $f'(2) = 10$.

ii) $f(6)$ since f decreases from $x = 6$ to $x = 12$.

b)



Chapter 1 to 3 Review **Question 16 Page 209**

a) $f'(x) = 4x^3 - 12x^2$

$0 = 4x^2(x-3)$

$x = 0, x = 3$

When $x < 0$, $f'(x) < 0$.
When $0 < x < 3$, $f'(x) < 0$.
When $x > 3$, $f'(x) > 0$.

Therefore, (0, 0) is a point of inflection and (3, –27) is a local minimum.

b) $\frac{dy}{dx} = 6x^2 - 6x - 11$

$0 = 6x^2 - 6x - 11$

$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(6)(-11)}}{2(6)}$

$x = \frac{3 \pm 5\sqrt{3}}{6}$

$x \approx 1.94, x \approx -0.94$

When $x < -0.94$, $f'(x) > 0$.
When $-0.94 < x < 1.94$, $f'(x) < 0$.
When $x > 1.94$, $f'(x) > 0$.

Therefore, (–0.94, 12) is a local maximum and (1.94, –12) is a local minimum.

c) $h'(x) = 4x^3 - 15x^2 + 2x + 21$

$0 = (x-3)(4x^2 - 3x - 7)$

$0 = (x-3)(4x-7)(x+1)$

$x = -1, x = \frac{7}{4}, x = 3$

When $x < -1$, $f'(x) < 0$.

When $-1 < x < \frac{7}{4}$, $f'(x) > 0$.

When $\frac{7}{4} < x < 3$, $f'(x) < 0$.

When $x > 3$, $f'(x) > 0$.

Therefore, (–1, –32) is a local minimum, $\left(\frac{7}{4}, \frac{1125}{256}\right)$ is a local maximum, and (3, 0) is a local minimum.

d) $g'(x) = 12x^3 - 24x^2 - 12x + 24$

$0 = (x-1)(12x^2 - 12x - 24)$

$0 = 12(x-1)(x-2)(x+1)$

$x = -1, x = 1, x = 2$

When $x < -1$, $f'(x) < 0$.
When $-1 < x < 1$, $f'(x) > 0$.
When $1 < x < 2$, $f'(x) < 0$.
When $x > 2$, $f'(x) > 0$.

Therefore, (–1, –28) is a local minimum, (1, 4) is a local maximum, and (2, –1) is a local minimum.

Chapter 1 to 3 Review **Question 17 Page 209**

a) Follow the six-step plan.

$f(x) = x^3 - 9x^2 + 15x + 4$

$f'(x) = 3x^2 - 18x + 15$

$f''(x) = 6x - 18$

Step 1:Since this is a polynomial function, the domain is $\mathbb{R}$.

Step 2: $f(0) = 4$. The y-intercept is 4.
Since $x^3 - 9x^2 + 15x + 4 = 0$ is not factorable, do not look for x-intercepts at this time.

Step 3: Find the critical numbers.

$3x^2 - 18x + 15 = 0$

$3(x^2 - 6x + 5) = 0$

$3(x-5)(x-1) = 0$

$x = 5, x = 1$

Use the second derivative test to classify the critical points.

$f''(1) = -12$

$f''(5) = 12$

Therefore (5, –21) is a local minimum point and (1, 11) is a local maximum.

Step 4: Find the possible points of inflection.

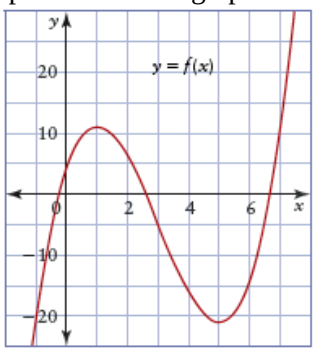
$6x - 18 = 0$

$x = 3$

Have already tested the intervals around $x = 3$.
Therefore (3, –5) is a point of inflection.

Step 5: From Step 3, f is decreasing for $1 < x < 5$ and increasing for $x < 1$ and $x > 5$.
From Step 4, f is concave down for $x < 3$ and concave up for $x > 3$.

Step 6: Sketch the graph.



b) Follow the six-step plan.

$f(x) = x^3 - 12x^2 + 36x + 5$

$f'(x) = 3x^2 - 24x + 36$

$f''(x) = 6x - 24$

Step 1: Since this is a polynomial function, the domain is $\mathbb{R}$.

Step 2: $f(0) = 5$. The y-intercept is 5.

Since $x^3 - 12x^2 + 36x + 5 = 0$ is not factorable, do not look for x-intercepts at this time.

Step 3: Find the critical numbers.

$3x^2 - 24x + 36 = 0$

$3(x^2 - 8x + 12) = 0$

$3(x - 6)(x - 2) = 0$

$x = 2, x = 6$

Use the second derivative test to classify the critical points.

$f''(2) = -12$

$f''(6) = 12$

Therefore (6, 5) is a local minimum point and (2, 37) is a local maximum.

④ Find the possible points of inflection.

$6x - 24 = 0$

$x = 4$

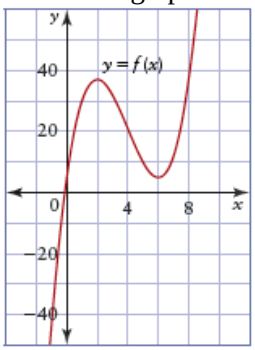
Have already tested the intervals around $x = 4$.

Therefore (4, 21) is a point of inflection.

⑤ From ③, f is decreasing for $2 < x < 6$ and increasing for $x < 2$ and $x > 6$.

From ④, f is concave down for $x < 4$ and concave up for $x > 4$.

⑥ Sketch the graph.



c) Follow the six-step plan.

$$f(x) = \frac{2x}{x^2 - 5x + 4}$$
$$= 2x(x^2 - 5x + 4)^{-1}$$

$$f'(x) = 2(x^2 - 5x + 4)^{-1} + 2x(-1)(x^2 - 5x + 4)^{-2}(2x - 5)$$
$$= (x^2 - 5x + 4)^{-2}(2x^2 - 10x + 8 - 4x^2 + 10x)$$
$$= (-2x^2 + 8)(x^2 - 5x + 4)^{-2}$$

$$f''(x) = (-4x)(x^2 - 5x + 4)^{-2} + (-2x^2 + 8)(-2)(x^2 - 5x + 4)^{-3}(2x - 5)$$
$$= (x^2 - 5x + 4)^{-3}(-4x^3 + 20x^2 - 16x + 8x^3 - 20x^2 - 32x + 80)$$
$$= (4x^3 - 48x + 80)(x^2 - 5x + 4)^{-3}$$

① Since this is a rational function, the domain is $\square$ except where there are vertical asymptotes.

$$x^2 - 5x + 4 = 0$$
$$(x - 4)(x - 1) = 0$$
$$x = 4, x = 1$$

There are vertical asymptotes at $x = 1$ and $x = 4$.

② $f(0) = 0$. The y-intercept is 0.
When $y = 0$, $x = 0$ is the only solution. The x-intercept is 0.

③ Find the critical numbers.

$$0 = (-2x^2 + 8)(x^2 - 5x + 4)^{-2}$$
$$0 = -2x^2 + 8$$
$$x^2 = 4$$
$$x = 2, x = -2$$

Use the second derivative test to classify the critical points.
$f''(2) = -2$
$f''(-2) = 0.003$
Therefore (–2, –0.22) is a local minimum point and (2, –2) is a local maximum.

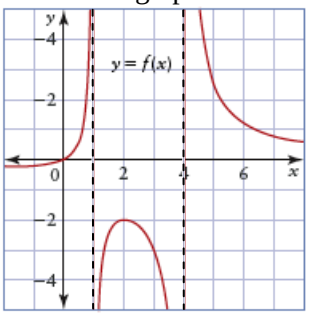
④ Find the possible points of inflection.

$$4x^3 - 48x + 80(x^2 - 5x + 4)^{-3} = 0$$
$$4x^3 - 48x + 80 = 0$$
$$4(x^3 - 12x + 20) = 0$$

This cubic equation does have a root but it is not readily factorable. There will be at least one point of inflection.

⑤ From ③, f is decreasing for $2 < x < 4$ and increasing for $1 < x < 2$.
From ④, f is concave down for $1 < x < 4$.

⑥ Sketch the graph.



d) Follow the six-step plan.

$$f(x) = 3x^3 + 7x^2 + 3x - 1$$
$$f'(x) = 9x^2 + 14x + 3$$
$$f''(x) = 18x + 14$$

① Since this is a polynomial function, the domain is $\mathbb{R}$.

② $f(0) = -1$. The y-intercept is –1.
For x-intercepts, let $y = 0$.

$$3x^3 + 7x^2 + 3x - 1 = 0$$

Try factoring using the factor theorem.
$f(1) = 12$
$f(-1) = 0$

Therefore, $(x + 1)$ is a factor.

$$(x + 1)(3x^2 + 4x - 1) = 0$$

$x = -1$ and $x = \dfrac{-4 \pm \sqrt{28}}{6}$ or $x \doteq -1.5$, $x \doteq 0.2$

There are x-intercepts at –1.5, –1, and 0.2.

③ Find the critical numbers.

$0 = 9x^2 + 14x + 3$

$x = \dfrac{-14 \pm \sqrt{88}}{18}$

$x \doteq -1.3,\ x \doteq 0.3$

Use the second derivative test to classify the critical points.

$f''(-0.3) = 8.6$

$f''(-1.3) = -9.4$

Therefore (–0.3, –1.4) is a local minimum point and (–1.3, 0.4) is a local maximum.

④ Find the possible points of inflection.

$18x + 14 = 0$

$x = -\dfrac{7}{9}$

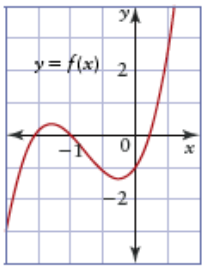
Have already tested the intervals around $x = -\dfrac{7}{9}$.

Therefore (–0.8, –0.5) is a point of inflection.

⑤ From ③, f is decreasing for $-1.3 < x < -0.3$ and increasing for $x < -1.3$ and $x > -0.3$.

From ④, f is concave down for $x < -\dfrac{7}{9}$ and concave up for $x > -\dfrac{7}{9}$.

⑥ Sketch the graph.



e) Follow the six-step plan.

$f(x) = x^4 - 5x^3 + x^2 + 21x - 18$

$f'(x) = 4x^3 - 15x^2 + 2x + 21$

$f''(x) = 12x^2 - 30x + 2$

① Since this is a polynomial function, the domain is $\mathbb{R}$.

② $f(0) = -18$. The y-intercept is –18.

For x-intercepts, let $y = 0$.

$x^4 - 5x^3 + x^2 + 21x - 18 = 0$

Try factoring using the factor theorem.
$f(1) = 0$

Therefore, $(x - 1)$ is a factor.
$(x-1)(x^3 - 4x^2 - 3x + 18) = 0$

Try factoring again using the factor theorem.
$f(3) = 0$

Therefore, $(x - 3)$ is a factor.
$(x-1)(x-3)(x^2 - x - 6) = 0$
$(x-1)(x-3)(x-3)(x+2) = 0$
$x = 1, x = 3, x = -2$

There are x-intercepts at –2, 1, and 3.

③ Find the critical numbers.
$4x^3 - 15x^2 + 2x + 21 = 0$

Use the factor theorem.
$f(-1) = 0$

Therefore, $(x + 1)$ is a factor.
$(x+1)(4x^2 - 19x + 21) = 0$
$(x+1)(x-3)(4x-7) = 0$
$x = -1, x = 3, x = 1.75$

Use the second derivative test to classify the critical points.
$f''(-1) = 44$
$f''(3) = 20$
$f''(1.75) = -13.75$

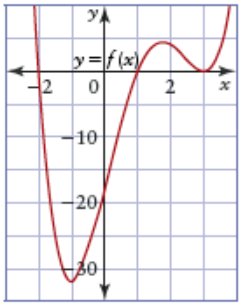
Therefore (–1, –32) and (3, 0) are local minimum points and (1.75, 4.4) is a local maximum.

④ Find the possible points of inflection.
$0 = 12x^2 - 30x + 2$
$0 = 6x^2 - 15x + 1$
$x = \dfrac{15 \pm \sqrt{201}}{12}$
$x \doteq 0.1, x \doteq 2.4$

Have already tested the intervals around these values.
Therefore (0.1, –16.6) and (2.4, 2.1) are points of inflection.

⑤ From ③, f is decreasing for $x < -1$ and $1.75 < x < 3$ and increasing for $1 < x < 1.75$ and $x > 3$.
From ④, f is concave down for $0.1 < x < 2.4$ and concave up for $x < 0.1$ and $x > 2.4$.

⑥ Sketch the graph.



Chapter 1 to 3 Review **Question 18 Page 209**

The volume of a cylinder is $V = \pi r^2 h$

Use $V = 900$ to find an equation for h in term of r.

$$900 = \pi r^2 h$$

$$h = \frac{900}{\pi r^2}$$

The surface area of a cylinder is S.A. $= 2\pi r^2 + 2\pi rh$.

Substitute the equation for h into S.A. to find the equation for the surface area in terms of r.

$$\text{S.A.} = 2\pi r^2 + 2\pi r\left(\frac{900}{\pi r^2}\right)$$

$$= 2\left(\pi r^2 + \frac{900}{r}\right)$$

The metal costs \$15.50 per m^2.

$$\frac{15.50}{(1)^2} = \frac{15.50}{(100)^2}$$

$$= 0.00155$$

The cost is \$0.00155/cm^2.

The function of cost is:

$$C(r) = 0.00155(\text{SA})$$

$$= 0.0031\left(\pi r^2 + \frac{900}{r}\right)$$

Find $C'(r)$ and set it to zero to find the radius at which cost will be optimized.

$$C'(r) = 0.0031\left(2\pi r - \frac{900}{r^2}\right)$$

$$0 = 2\pi r^3 - 900$$

$$r = \sqrt[3]{\frac{450}{\pi}}$$

The cost will be minimized when $r = \sqrt[3]{\frac{450}{\pi}}$ cm and $h = 2\sqrt[3]{\frac{450}{\pi}}$ cm.

$$C\left(\sqrt[3]{\frac{450}{\pi}}\right) = 0.0031\left(\pi\left(\sqrt[3]{\frac{450}{\pi}}\right)^2 + \frac{900}{\left(\sqrt[3]{\frac{450}{\pi}}\right)}\right)$$

$$\approx 0.0031(86.005 + 172.011)$$

$$\approx 0.80$$

The cost of making the can is \$0.80.

Chapter 1 to 3 Review **Question 19 Page 209**

For the field to be rectangular, it must be split up into 4 equal rectangles side by side.
Let *l* represent the length of the whole field and *w* represent the width.
Let *P* represent the amount of fence needed.

$$P = 2l + 5w$$

$$6000 = 2l + 5w$$

$$l = \frac{6000 - 5w}{2}$$

The area of the whole field is:

$$A = l \times w$$

$$= \left(\frac{6000 - 5w}{2}\right) w$$

$$= 3000w - \frac{5}{2}w^2$$

Find $A'(w)$ and set it to zero to find the width of the field that will optimize area.

$$A'(w) = 3000 - 5w$$

$$0 = 3000 - 5w$$

$$w = 600$$

The width of the field is 600 m, which is also the width of each plot of land.

$$\frac{6000 - 5(600)}{2} = 1500$$

The length of the whole field is 1500 m.

Since the field is split up into four congruent plots of land, each side length is 375 m.
Therefore, the dimensions of the land are 375 m by 600 m.

Chapter 4 **Derivatives of Sinusoidal Functions**

Chapter 4 Prerequisite Skills

Chapter 4 Prerequisite Skills **Question 1 Page 212**

a) $360° = 2\pi$ rad

b) $90° = \frac{\pi}{2}$ rad

c) $-45° = -\frac{\pi}{4}$ rad

d) $29.5° = \frac{59\pi}{360}$ rad

e) $115° = \frac{23\pi}{36}$ rad

f) $240° = \frac{4\pi}{3}$ rad

Chapter 4 Prerequisite Skills **Question 2 Page 212**

a) $a = 5\pi$
$\doteq 15.7$

The arc length is approximately 15.7 cm.

b) $a = 5(2.0)$
$= 10.0$

The arc length is 10.0 cm.

c) $a = 5\left(\frac{\pi}{3}\right)$
$\doteq 5.2$

The arc length is $\frac{5\pi}{3}$ cm or approximately 5.2 cm.

d) $a = 5\left(\frac{11.4}{360}\right)(2\pi)$
$\doteq 1.0$

The arc length is approximately 1.0 cm.

e) $a = \frac{5\pi}{2}$

The arc length is $\frac{5\pi}{2}$ cm or approximately 7.85 cm.

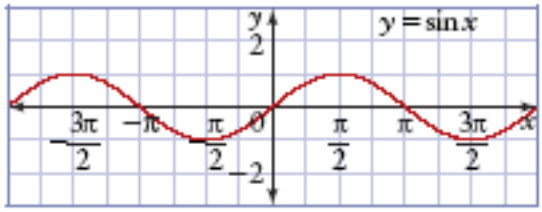
f) $a = 5\left(\frac{173}{360}\right)(2\pi)$

$= \frac{173\pi}{36}$

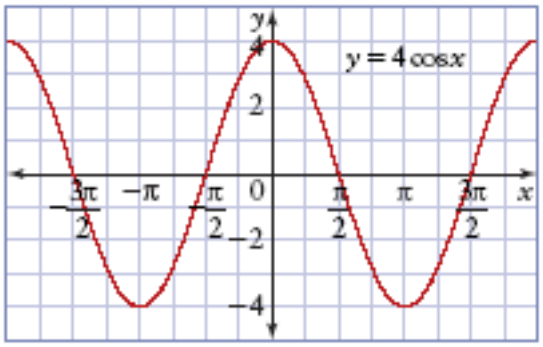
The arc length is $\frac{173\pi}{36}$ cm or approximately 15.10 cm.

Chapter 4 Prerequisite Skills **Question 3 Page 212**

a)



b)



Chapter 4 Prerequisite Skills **Question 4 Page 212**

a) amplitude: 1
period: 2π

b) amplitude: 4
period: 2π

Chapter 4 Prerequisite Skills **Question 5 Page 212**

a) The graph of $f(x) = \cos x$ is horizontally compressed by a factor of 2 and vertically stretched by a factor of 3 to obtain the graph of $y = 3f(2x)$.

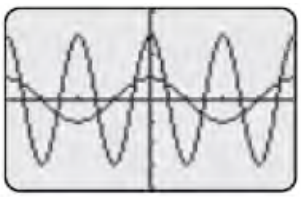
b) i) The minimum value is $3(-1) = -3$.

ii) The maximum value is $3(1) = 3$.

c) i) $\{x \mid x = k\pi, k \in \mathbb{Z}\}$

ii) $\{x \mid x = k\pi + \frac{\pi}{2}, k \in \mathbb{Z}\}$

d)



Chapter 4 Prerequisite Skills **Question 6 Page 212**

a) Graph A: maximum value 3, minimum value –3.
Graph B: maximum value 3, minimum value 1.

b) Answers will may vary. For example:

Graph A: $y = 3\sin\left(x - \frac{\pi}{2}\right)$

Graph B: $y = \cos\left(x - \frac{\pi}{2}\right) + 2$

c) Answers will vary. For example:

Sine and cosine functions are periodic, so there are many possible solutions.

Graph A: $y = 3\sin\left(x + \frac{(2k+1)\pi}{2}\right)$ for $k \in \mathbb{Z}$ has the same graph.

Graph B: $y = \cos\left(x + \frac{(2k+1)\pi}{2}\right) + 2$ for $k \in \mathbb{Z}$ has the same graph.

Chapter 4 Prerequisite Skills **Question 7 Page 212**

Use the $\left(1, 1, \sqrt{2}\right)$ and $\left(2, 1, \sqrt{3}\right)$ triangles.

a) $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

b) $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

c) $\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{3}\right) = 1 + \frac{1}{2}$

$= \frac{3}{2}$

d) $\sin^2\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{6}\right) = \left(\frac{1}{\sqrt{2}}\right)^2 - \frac{1}{2}$

$= 0$

e) $\sec\left(\frac{\pi}{4}\right) = \sqrt{2}$

f) $\cot\left(\frac{\pi}{2}\right) = 0$

g) $\csc\left(\frac{\pi}{3}\right) = \frac{2}{\sqrt{3}}$

h) $\sec^2\left(\frac{\pi}{4}\right) = 2$

Chapter 4 Prerequisite Skills **Question 8 Page 213**

a) $\frac{dy}{dx} = 5$

b) $\frac{dy}{dx} = -6x^2 + 8x$

c) $\frac{dy}{dx} = \frac{1}{2}(t^2 - 1)^{-\frac{1}{2}}(2t)$

d) $\frac{dy}{dx} = 2\left[(x^{-2})\left(\frac{1}{2}(x-3)^{-\frac{1}{2}}(1)\right) + (x-3)^{\frac{1}{2}}(-2x^{-3})\right]$

Chapter 4 Prerequisite Skills **Question 9 Page 213**

a) $f(g(x)) = (3x+4)^2$

$\frac{d}{dx}[(f(g(x))] = \frac{d}{dx}\left[(3x+4)^2\right]$

$= 2(3x+4)(3)$

b) $g(f(x)) = 3x^2 + 4$

$\frac{d}{dx}[g(f(x))] = \frac{d}{dx}(3x^2+4)$

$= 6x$

c) $f(f(x)) = (x^2)^2$

$\frac{d}{dx}[f(f(x))] = \frac{d}{dx}(x^4)$

$= 4x^3$

d) $f(x)g(x) = (x^2)(3x+4)$

$= 3x^3 + 4x^2$

$\frac{d}{dx}[f(x)g(x)] = 9x^2 + 8x$

Chapter 4 Prerequisite Skills **Question 10 Page 213**

To find the slope, differentiate the given function, $y = -3x^2 + 5x - 11$, with respect to x.

$\frac{dy}{dx} = -6x + 5$

The value of the derivative at $x = -4$, gives the slope of the function at that point.

$-6(-4) + 5 = 29$

Therefore, the slope of the graph of $y = -3x^2 + 5x - 11$ at $x = -4$ is. 29.

Chapter 4 Prerequisite Skills **Question 11 Page 213**

The slope of a line tangent to a curve, y, is given by $\frac{dy}{dx}$.

$\frac{dy}{dx} = x + 6$

The value of the derivative at $x = -2$, gives the slope of the function at that point.

$-2+6=4$

At $x=-2$, slope = 4 and $y=-10$.
The equation of the tangent is of the form $y=mx+b$. Find b.

$$b=-10+2(4)$$
$$=-2$$

Therefore, the equation of the tangent is $y=4x-2$.

Chapter 4 Prerequisite Skills **Question 12 Page 213**

At local maxima and minima, $\frac{dy}{dx}=0$.

$$\frac{dy}{dx}=3x^2+10x+3$$
$$0=3x^2+10x+3$$
$$x=\frac{-10\pm\sqrt{100-36}}{6}$$
$$x=-3 \text{ or } x=-\frac{1}{3}$$

At $x=-3$, $y=6$ and at $x=-\frac{1}{3}$, $y=-\frac{94}{27}$.
Local maximum point: $(-3, 6)$.
Local minimum point: $\left(-\frac{1}{3},-\frac{94}{27}\right)$.

Since the y-values correspond to both critical points it is not necessary to use the second derivative test to determine if the points are local maxima or minima.

Chapter 4 Prerequisite Skills **Question 13 Page 213**

a) $\sin(a+b)=(\sin a)(\cos b)+(\cos a)(\sin b)$
Therefore, $x=\sin a$ and $y=\cos a$.

b) $\sin(a-b)=(\sin a)(\cos b)-(\cos a)(\sin b)$
Therefore, $x=\sin a$.

c) $\cos(a+b)=(\cos a)(\cos b)-(\sin a)(\sin b)$
Therefore, $x=\sin a$ and $y=\cos a$.

Chapter 4 Prerequisite Skills **Question 14 Page 213**

Answers may vary. For example:

a) Use the definitions of $\sin\ \theta = \frac{y}{r}$, $\cos\ \theta = \frac{x}{r}$, and the Pythagorean theorem.

L.S. $= \sin^2\theta$
$= (\sin\ \theta)^2$
$= \left(\frac{y}{r}\right)^2$
$= \frac{y^2}{r^2}$

R.S. $= 1-\cos^2\theta$
$= 1-(\cos\ \theta)^2$
$= 1-\left(\frac{x}{r}\right)^2$
$= \frac{r^2}{r^2}-\left(\frac{x}{r}\right)^2$
$= \frac{r^2}{r^2}-\frac{x^2}{r^2}$
$= \frac{r^2-x^2}{r^2}$
$= \frac{y^2}{r^2}$

Since L.S. = R.S., the identity has been proven.

b) L.S. $= \tan(-\theta)\cos(-\theta)$
$= (-\tan\ \theta)(\cos\ \theta)$
$= \left(\frac{-\sin\ \theta}{\cos\ \theta}\right)(\cos\ \theta)$
$= -\sin\ \theta$

R.S. $= -\sin\ \theta$

Since L.S. = R.S., the identity has been proven.

c) L.S. $= \cot\ \theta$
$= \frac{1}{\tan\ \theta}$
$= \frac{1}{\left(\frac{\sin\ \theta}{\cos\ \theta}\right)}$
$= \frac{\cos\ \theta}{\sin\ \theta}$

R.S. $= \frac{\cos\theta}{\sin\theta}$

Since L.S. = R.S., the identity has been proven.

d) L.S. $= \cot\ \theta$ R.S. $= \dfrac{\csc\ \theta}{\sec\ \theta}$

$= \dfrac{\cos\ \theta}{\sin\ \theta}$ Using the result of part c).

$= \dfrac{\left(\dfrac{1}{\sec\ \theta}\right)}{\left(\dfrac{1}{\csc\ \theta}\right)}$

$= \dfrac{\csc\ \theta}{\sec\ \theta}$

Since L.S. = R.S., the identity has been proven.

Chapter 4 Prerequisite Skills **Question 15 Page 213**

a) $\dfrac{\sin x}{\tan x} = \dfrac{\sin x}{\left(\dfrac{\sin x}{\cos x}\right)}$

$= \cos x$

b) $\dfrac{1-\sin^2\theta}{\cos^2\theta} = \dfrac{\cos^2\theta}{\cos^2\theta}$

$= 1$

c) $\dfrac{\sin x}{\sin^2 x} = \dfrac{1}{\sin x}$

$= \csc x$

Chapter 4 Prerequisite Skills **Question 16 Page 213**

a) L.S. $= \cos\left(\theta - \dfrac{\pi}{2}\right)$ R.S. $= \sin\theta$

$= \cos\theta\cos\dfrac{\pi}{2} + \sin\theta\sin\dfrac{\pi}{2}$

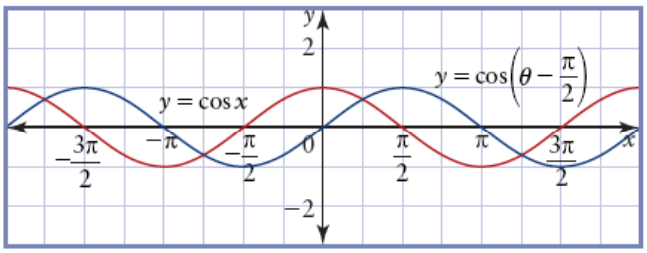
$= \cos\theta(0) + \sin\theta(1)$

$= 0 + \sin\theta$

$= \sin\theta$

Since L.S. = R.S., the identity has been proven.

b)



Chapter 4 Prerequisite Skills **Question 17 Page 213**

a) L.S. $= \sin\left(\theta + \pi\right)$ R.S. $= -\sin\theta$

$= \sin\theta\cos\pi + \cos\theta\sin\pi$

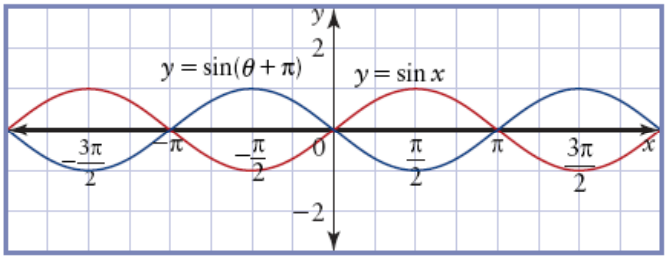
$= \sin\theta(-1) + \cos\theta(0)$

$= -\sin\theta + 0$

$= -\sin\theta$

Since L.S. = R.S., the identity has been proven.

b)



Chapter 4 Section 1 **Instantaneous Rates of Change of Sinusoidal Functions**

Chapter 4 Section 1 **Question 1 Page 217**

a) i) $\left\{x \middle| x = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$

ii) $\left\{x \middle| x = \frac{3}{2}\pi + 2k\pi, k \in \mathbb{Z}\right\}$

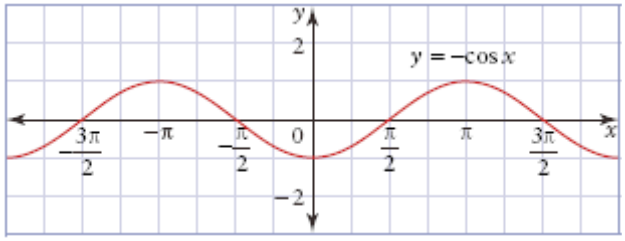
iii) $\left\{x \middle| x = \frac{1}{2}\pi + 2k\pi, k \in \mathbb{Z}\right\}$

b) i) The curve is concave up for … $(-2\pi \le x \le -\pi)$, $(0 \le x \le \pi)$ …

ii) The curve is concave down for … $(-3\pi \le x \le -2\pi)$, $(-\pi \le x \le 0)$, $(\pi \le x \le 2\pi)$, …

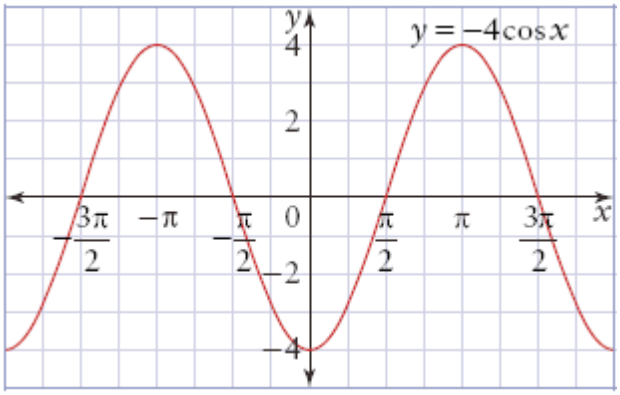
c) The maximum value of the slope is 1 at $(2k+1)\pi$.
The minimum value of the slope is -1 at $(2k)\pi$.

d)

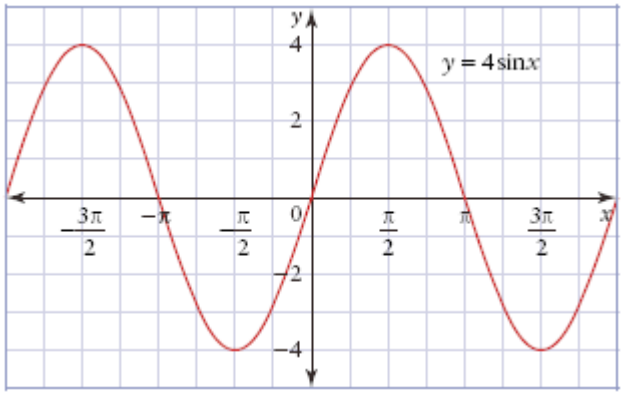


Chapter 4 Section 1 **Question 2 Page 217**

a)



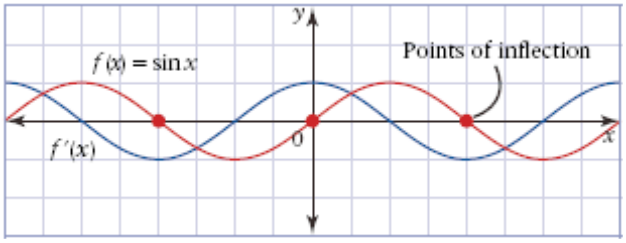
b)



Chapter 4 Section 1 **Question 3 Page 217**

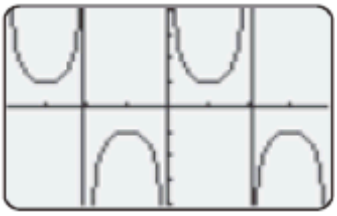
Answers may vary. For example:

Yes. A sinusoidal curve does have points of inflection. The points of inflection will occur at points where the first derivative is a local maximum or a local minimum.



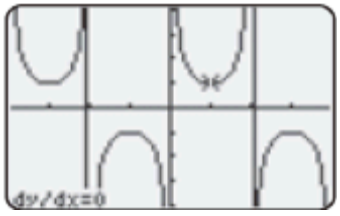
Chapter 4 Section 1 **Question 4 Page 217**

a)

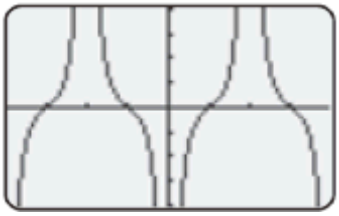


b) Answers may vary. For example:

Use a graphing calculator and the Value function.
The instantaneous rate of change is −2.140 787 at the point (−5.628 687, 1.642 679 6).
The instantaneous rate of change is 2.140 787 at the point (−2.487 094, −1.642 679 6).
The instantaneous rate of change is 0 at the point $\left(\frac{\pi}{2}, 1\right)$.



c)

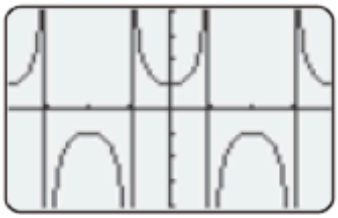


Answers may vary. For example:

The graph of the instantaneous rate of change of $y = \csc x$ as a function of x has points of inflection at the points where the graph of $y = \csc x$ has local maximum points and local minimum points. Both graphs have vertical asymptotes at the same x-values.

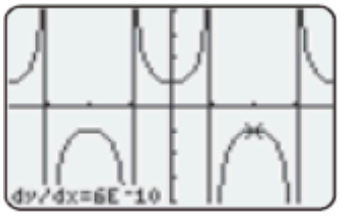
Chapter 4 Section 1 **Question 5 Page 217**

a)

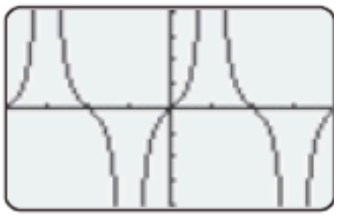


b) Answers may vary. For example:

Use a graphing calculator and the Value function.
The instantaneous rate of change is −1.414 216 9 at the point (−3.926 991, −1.414 214).
The instantaneous rate of change is −8.203 512 at the point (−1.916 297 9, −2.952 739).
The instantaneous rate of change is 0 at the point (π , −1).



c)

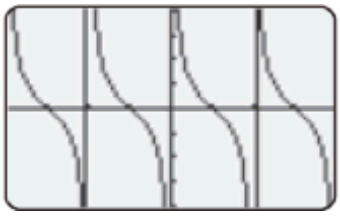


Answers may vary. For example:

The graph of the instantaneous rate of change of $y = \sec x$ as a function of x has points of inflection at the points where the graph of $y = \sec x$ has local maximum points and local minimum points. Both graphs have vertical asymptotes at the same x-values.

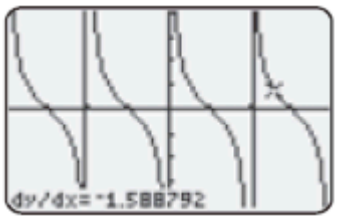
Chapter 4 Section 1 **Question 6 Page 217**

a)

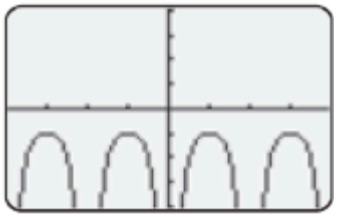


b) Answers may vary. For example:

Use a graphing calculator and the Value function.
The instantaneous rate of change is −4.000 015 at the point (−2.617 994, 1.732 051 3).
The instantaneous rate of change is −2.000 003 at the point (2.356 194 5, −1).
The instantaneous rate of change is −2.698 402 at the point (3.796 091 1, 1.303 225 4).



c)



Answers may vary. For example:

The graph of the instantaneous rate of change of $y = \cot x$ as a function of x has local minimum points where the graph of $y = \cot x$ has points of inflection. Both graphs have vertical asymptotes at the same x-values.

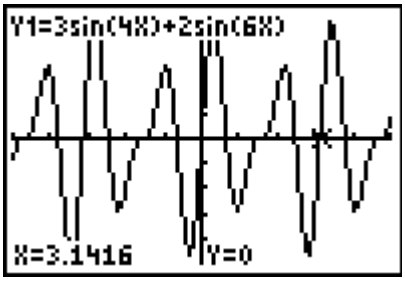
Chapter 4 Section 1 **Question 7 Page 217**

C is the correct answer.

The period of the function $y = \sin 4x$ is $\frac{2\pi}{4} = \frac{\pi}{2}$.

The period of the function $y = \sin 6x$ is $\frac{2\pi}{6} = \frac{\pi}{3}$.

Therefore, the period of the function $y = 3\sin 4x + 2\sin 6x$ is the LCM of $\frac{\pi}{2}$ and $\frac{\pi}{3}$, which is π.



The graph confirms that the period is π.

Chapter 4 Section 2 **Derivatives of the Sine and Cosine Functions**

Chapter 4 Section 2 **Question 1 Page 225**

a) $y = \sin x$; derivative: B $\frac{dy}{dx} = \cos x$

b) $y = \cos x$; derivative: C $\frac{dy}{dx} = -\sin x$

c) $y = -\sin x$; derivative: D $\frac{dy}{dx} = -\cos x$

d) $y = -\cos x$; derivative: A $\frac{dy}{dx} = \sin x$

Chapter 4 Section 2 **Question 2 Page 225**

a) $\frac{dy}{dx} = 4\cos x$

b) $\frac{dy}{dx} = -\pi \sin x$

c) $f'(x) = 3\sin x$

d) $g'(x) = \frac{1}{2}\cos x$

e) $f'(x) = 0.007\cos x$

Chapter 4 Section 2 **Question 3 Page 226**

a) $\frac{dy}{dx} = -\sin x - \cos x$

b) $\frac{dy}{dx} = \cos x - 2\sin x$

c) $\frac{dy}{dx} = 2x - 3\cos x$

d) $\frac{dy}{dx} = -\pi \sin x + 2 + 2\pi \cos x$

e) $\frac{dy}{dx} = 5\cos x - 15x^2$

f) $\frac{dy}{dx} = -\sin x + 7\pi\cos x - 3$

Chapter 4 Section 2 **Question 4 Page 226**

a) $f'(\theta) = 3\sin\theta - 2\cos\theta$

b) $f'(\theta) = \frac{\pi}{2}\cos\theta + \pi\sin\theta$

c) $f'(\theta) = -15\sin\theta + 1$

d) $f'(\theta) = -\frac{\pi}{4}\sin\theta - \frac{\pi}{3}\cos\theta$

Chapter 4 Section 2 **Question 5 Page 226**

a) To find the slope, differentiate the given function with respect to *x*.

$\frac{dy}{dx} = 5\cos x$

The value of the derivative at $x = \frac{\pi}{2}$, gives the slope of the function at that point.

Therefore, the slope of the graph of $y = 5\sin x$ at $x = \frac{\pi}{2}$ is $5\cos\frac{\pi}{2} = 0$.

b) Answers may vary. For example:

From part a), the slope of the graph of $y = 5\sin x$ at $x = \frac{\pi}{2}$ is $5\cos\frac{\pi}{2} = 0$. Similarly, the slope of the graph of $y = \sin x$ at $x = \frac{\pi}{2}$ is $\cos\frac{\pi}{2} = 0$. The derivative functions, $5\cos x$ and $\cos x$, of both these curves, $y = 5\sin x$ and $y = \sin x$, cross the *x*-axis at $x = \frac{\pi}{2}$.

Therefore, the derivatives (slopes) of both functions will be 0 at $x = \frac{\pi}{2}$.

.

Chapter 4 Section 2 **Question 6 Page 226**

To find the slope, differentiate the given function, $y = 2\cos\theta$, with respect to θ.

$$\frac{dy}{d\theta} = -2\sin\theta$$

The value of the derivative at $\theta = \frac{\pi}{6}$, gives the slope of the function at that point.

Therefore, the slope of the graph of $y = 2\cos\theta$ at $\theta = \frac{\pi}{6}$ is $-2\sin\frac{\pi}{6} = -1$.

Chapter 4 Section 2 **Question 7 Page 226**

a) Answers may vary. For example:

Substitute $\frac{\pi}{3}$ for x in $y = \cos x$.

$$y = \cos\left(\frac{\pi}{3}\right)$$
$$= \frac{1}{2}$$

Therefore, $\left(\frac{\pi}{3}, \frac{1}{2}\right)$ is a point on the curve $y = \cos x$.

b) To find the equation of the tangent line, you need its slope and a point on the line.
To find the slope, differentiate the given function, $y = \cos x$, with respect to x.

$$\frac{dy}{dx} = -\sin x$$

The value of the derivative at $x = \frac{\pi}{3}$, gives the slope of the function at that point.

Therefore, the slope of the graph of $y = \cos x$ at $x = \frac{\pi}{3}$ is $-\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$.

Front part a). $\left(\frac{\pi}{3}, \frac{1}{2}\right)$ is a point on the line.

Use the slope-point form of the line to find the equation of the tangent line.

$$y - y_1 = \left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}} (x - x_1)$$

$$y - \frac{1}{2} = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)$$

$$y = -\frac{\sqrt{3}}{2}x + \frac{\sqrt{3}}{6} + \frac{1}{2}$$

Therefore, the equation of the tangent is $y = -\frac{\sqrt{3}}{2}x + \frac{\sqrt{3}}{6} + \frac{1}{2}$.

Chapter 4 Section 2 **Question 8 Page 226**

Slope is $\frac{dy}{dx} = -4\cos x$.

If the equation of the tangent is $y = mx + b$ then, for $x = \frac{\pi}{4}$, $y = -2\sqrt{2}$, $m = -2\sqrt{2}$.

Therefore:

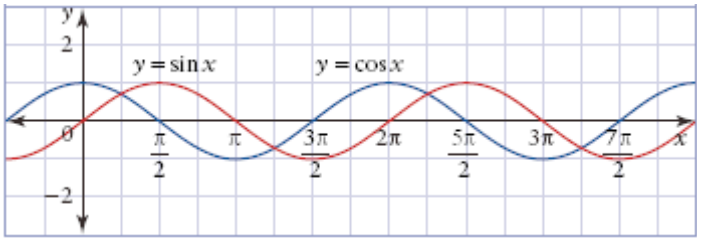
$b = -2\sqrt{2} + 2\sqrt{2}\left(\frac{\pi}{4}\right)$

$$y = -2\sqrt{2}x - 2\sqrt{2} + \frac{\sqrt{2}\pi}{2}$$

So, the equation of the tangent is $y = -2\sqrt{2}x + \frac{\sqrt{2}\pi}{2} - 2\sqrt{2}$.

Chapter 4 Section 2 **Question 9 Page 226**

a)



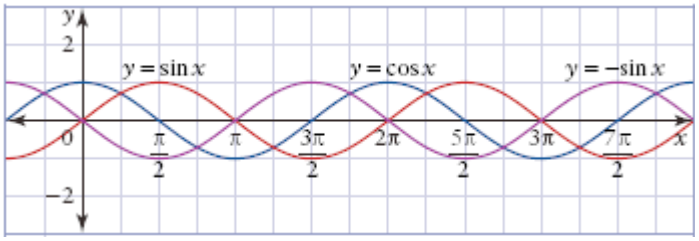
b) Answers may vary. For example:

The graph of $y = \cos x$ is the graph of $y = \sin x$ shifted horizontally $\frac{\pi}{2}$ units to the left.

c) $y = \sin x$

$$\frac{dy}{dx} = \cos x$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = -\sin x$$

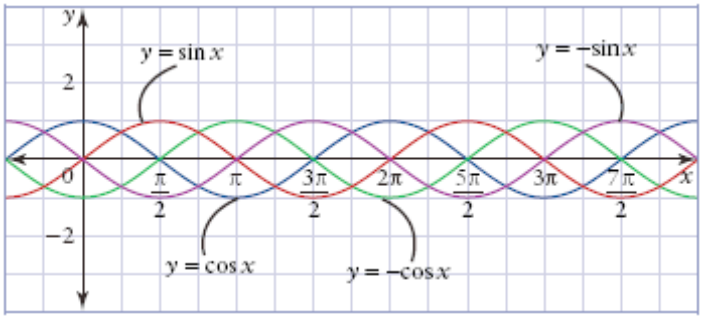


Answers may vary. For example:

The graph of $y = -\sin x$ is the graph of the first derivative, $y = \cos x$, shifted horizontally $\frac{\pi}{2}$ units to the left, and the graph of $y = \sin x$ shifted horizontally π units to the left.

d) Answers may vary. For example:

The graph of the third derivative is the graph of the second derivative, $y = -\sin x$, shifted horizontally $\frac{\pi}{2}$ units to the left; which is the same as the graph of the first derivative, $y = \cos x$, shifted horizontally ∂ units to the left, as well as the graph of $y = \sin x$ shifted horizontally $\frac{3\pi}{2}$ units to the left. The third derivative is $\frac{d^3y}{dx^3} = \frac{d}{dx}(-\sin x) = -\cos x$.



e) Answers may vary. For example:

i) The fourth derivative of $y = \sin x$ is the graph of $y = \sin x$ shifted horizontally $\frac{4\pi}{2}$ units to the left and will be the same as the graph of $y = \sin x$. The fourth derivative of $y = \sin x$ is $\frac{d^4y}{dx^4} = \sin x$.

ii) The tenth derivative of $y = \sin x$ is the graph of $y = \sin x$ shifted horizontally $\frac{10\pi}{2}$ units to the left and will be the same as the graph of the sixth derivative and the second derivative of $y = \sin x$.

The tenth derivative of $y = \sin x$ is $\frac{d^{10}y}{dx^{10}} = -\sin x$.

Chapter 4 Section 2 **Question 10 Page 226**

$\frac{d^{15}y}{dx^{15}} = \sin x$

Explanations may vary. For example:

The first five derivatives of $y = \cos x$ are:

$\frac{dy}{dx} = -\sin x$, $\frac{d^2y}{dx^2} = -\cos x$, $\frac{d^3y}{dx^3} = \sin x$, $\frac{d^4y}{dx^4} = \cos x$, and $\frac{d^5y}{dx^5} = -\sin x$.

The fourth, eighth, and twelfth derivatives will be the same as the original function, $y = \cos x$.

The fifth, ninth, and thirteenth derivatives will be the same as the first derivative, $\frac{dy}{dx} = -\sin x$.

The sixth, tenth, and fourteenth derivative will be the same as the second derivative, $\frac{d^2y}{dx^2} = -\cos x$.

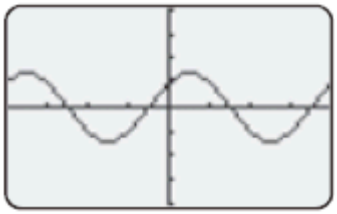
The seventh, eleventh, and fifteenth derivative will be the same as the third derivative, $\frac{d^3y}{dx^3} = \sin x$.

Therefore, the fifteenth derivative of $y = \cos x$ is $\frac{d^{15}y}{dx^{15}} = \sin x$.

Chapter 4 Section 2 **Question 11 Page 226**

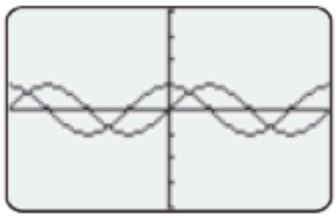
Answers may vary. For example:

Use a graphing calculator to graph the function $y = \sin x + \cos x$.



The derivative of $y = \sin x + \cos x$ is $\frac{dy}{dx} = \cos x - \sin x$.

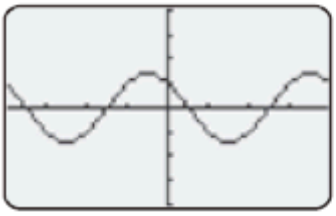
Use a graphing calculator to graph the functions: $y = \cos x$ and $y = -\sin x$ in the same viewing screen and display the table of values for the two functions.



X	Y_1	Y_2
-3	-.99	.14112
-2	-.4161	.9093
-1	.5403	.84147
0	1	0
1	.5403	-.8415
2	-.4161	-.9093
3	-.99	-.1411

X= -3

Graph the function $y = \cos x - \sin x$ and display the table of values for the function.



X	Y_1
-3	-.8489
-2	.49315
-1	1.3818
0	1
1	-.3012
2	-1.325
3	-1.131

X= -3

The *y*-values for the derivative function $\frac{dy}{dx} = \cos x - \sin x$ of the function $y = \sin x + \cos x$ are the sum of the *y*-values for the derivative of $y = \sin x$, $\frac{dy}{dx} = \cos x$ and the derivative of $y = \cos x$, $\frac{dy}{dx} = -\sin x$. This shows that the sum differentiation rule holds true for the sinusoidal function $y = \sin x + \cos x$. Using a similar method it can be shown that the difference differentiation rule will hold true for the sinusoidal function $y = \sin x - \cos x$.

Chapter 4 Section 2 **Question 12 Page 226**

a) Answers may vary. For example:

$$\text{Slope} = \frac{dy}{dx}$$
$$= \sin x$$

If $\frac{dy}{dx} = -1$,

$$\sin x = -1$$
$$x = \left((2k-1)\pi + \frac{\pi}{2} \right), k \in \mathbb{Z}$$

For all $k \in \mathbb{Z}$, if $y = mx + b$ is the equation of a tangent, then at $x = \left((2k-1)\pi + \frac{\pi}{2} \right)$,

$$y = -\cos\left((2k-1)\pi + \frac{\pi}{2} \right) \qquad \frac{dy}{dx} = m$$
$$= 0 \qquad = -1$$

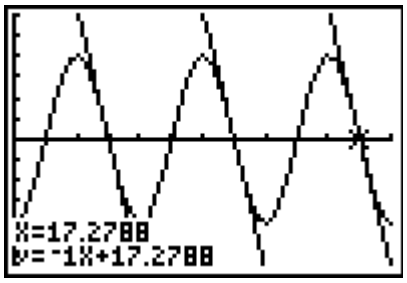
$$b = \left((2k-1)\pi + \frac{\pi}{2} \right)$$

$$y = -x + \left((2k-1)\pi + \frac{\pi}{2} \right)$$

If $k = 1$, this gives $y = -x + \frac{3\pi}{2}$.

b) Yes, there is more than one solution.

Since the function $y = -\cos x$ is periodic, there will be an infinite number of solutions, as shown from the tangent equation $y = -x + \left((2k-1)\pi + \frac{\pi}{2} \right),\ k \in \mathbb{Z}$. Each value of k, where k is an integer, will provide a different equation of the tangent line. Examples are shown below.



Chapter 4 Section 2 **Question 13 Pages 226-227**

a) From the graph: maximum height: 18 m; minimum height: 2 m

b) Since the model has maxima at (0, 18) and (4π , 18) and minimum at (2π , 2) it has the form $y = 8\cos(bx - c) + 10$. (It is translated vertically by 10 m, and expanded vertically by a factor of 8.)

Use the maximum and minimum points.

$18 = 8\cos(-c) + 10$ ①

$2 = 8\cos\left(b(2\pi) - c\right) + 10$ ②

From ①:

$\cos(-c) = 1$

$c = -2k\pi,\ k \in \mathbb{Z}$

Need only one equation that relates vertical and horizontal positions, so let $k = 0$, $c = 0$.

From ②:

$$-8 = 8\cos\left(b(2\pi) - c\right)$$
$$-1 = \cos\left(b(2\pi)\right)$$
$$2b\pi = (2h+1)\pi$$
$$b = \frac{(2h+1)}{2},\ h \in \mathbb{Z}$$

Need only one equation that relates the vertical and horizontal positions, so let $h = 0,\ b = \frac{1}{2}$.

Therefore, an equation that models the vertical and the horizontal position is $y = 8\cos\left(\frac{1}{2}x\right) + 10$.

c) For $y = 8\cos\left(\frac{1}{2}x\right) + 10$:

$$\frac{dy}{dx} = -8\sin\left(\frac{1}{2}x\right)\left(\frac{1}{2}\right)$$
$$= -4\sin\left(\frac{1}{2}x\right)$$

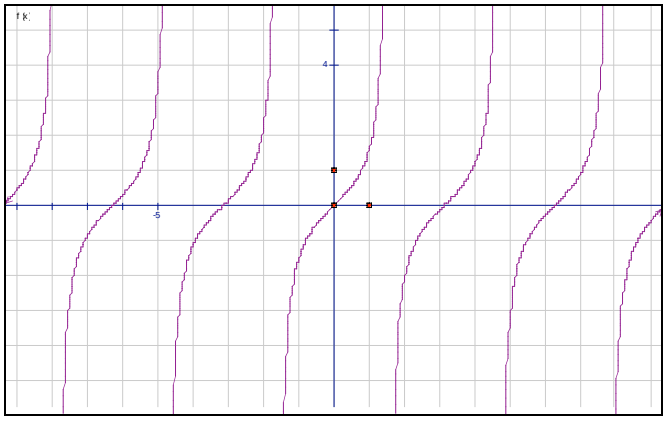
Since $-1 \le \sin\left(\frac{1}{2}x\right) \le 1$, $\frac{dy}{dx}$ has a maximum value of (–4)(–1) = 4 at $x = 3\pi$.

Chapter 4 Section 2 **Question 14 Page 227**

Solutions for Achievement Checks are shown in the Teacher Resource.

Chapter 4 Section 2 **Question 15 Page 227**

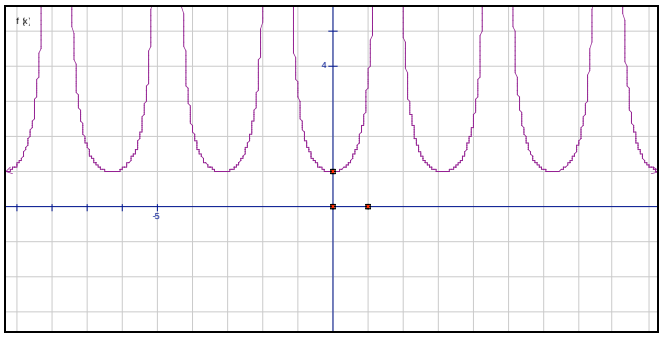
a) Yes the function $y = \tan x$ is periodic with period π.



b) Answers may vary. For example:

The graph of the derivative of $y = \tan x$ will have the same asymptotes as the graph of $y = \tan x$. The graph of the derivative of $y = \tan x$ will also have local minimum points for x-values where the function $y = \tan x$ crosses the x-axis and has points of inflection. For intervals where the graph of $y = \tan x$ is increasing and concave down, the derivative will be decreasing and concave up. For intervals where the graph of $y = \tan x$ is increasing and concave up, the derivative will be increasing and concave up.

c)



Answers may vary. For example:

Yes. The results were as I expected. The derivative of $y = \tan x$ is $y' = \sec^2 x$. The derivative function is positive for all values of x for which it is defined and will have local minimum values for values of x for which:

$$1 = \sec^2 x$$

$$\frac{1}{\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x}$$

$$1 = 1 + \tan^2 x$$

$$0 = \tan^2 x$$

$$x = k\pi, \ k \in \mathbb{Z}$$

Chapter 4 Section 2 **Question 16 Page 227**

Answers may vary. For example:

a) i) As $x \to \frac{\pi}{2}$ from the left, the graph of the derivative of $y = \tan x$ becomes large and positive.

ii) As $x \to \frac{\pi}{2}$ from the right, the graph of the derivative of $y = \tan x$ becomes large and positive.

b) This implies that the value of the derivative of $y = \tan x$ at $x = \frac{\pi}{2}$ is not defined and there is discontinuity at $x = \frac{\pi}{2}$. Therefore, the derivative of $y = \tan x$ does not exist at $x = \frac{\pi}{2}$.

Chapter 4 Section 2 Question 17 Page 227

a) Answers may vary. For example:

This sketch illustrates that $y = \cos x$ is the derivative of $y = \sin x$. The slope of the tangent line at the point $(-7.43, -0.93)$ is 0.37. The equation of the tangent line to the function $y = \sin x$ at the point x_p is represented by $h(x) = y_p + f'(x_p) \cdot (x - x_p)$. The graph of $y = \cos x$ is the graph of $y = \sin x$ translated horizontally $\frac{\pi}{2}$ units to the left.

b) Answers may vary. For example:

If the **Animate P** button is pressed, the point P will move along the curve $y = \sin x$ from left to right and the green tangent line will move along the curve as well. The slope of the tangent line will increase to a local maximum value at the first point of inflection on the *x*-axis and then become 0 at the local maximum value where the line becomes horizontal. The slope will decrease to a local minimum value at the second point of inflection on the *x*-axis and then become 0 at the local minimum value where the tangent line becomes horizontal. As the point continues to travel to the right on the curve, the tangent line will continue in the same pattern.

c) The point P moves along the sine curve and the tangent to the curve at point P is shown.

d) Answers may vary.

e) Answers may vary.

Chapter 4 Section 2 Question 18 Page 227

Answers may vary. For example:

Consider the reciprocal trigonometric function $y = \csc x$. The derivative is $\frac{dy}{dx} = -\csc x \cot x$.

a) domain:
function: $x \in \mathbb{R}$, $x \neq n\pi$, $n \in \mathbb{Z}$
derivative: $x \in \mathbb{R}$, $x \neq n\pi$, $n \in \mathbb{Z}$

range:
function: $y \in (-\infty, -1]$ or $[1, \infty)$
derivative: $y \in (-\infty, \infty)$

b) No maximum or minimum values for the function or the derivative.
No local minimum or maximum values for the derivative.

local maximum values:

function: $\left\{ x \left| x = \frac{3\pi}{2} + 2k\pi, k \in \mathbb{Z} \right. \right\}$

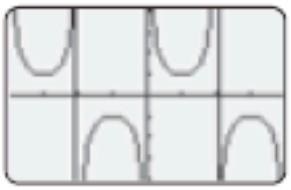
local minimum values

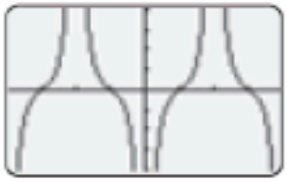
function: $\left\{ x \left| x = \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z} \right. \right\}$

c) function: periodic with period 2π
derivative: periodic with period 2π

d) function: vertical asymptotes at $x = n\pi$, $n \in \mathbb{Z}$
derivative: vertical asymptotes at $x = n\pi$, $n \in \mathbb{Z}$

e) function derivative





Chapter 4 Section 2 **Question 19 Page 227**

Answer may vary. For example:

Let $f(x) = \sin x$. Then $f'(x) = \cos x$.

$f'(x) \doteq \frac{f(y) - f(x)}{y - x}$ for $y \doteq x$ implies that $f(y) - f(x) \doteq f'(x) \times (y - x)$ for $y \doteq x$.

$$37° - 36° = 1°$$
$$= \frac{\pi}{180} \text{ rad}$$

Therefore,

$$\sin 37° - \sin 36° \doteq \cos 36° \times \frac{\pi}{180}$$
$$= \frac{1 + \sqrt{5}}{4} \times \frac{\pi}{180}$$
$$= \frac{\left(1 + \sqrt{5}\right)\pi}{720}$$

Chapter 4 Section 3 **Differentiation Rules for Sinusoidal Functions**

Chapter 4 Section 3 **Question 1 Page 231**

a) $\frac{dy}{dx} = 4\cos 4x$

b) $\frac{dy}{dx} = \pi \sin(-\pi x)$

c) $f'(x) = 2\cos(2x + \pi)$

d) $f'(x) = \sin(-x - \pi)$

Chapter 4 Section 3 **Question 2 Page 231**

a) $\frac{dy}{d\theta} = -6\cos(3\theta)$

b) $\frac{dy}{d\theta} = 5\sin\left(5\theta - \frac{\pi}{2}\right)$

c) $f'(\theta) = -\pi \sin(2\pi\theta)$

d) $f'(\theta) = -6\cos(2\theta - \pi)$

Chapter 4 Section 3 **Question 3 Page 231**

a) $\frac{dy}{dx} = 2\sin x \cos x$

b) $\frac{dy}{dx} = -\cos^2 x \sin x$

c) $f'(x) = -2\sin 2x$

d) $f'(x) = -6\cos^2 x \sin x - 4\cos^3 x \sin x$

Chapter 4 Section 3 **Question 4 Page 231**

a) $\frac{dy}{dt} = 12\sin(2t - 4)\cos(2t - 4) + 12\cos(3t + 1)\sin(3t + 1)$

b) $f'(t) = 2t\cos(t^2 + \pi)$

c) $\frac{dy}{dt} = \left[-\sin\left(\sin t\right)\right]\left[\cos t\right]$

d) $f'(t) = -2\left[\sin(\cos t)\right]\left[\cos(\cos t)\right]\left[\sin t\right]$

Chapter 4 Section 3 **Question 5 Page 231**

a) $\frac{dy}{dx} = -2x\sin 2x + \cos 2x$

b) $f'(x) = -3x^2\cos(3x - \pi) - 2x\sin(3x - \pi)$

c) $\frac{dy}{d\theta} = -2\sin^2\theta + 2\cos^2\theta$

d) $f'(\theta) = -2\sin^3\theta\cos\theta + 2\sin\theta\cos^3\theta$

e) $f'(t) = 18t(\sin^2(2t - \pi))(\cos(2t - \pi)) + 3(\sin^3(2t - \pi))$

f) $\frac{dy}{dx} = -2x^{-1}\cos x\sin x - x^{-2}\cos^2 x$

Chapter 4 Section 3 **Question 6 Page 231**

Answers may vary. For example:

a) The derivatives of each of the functions are the same: $\frac{dy}{dx} = \cos x$.

b) The equations of the three functions are $y = \sin x$ (middle), $y = \sin x + 3$ (top), and $y = \sin x - 2$ (bottom). The graph in the middle is a sinusoidal function with an amplitude of 1 and a period of 2π, a local maximum at $\left(\frac{\pi}{2}, 1\right)$ and a local minimum at $\left(\frac{3\pi}{2}, -1\right)$.
Therefore the equation of this function is $y = \sin x$.

The highest placed function is also a sinusoidal function with an amplitude of 1 and a period of 2π. The graph is congruent to $y = \sin x$ and has been vertically translated up 3 units. The equation of this function is $y = \sin x + 3$.

The lowest placed function is also a sinusoidal function with an amplitude of 1 and a period of 2π. The graph is congruent to $y = \sin x$ and has been vertically translated down 2 units. The equation of this function is $y = \sin x - 2$.

Chapter 4 Section 3 **Question 7 Page 231**

The slope of the function $y = 2\cos x \sin 2x$ is given by its derivative with respect to x.

$$y = 2\cos x \sin 2x$$

$$\frac{dy}{dx} = 2(-\sin x)(\sin 2x) + 2(\cos x)(\cos 2x)(2)$$

$$\frac{dy}{dx} = -2\sin x \sin 2x + 4\cos x \cos 2x$$

At $x = \frac{\pi}{2}$,

$$\begin{aligned}\frac{dy}{dx} &= -2\sin\frac{\pi}{2}\sin\pi + 4\cos\frac{\pi}{2}\cos\pi \\ &= (-2)(1)(0) + (4)(0)(-1) \\ &= 0\end{aligned}$$

Therefore, the slope of the function $y = 2\cos x \sin 2x$ at $x = \frac{\pi}{2}$ is 0.

Chapter 4 Section 3 **Question 8 Page 231**

To find the equation of the tangent line, you need its slope and a point on the line.
To find the slope, differentiate the given function, $y = x^2 \sin 2x$, with respect to x.

$$\frac{dy}{dx} = 2x\sin 2x + 2x^2\cos 2x$$

The value of the derivative at $x = -\partial$, gives the slope of the function at that point.

$$\begin{aligned}\frac{dy}{dx}\Big|_{x=-\pi} &= 2(-\pi)\sin(-2\pi) + 2(-\pi)^2\cos(-2\pi) \\ &= 0 + 2\pi^2 \\ &= 2\pi^2\end{aligned}$$

Substitute $x = -\pi$ into the original function to get $y = 0$.

Use the slope-point form of the line to get the equation for the tangent line.

$$y - y_1 = \frac{dy}{dx}\Big|_{x=-\pi}(x - x_1)$$

$$y - 0 = 2\pi^2(x - (-\pi))$$

$$y = 2\pi^2 x + 2\pi^3$$

Therefore, the equation of the line tangent to $y = x^2 \sin 2x$ at $x = -\pi$ is $y = 2\pi^2 x + 2\pi^3$.

Chapter 4 Section 3 **Question 9 Page 231**

Answers may vary. For example:

a) An odd function is one for which satisfies $f(-x) = f(x)$.

$$\begin{aligned} f(x) &= \sin x \\ f(-x) &= \sin(-x) \\ &= -\sin x \\ &= -f(x) \end{aligned}$$

Since, $f(-x) = -f(x)$, the function $y = \sin x$ is an odd function.

b) Since $y = \sin x$ is an odd function, $\sin(-x) = -\sin(x)$.

Therefore,

$$\begin{aligned} \frac{dy}{dx}(\sin(-x)) &= \frac{dy}{dx}(-\sin x) \\ &= -\cos x \end{aligned}$$

Chapter 4 Section 3 **Question 10 Page 231**

Answers may vary. For example:

a) An even function is one which satisfies $f(-x) = -f(x)$.

$$\begin{aligned} f(x) &= \cos x \\ f(-x) &= \cos(-x) \\ &= -\cos(x) \\ &= -f(x) \end{aligned}$$

Since $f(-x) = f(x)$, the function $y = \cos x$ is an even function.

b) Since $y = \cos x$ is an even function, $\cos(-x) = \cos(x)$.

$$\begin{aligned} \frac{dy}{dx}(\cos(-x)) &= \frac{dy}{dx}(\cos x) \\ &= -\sin x \end{aligned}$$

Chapter 4 Section 3 **Question 11 Page 231**

a) Find the derivative of $y = \sin^2 x + \cos^2 x$.

$$y = \sin^2 x + \cos^2 x$$
$$\frac{dy}{dx} = 2\sin x \cos x + 2\cos x(-\sin x)$$
$$= 2\sin x \cos x - 2\sin x \cos x$$
$$= 0$$

Since $\frac{dy}{dx} = 0$, $y = \sin^2 x + \cos^2 x$ is a constant function.

b) If $y = \sin^2 x + \cos^2 x$, then $y = 1$, using the trigonometric identity $\sin^2 x + \cos^2 x = 1$.
Therefore $y = \sin^2 x + \cos^2 x$ is a constant function.

Chapter 4 Section 3 **Question 12 Page 232**

$$\frac{dy}{dx} = x^2(-\sin x) + 2x\cos x$$
$$\frac{d^2y}{dx^2} = -x^2\cos x - 2x\sin x - 2x\sin x + 2\cos x$$
$$\frac{d^2y}{dx^2} = -x^2\cos x - 4x\sin x + 2\cos x$$

Chapter 4 Section 3 **Question 13 Page 232**

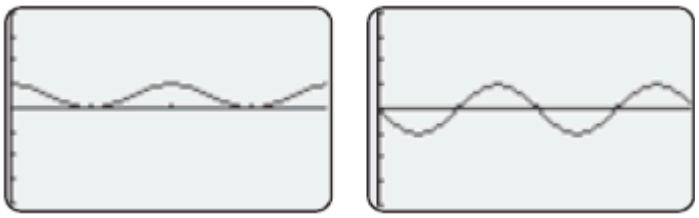
a) Answers may vary. For example:

For the function $f(x) = \cos^2 x$, all values in the range will be greater than or equal to zero. On the interval $0 \le x < 2\pi$, the zeros of this function are the same as the zeros of the function $f(x) = \cos x$, $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

The derivative of this function is $f'(x) = -2\sin x \cos x$. On the interval $0 \le x < 2\pi$, the zeros of the derivative function are the same as the zeros of the function $f(x) = \cos x$, i.e., $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, and the zeros of the function $f(x) = \sin x$, i.e., 0 and π.

Therefore the function $f(x) = \cos^2 x$ will have half as many zeros as its derivative $f'(x) = -2\sin x \cos x$.

b)



Chapter 4 Section 3 **Question 14 Page 232**

Answers may vary. For example:

A composite function is $y = \sin(x^3)$.

First derivative: $\frac{dy}{dx} = \left[\cos(x^3)\right](3x^2)$.

Second derivative: $\frac{d^2y}{dx^2} = 6x\cos x^3 - 9x^4 \sin x^3$.

Chapter 4 Section 3 **Question 15 Page 232**

a) $y = \frac{1}{\sin x}$

b) $y = (\sin x)^{-1}$

c) $y = (\sin x)^{-1}$

$$\frac{dy}{dx} = -(\sin x)^{-2}\cos x$$
$$= -\frac{\cos x}{(\sin x)^2}$$
$$= -\csc x \cot x$$

d) Domain of $y = \csc x$: $x \in \mathbb{R}$, $x \neq n\pi$, $n \in \mathbb{Z}$.

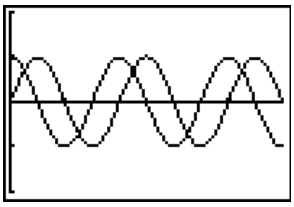
Domain of the derivative of $y = \csc x$, $\frac{dy}{dx} = -\csc x \cot x$: $x \in \mathbb{R}$, $x \neq n\pi$, $n \in \mathbb{Z}$.

Chapter 4 Section 3 **Question 16 Page 232**

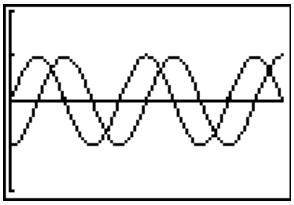
Answers may vary. For example:

A horizontal shift of a sinusoidal function will result in a similar shift of the derivative of that function. If the function $y = \cos x$ is shifted horizontally $\frac{3\pi}{2}$ units to the right then its derivative, $y = -\sin x$, will also shift $\frac{3\pi}{2}$ units to the right.

Here are graphs of $y = \cos x$ and $y = \cos\left(x + \frac{3\pi}{2}\right)$.



Here are graphs of the derivatives, $y = -\sin x$ and $y = -\sin\left(x + \frac{3\pi}{2}\right)$.



Chapter 4 Section 3 **Question 17 Page 232**

Answers may vary. For example:

a) I used a graphing calculator and systemic trial to determine that the function that models the roller coaster segment on the left is a piecewise sinusoidal function. On the interval $0 \le x \le \pi$, the function that models the roller coaster is $y = 0.25\sin^2 2x + 4$. On the interval $\pi < x \le 2\pi$, the function that models the roller coaster is $y = 2\sin x + 4$.

I also used a graphing calculator and systemic trial to determine that the function that models the roller coaster segment on the right is a piecewise sinusoidal function. On the interval $0 \le x \le \pi$, the function that models the roller coaster is $y = 3\sin 2x + 4$. On the interval $\pi < x \le 2\pi$, the function that models the roller coaster is $y = -3\sin 2x + 4$.

b) Maximum slope of the roller coaster segment on the left occurs when $x = \frac{7\pi}{4}$.

$$y' = 2\cos\left(\frac{7\pi}{4}\right)$$
$$= 2\frac{\sqrt{2}}{2}$$
$$= \sqrt{2}$$

When $x = \frac{7\pi}{4}$, the slope is $\sqrt{2}$.

Maximum slope of the roller coaster segment on the right occurs when $x = 1.5\pi$.

$$y' = -3\left(\cos 2\left(\frac{3\pi}{2}\right)\right)(2)$$
$$= -6(-1)$$
$$= 6$$

When $x = 1.5\pi$, the slope is 6.

Chapter 4 Section 3 **Question 18 Page 232**

a) $y' = \frac{d}{dx}(\sec x)$
$= \frac{d}{dx}(\cos x)^{-1}$
$= -(\cos x)^{-2}\sin x$
$= \frac{1}{\cos x}\left(\frac{\sin x}{\cos x}\right)$
$= \sec x \tan x$

b) $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sec x}{\cos^2 x}\right)$
$= \frac{d}{dx}(\sec^3 x)$
$= 3\sec^2 x(\sec x \tan x)$
$= 3\tan x \cos^{-3} x$

Chapter 4 Section 3 **Question 19 Page 232**

Answers may vary. For example:

$$\begin{aligned} y &= \tan x \\ &= \frac{\sin x}{\cos x} \\ &= (\sin x)(\cos x)^{-1} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= ((\sin x)(-(\cos x)^{-2})(\sin x)) + (\cos x)^{-1}(\cos x) \\ &= \frac{\sin^2 x}{\cos^2 x} + \frac{\cos x}{\cos x} \\ &= \tan^2 x + 1 \end{aligned}$$

Therefore, $\frac{dy}{dx} = 1 + \tan^2 x$.

Chapter 4 Section 3 **Question 20 Page 232**

Answers may vary. For example:

$$\begin{aligned} y &= \cot x \\ &= \frac{\cos x}{\sin x} \\ &= (\cos x)(\sin x)^{-1} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= ((\cos x)(-(\sin x)^{-2})(\cos x)) + (\sin x)^{-1}(-\sin x) \\ &= -\frac{\cos^2 x}{\sin^2 x} - \frac{\sin x}{\sin x} \\ &= -\frac{\cos^2 x}{\sin^2 x} - \frac{\sin^2 x}{\sin^2 x} \\ &= \frac{-(\cos^2 x + \sin^2 x)}{\sin^2 x} \\ &= \frac{-1}{\sin^2 x} \\ &= -\csc^2 x \end{aligned}$$

Therefore, $\frac{dy}{dx} = -\csc^2 x$.

Chapter 4 Section 3 **Question 21 Page 232**

Answers may vary. For example:

$$y = \cos^3 5x$$

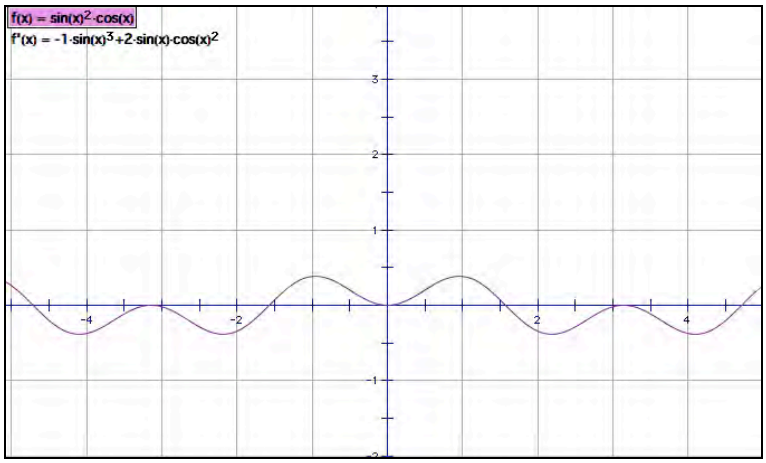
$$\frac{dy}{dx} = 3(\cos^2 5x)(-\sin 5x)(5)$$

$$= -15(\cos^2 5x)(-\sin 5x)$$

Chapter 4 Section 3 **Question 22 Page 232**

a) $f'(x) = (2\sin x)(\cos^2 x) - \sin^3 x$

b)



c) Yes. The software produced the same equation as the one in part a).

Chapter 4 Section 3 **Question 23 Page 232**

The correct answer is D.

The given infinite series is a geometric series with a common ratio $-\tan^2 x$. The sum of an infinite geometric series is given by Sum = $\frac{a}{1-r}$, where a is the first term and r is the common ratio.

$$S(x) = \frac{1}{1-(-\tan^2 x)}$$

$$= \frac{1}{\sec^2 x}$$

$$= \cos^2 x$$

Therefore:

$$S'(x) = -2\cos x \sin x$$

$$= -\sin 2x$$

Chapter 4 Section **Applications of Sinusoidal Functions and Their Derivatives**

Chapter 4 Section 4 **Question 1 Page 241**

a) $I(t) = 60\cos t + 25$ has a maximum value of 85 A when $\cos t = 1$ and a minimum value of –35 A when $\cos t = -1$.

Maximum current: 85 A at times t, in seconds, $\{t \mid t = 2k\pi,\ k \in \mathbb{Z},\ k \geq 0\}$
Minimum current: −35 A at times t, in seconds, $\{t \mid t = (2k+1)\pi,\ k \in \mathbb{Z},\ k \geq 0\}$

b) i) It is 2π since the function $I(t) = 60\cos t + 25$ has the same period as $\cos t$.
$T = 2\pi$ s

ii) The frequency is the reciprocal of the period.

Therefore, $f = \frac{1}{2\pi}$ Hz.

iii) The amplitude is the given by:

$$A = \frac{85 - (-35)}{2}$$
$$= 60$$

The amplitude is 60 A.

Chapter 4 Section 4 **Question 2 Page 241**

a) $V(t) = 170\sin(120\pi t)$
$V'(t) = 170\left(\cos(120\pi t)\right)120\pi$

To find the maximum and minimum voltage, set the first derivative of voltage to zero. This will provide you with the complete set of critical points.

$V'(t) = 0$ when $\cos(120\pi t) = 0$.

$$120\pi t = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$
$$t = \ldots, -\frac{3}{240}, -\frac{1}{240}, \frac{1}{240}, \frac{3}{240}, \ldots$$

Since time cannot be negative:

$$t = \left\{\frac{2k+1}{240}, k \geq 0, k \in \mathbb{Z}\right\}.$$

To determine the set of maxima and minima, consider $t = \frac{1}{240}$ and $t = \frac{3}{240}$.

At $t = \frac{1}{240}$,

$$V(t) = 170\sin\left(120\pi \cdot \frac{1}{240}\right)$$
$$= 170\sin\frac{\pi}{2}$$
$$= 170$$

This value for $V(t)$ occurs at $t = \left\{\frac{4k+1}{240},\ k \in \mathbb{Z},\ k \geq 0\right\}$.

At $t = \frac{3}{240}$,

$$V(t) = 170\sin\left(120\pi \cdot \frac{3}{240}\right)$$
$$= 170\sin\frac{3\pi}{2}$$
$$= -170$$

This value for $V(t)$ occurs at $t = \left\{\frac{4k+3}{240},\ k \in \mathbb{Z},\ k \geq 0\right\}$.

Maximum voltage: 170 V at times t, in seconds, $t = \left\{\frac{4k+1}{240},\ k \in \mathbb{Z},\ k \geq 0\right\}$.

Minimum voltage: −170 V at times t, in seconds, $t = \left\{\frac{4k+3}{240},\ k \in \mathbb{Z},\ k \geq 0\right\}$.

b) i) The period is $\frac{2\pi}{120\pi} = \frac{1}{60}$ s since the function $V(t) = 170\sin 120\pi t$ has the same period as $\sin 120\pi t$.

ii) $f = 60$ Hz

iii) The amplitude is given by:

$$A = \frac{170 - (-170)}{2}$$
$$= 170$$

The amplitude is 170 V.

Chapter 4 Section 4 **Question 3 Page 241**

a) Note that the length of the pendulum and the horizontal displacement are measured in centimetres and acceleration due to gravity is measured is meters per square seconds. Convert 50 cm and 8 cm into 0.5 m and 0.08 m respectively.

$$\begin{aligned} T &= 2\pi\sqrt{\frac{l}{g}} \\ &= 2\pi\sqrt{\frac{0.5}{9.8}} \\ &\doteq 1.42 \text{ s} \end{aligned}$$

b) $$\begin{aligned} h(t) &= A\cos\frac{2\pi t}{T} \\ &= 8\cos\frac{2\pi t}{1.42} \\ &\doteq 8\cos 1.4\pi t \end{aligned}$$

Based on this equation, the horizontal position is measured in centimetres.

c) $$\begin{aligned} v(t) &= h'(t) \\ &= 8(-1.4\pi)(\sin 1.4\pi) \\ &= -11.2\pi\sin 1.4\pi t \end{aligned}$$

Based on this equation, the velocity is measured in centimetres per second.

d) $$\begin{aligned} a(t) &= v'(t) \\ &= -11.2\pi(1.4\pi)\cos 1.4\pi t \\ &= -15.68\pi^2\cos 1.4\pi t \end{aligned}$$

Based on this equation, the acceleration is measured in centimetres per square seconds.

Chapter 4 Section 4 **Question 4 Page 241**

a) $v(t) = -11.2\pi\sin 1.4\pi t$ is maximised when $\sin 1.4\pi t$ is minimized. Since $-1 \le \sin 1.4\pi t \le 1$, the minimum possible value of $\sin 1.4\pi t$ is –1.

$$\begin{aligned} 1.4\pi t &= \frac{3\pi}{2} \\ t &= \frac{3}{2.8} \\ t &\doteq 1.1 \end{aligned}$$

$v(1.1) = 35.19$

Maximum velocity of the bob, 35.2 cm/s, first occurs at time t = 1.1 s.

b) $a(t) = -15.68\pi^2 \cos 1.4\pi t$ is maximised when $\cos 1.4\pi t$ is minimized. Since $-1 \le \cos 1.4\pi t \le 1$, the minimum possible value of $\cos 1.4\pi t$ is –1.

$1.4\pi t = \pi$

$$t = \frac{1}{1.4}$$

$$t \doteq 0.71$$

$a(0.71) \doteq 154.8$

Maximum acceleration of the bob, 154.8 cm/s^2, first occurs at time t = 0.71 s.

c) i) Answers may vary. For example:

Displacement equals zero when $h(t) = 0$.

$$0 = A\cos\left(\frac{2\pi t}{T}\right)$$

$$\frac{2\pi t}{T} = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$t = \frac{T}{4} \text{ or } \frac{3T}{4}$$

That is, at the $\frac{1}{4}$ and $\frac{3}{4}$ way point of each complete oscillation.

ii) The velocity equals zero when $v(t) = 0$.

$$0 = -A\left(\frac{2\pi}{T}\right)\sin\left(\frac{2\pi t}{T}\right)$$

$$\frac{2\pi t}{T} = 0 \text{ or } \pi \text{ or } 2\pi$$

$$t = 0 \text{ or } \frac{T}{2} \text{ or } T$$

That is, at the beginning point, middle point, and end point of each complete oscillation.

iii) The acceleration equals zero when $a(t) = 0$.

$$0 = -A\left(\frac{2\pi}{T}\right)^2 \cos\left(\frac{2\pi t}{T}\right)$$

$$\frac{2\pi t}{T} = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$t = \frac{T}{4} \text{ or } \frac{3T}{4}$$

That is, the acceleration equals zero: at the $\frac{1}{4}$ and $\frac{3}{4}$ way point of each complete oscillation.

d) Answers may vary. For example:

The acceleration is equal to zero when the pendulum is in a vertical position and its displacement equals zero. The velocity will equal zero when the displacement of the pendulum is at a maximum.

Chapter 4 Section 4 **Question 5 Page 241**

a) The period of each oscillation is 1 s. Frequency of the oscillating spring is given by:

$$f = \frac{1}{T}$$
$$= 1$$

The frequency is 1 Hz.

b) To get a simplified expression for the position of the marble as a function of time, substitute the values of f and A into the function $h(t)$.

$$h(t) = A\cos 2\pi ft$$
$$= 10\cos 2\pi(1)t$$
$$= 10\cos 2\pi t$$

c) Velocity is defined as the rate of change of displacement.

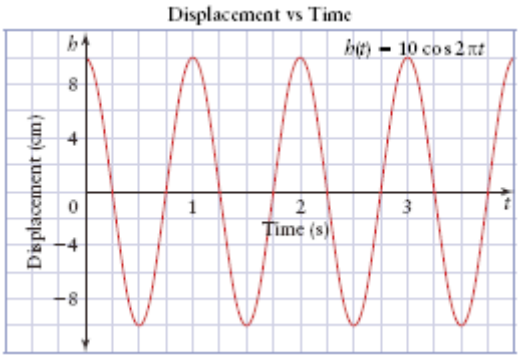
$$v(t) = h'(t)$$
$$= -20\pi \sin 2\pi t$$

d) Acceleration is defined as the rate of change of velocity.

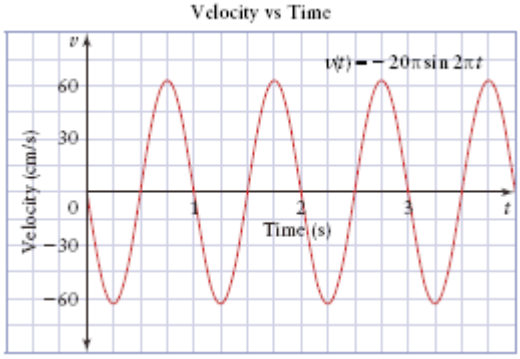
$$a(t) = v'(t)$$
$$= -40\pi^2 \cos 2\pi t$$

Chapter 4 Section 4 **Question 6 Page 242**

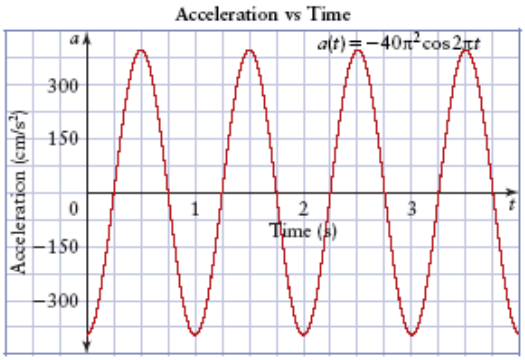
a) i)



ii)



iii)



b) Answers may vary. For example:

Similarities: The graphs of displacement versus time, velocity versus time, and acceleration versus time are all sinusoidal functions. The three graphs have the same period. The graphs of displacement versus time and acceleration versus time have the same zeros.

Differences: The three graphs have different amplitudes. The three graphs are graphs of a sine function shifted horizontally to the left or horizontally to the right.

c) Answers may vary. For example:

Maximum value(s) for displacement: The maximum displacement is 10 cm at the beginning point, middle point and end point of each complete oscillation.

Minimum value(s) for displacement: It is 0 cm at the $\frac{1}{4}$ way and $\frac{3}{4}$ way point of each complete oscillation.

These values make sense because when the bob is at its greatest displacement it will be at rest, but will quickly accelerate from rest.

Chapter 4 Section 4 **Question 7 Page 242**

a) Velocity of the piston head as a function of time:

$$\begin{aligned} v(t) &= h'(t) \\ &= -0.65\sin 13t \end{aligned}$$

b) First find all the critical points of the function $v(t)$.

$$\begin{aligned} v'(t) &= a(t) \\ &= \frac{d}{dt}[-0.65\sin 13t] \\ &= -0.65(13)\cos 13t \end{aligned}$$

Solve $v'(t) = 0$.

$$\begin{aligned} v'(t) &= 0 \\ -0.65(13)\cos 13t &= 0 \\ \cos 13t &= 0 \\ 13t &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \\ t &= \frac{\pi}{26}, \frac{3\pi}{26}, \frac{5\pi}{26}, \ldots \end{aligned}$$

Negative values of time have no meaning in this situation.

Evaluate $v(t)$ at $t = \frac{\pi}{26}$ and $t = \frac{3\pi}{26}$.

$$\begin{aligned} v\left(\frac{\pi}{26}\right) &= -0.65\sin\left(13\frac{\pi}{26}\right) \\ &= -0.65\sin\left(\frac{\pi}{2}\right) \\ &= -0.65 \end{aligned} \qquad \begin{aligned} v\left(\frac{3\pi}{26}\right) &= -0.65\sin\left(13\frac{3\pi}{26}\right) \\ &= -0.65\sin\left(\frac{3\pi}{2}\right) \\ &= 0.65 \end{aligned}$$

Maximum velocity: 0.65 m/s at time t, in seconds, $\left\{t \mid t = \frac{(4k+3)\pi}{26},\ k \in \mathbb{Z},\ k \geq 0\right\}$.

Minimum velocity: −0.65 m/s at time t, in seconds, $\left\{t \mid t = \frac{(4k+1)\pi}{26},\ k \in \mathbb{Z},\ k \geq 0\right\}$.

Chapter 4 Section 4 **Question 8 Page 242**

a) The AC component is:

$$V_{AC}(t) = A\sin 2\pi ft$$
$$= 380\sin 120\pi t$$

The DC component is: 120 kV.

Therefore, the total voltage: $V(t) = 380\sin(120\pi t) + 120$.

b) $V(t) = 380\sin(120\pi t) + 120$

$$V'(t) = 380(120\pi)\cos(120\pi t)$$

The critical points are given by $V'(t) = 0$.

$$0 = 380(120\pi)\cos(120\pi t)$$
$$0 = \cos(120\pi t)$$
$$120\pi t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$
$$t = \frac{1}{240}, \frac{3}{240}, \frac{5}{240}, \ldots$$

$$V\left(\frac{1}{240}\right) = 380 + 120 \qquad V\left(\frac{3}{240}\right) = -380 + 120$$
$$= 500 \text{ kV} \qquad = -260 \text{ kV}$$

Maximum voltage: 500 kV at time $\left\{t \middle| t = \frac{4k+1}{240},\ k \in \mathbb{Z},\ k \geq 0\right\}$.

Minimum voltage: –260 kV at time $\left\{t \middle| t = \frac{4k+3}{240},\ k \in \mathbb{Z},\ k \geq 0\right\}$.

Chapter 4 Section 4 **Question 9 Page 242**

Answers may vary. For example:

$$y = \pi\sin\theta + 2\pi\cos\theta$$
$$\frac{dy}{d\theta} = \pi\cos\theta - 2\pi\sin\theta$$
$$\frac{d^2y}{d\theta^2} = \pi(-\sin\theta) - 2\pi\sin\theta$$
$$\frac{d^2y}{d\theta^2} = -\pi\sin\theta - 2\pi\sin\theta$$

Therefore:

$$\frac{d^2y}{d\theta^2} + y = -\pi\sin\theta - 2\pi\sin\theta + \pi\sin\theta + 2\pi\cos\theta$$
$$= 0$$

The function $y = \pi\sin\theta + 2\pi\cos\theta$ is a solution to the differential equation: $\frac{d^2y}{d\theta^2} + y = 0$.

Chapter 4 Section 4 **Question 10 Page 242**

Answers may vary. For example:

a) A function that satisfies the differential equation $\frac{d^2y}{dx^2} = -4y$ is $y = \sin(2x)$.

b) $y = \sin(2x)$

$$\frac{dy}{dx} = 2\cos(2x)$$
$$\frac{d^2y}{dx^2} = (2)^2(-\sin(2x))$$
$$= -4\sin(2x)$$
$$= -4y$$

Chapter 4 Section 4 **Question 11 Page 242**

Answers will vary. For example:

a) A differential equation that is satisfied by a sinusoidal function is the function $\frac{d^2y}{d\theta^2} = -9y$. The sinusoidal function is $y = \cos 3\theta$.

$$y = \cos 3\theta$$
$$\frac{dy}{d\theta} = (-\sin 3\theta)(3)$$
$$\frac{d^2y}{d\theta^2} = -3(\cos 3\theta)(3)$$
$$\frac{d^2y}{d\theta^2} = -9(\cos 3\theta)$$

Therefore, $\frac{d^2y}{d\theta^2} = -9y$.

b) Answers may vary. For example: I used the method of trial and error.

Chapter 4 Section 4 **Question 12 Page 242**

Solutions for Achievement Checks are shown in the Teacher Resource.

Chapter 4 Section 4 **Question 13 Page 242**

The displacement x in $U = \frac{1}{2}kx^2$ can be replaced by the expression for displacement in terms of t, i.e., $h(t) = A\cos 2\pi ft$ '

Therefore, $U = \frac{1}{2}k(A\cos 2\pi ft)^2$.

Chapter 4 Section 4 **Question 14 Page 242**

The velocity v in $K = \frac{kv^2T^2}{8\pi^2}$ can be replaced by the expression for velocity in terms of t, i.e.,

$$\begin{aligned} v(t) &= h'(t) \\ &= -A(2\pi f)\sin 2\pi ft \end{aligned}$$

$$\begin{aligned} K &= \frac{kv^2T^2}{8\pi^2} \\ &= \frac{k(-A(2\pi f)\sin 2\pi ft)^2T^2}{8\pi^2} \\ &= \frac{A^2 4\pi^2 f^2 k[\sin(2\pi t)]^2T^2}{8\pi^2} \end{aligned}$$

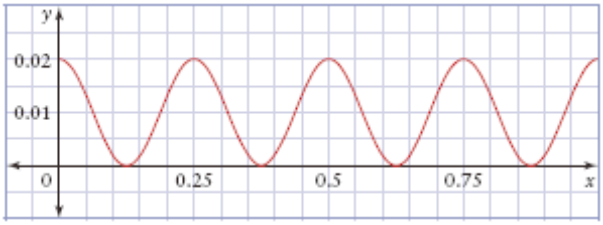
Therefore, $K = \frac{A^2k[\sin(2\pi t)]^2}{2}$.

Chapter 4 Section 4 **Question 15 Page 242**

a) $U = \frac{1}{2}k\left(A\cos 2\pi ft\right)^2$

$= 50(0.02)^2 \cos^2\left\{2\pi\left(\frac{1}{0.5}\right)t\right\}$

$= 0.02\cos^2 4\pi t$

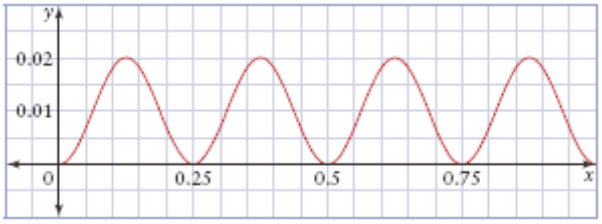


Maxima: 0.02 Nm at times t = 0, 0.25, 0.5,… s
Minima: 0 Nm at times t = 0.125, 0.375, 0.625,… s
Zeros: at times t = 0.125, 0.375, 0.625,… s

b) $K = \frac{A^2k[\sin(2\pi ft)]^2}{2}$

$= 50(0.02)^2\left\{\sin^2(2\pi\left(\frac{1}{0.5}\right)t)\right\}$

$= 0.02\sin^2 4\pi t$



Maxima: 0.02 Nm at time t = 0.125, 0.375, 0.625,… s
Minima: 0 Nm at time t = 0, 0.25, 0.5,… s
Zeros: at time t = 0, 0.25, 0.5,… s

c) Answers may vary. For example:

When the spring is either in a state of maximum extension or compression its potential energy is at a maximum and its kinetic energy is at a minimum. When the spring is in the same position as its resting position, its kinetic energy is at a maximum and its potential energy is at a minimum. The total energy is the sum of the potential energy and kinetic energy.

Chapter 4 Section 4 **Question 16 Page 242**

The correct answer is E.

Let $y = A\sin x + B\cos x$.

Then $\frac{dy}{dx} = A\cos x - B\sin x$ and $\frac{d^2y}{dx^2} = -A\sin x - B\cos x$.

If $\frac{dy}{dx} = 0$, $\tan x = \frac{A}{B}$.

Use $\sin x = \frac{A}{\sqrt{A^2 + B^2}}$ and $\cos x = \frac{B}{\sqrt{A^2 + B^2}}$,

$$\begin{aligned}\frac{d^2y}{dx^2} &= -\frac{A^2}{\sqrt{A^2 + B^2}} - \frac{B^2}{\sqrt{A^2 + B^2}} \\ &= -\sqrt{A^2 + B^2} \\ &< 0\end{aligned}$$

Therefore the local maximum value of y is:

$$\frac{A^2}{\sqrt{A^2 + B^2}} + \frac{B^2}{\sqrt{A^2 + B^2}} = \sqrt{A^2 + B^2}\ .$$

Chapter 4 Review

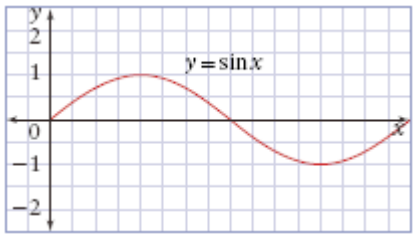
Chapter 4 Review **Question 1 Page 244**

a) i) $(0, -1), (\pi, 1), (2\pi, -1)$

ii) $(\pi, 1)$

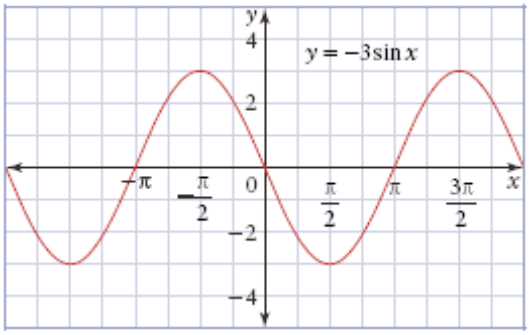
iii) $(2\pi, -1)$ or $(0, -1)$

b)

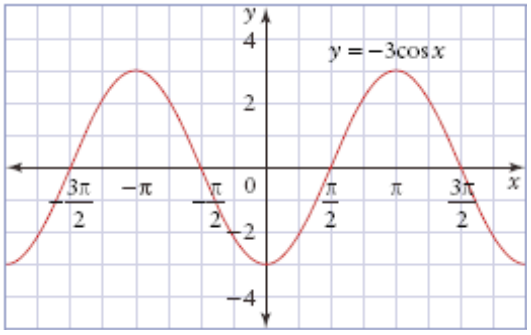


Chapter 4 Review **Question 2 Page 244**

a)



b)



Chapter 4 Review **Question 3 Page 244**

a) $\frac{dy}{dx} = -\sin x$

b) $f'(x) = -2\cos x$

c) $\frac{dy}{dx} = -\sin x - \cos x$

d) $f'(x) = 3\cos x + \pi \sin x$

Chapter 4 Review **Question 4 Page 244**

The slope of a function at a point is given by the value of its first derivative at that point.

$$y = 4\sin x$$

$$\frac{dy}{dx} = 4\cos x$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}} = 4\cos\frac{\pi}{3}$$

$$= 2$$

Therefore, the slope of the function $y = 4\sin x$ at $x = \frac{\pi}{3}$ is 2.

Chapter 4 Review **Question 5 Page 244**

a) To find the equation of the tangent line, find the slope and the y-coordinates at $\theta = \frac{\pi}{4}$.

$$y = 2\sin\theta + 4\cos\theta$$

$$\frac{dy}{d\theta} = 2\cos\theta - 4\sin\theta$$

$$\left.\frac{dy}{d\theta}\right|_{\theta=\frac{\pi}{4}} = \frac{2}{\sqrt{2}} - \frac{4}{\sqrt{2}}$$

$$= -\frac{2}{\sqrt{2}}$$

$$= -\sqrt{2}$$

At $\theta = \frac{\pi}{4}$, $y = 3\sqrt{2}$.

Use the slope-point form of a line to find the equation of the tangent line.

$$y - 3\sqrt{2} = -\sqrt{2}\left(\theta - \frac{\pi}{4}\right)$$

Therefore, the equation of the tangent is $y = -\sqrt{2}\theta + \frac{\sqrt{2}}{4}\pi + 3\sqrt{2}$.

b) Use the same approach as in part a).

$$y = 2\cos\theta - \frac{1}{2}\sin\theta$$

$$\frac{dy}{d\theta} = -2\sin\theta - \frac{1}{2}\cos\theta$$

$$\left.\frac{dy}{d\theta}\right|_{\theta=\frac{3\pi}{2}} = -2(-1) - \frac{1}{2}(0)$$

$$= 2$$

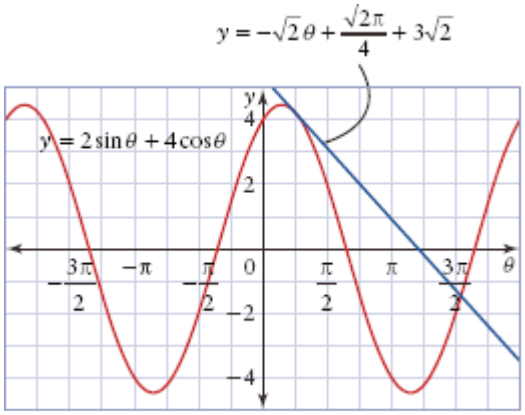
At $\theta = \frac{3\pi}{2}$, $y = \frac{1}{2}$.

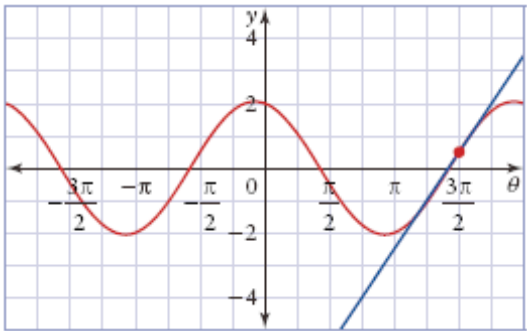
Use the slope-point form of a line to find the equation of the tangent.

$$y - \frac{1}{2} = 2\left(\theta - \frac{3\partial}{2}\right)$$

Therefore, the equation of the tangent is $y = 2\theta - 3\pi + \frac{1}{2}$.

c)





Chapter 4 Review **Question 6 Page 244**

a) $\frac{dy}{dx} = -2\cos x(-\sin x)$

$= \sin 2x$

b) $\frac{dy}{d\theta} = 2\cos 2\theta + 4\sin 2\theta$

c) $f'(\theta) = -\pi\cos(2\theta - \pi)$

d) $f'(x) = (\cos(\sin x))(\cos x)$

e) $f'(x) = \left[-\sin(\cos x)\right](-\sin x)$

$= \left[\sin(\cos x)\right](\sin x)$

f) $f'(\theta) = -7\sin(7\theta) + 5\sin(5\theta)$

Chapter 4 Review **Question 7 Page 244**

a) $\frac{dy}{dy} = 3x\cos x + 3\sin x$

b) $f'(t) = -4t^2\sin 2t + 4t\cos 2t$

c) $\frac{dy}{dt} = \pi^2 t(\cos(\pi t - 6)) + \pi\sin(\pi t - 6)$

d) $\frac{dy}{d\theta} = -\left[\sin(\sin\theta)\right](\cos\theta) - \left[\cos(\cos\theta)\right](\sin\theta)$

e) $f'(x) = -2\left[\cos(\sin x)\right]\left[\sin(\sin x)\right](\cos x)$

f) $f'(\theta) = -7(\sin 7\theta) + 10(\cos 5\theta)(\sin 5\theta)$

Chapter 4 Review **Question 8 Page 244**

a) Answers may vary. For example:

To find the slope, $f'(x)$, first find all the critical points at $f''(x) = 0$.

$f(x) = 2\cos 3x$

$f'(x) = -6\sin 3x$

$f''(x) = -18\cos 3x$

Let $f''(x) = 0$

$\cos 3x = 0$

$3x = ..., -\frac{3}{2}\pi, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, ...$

$x = ..., -\frac{3}{6}\pi, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{3\pi}{6}, ...$

$x = \frac{(2k+1)\pi}{6}, k \in \mathbb{Z}$

At, $x = ..., -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{3\pi}{6}, \frac{7\pi}{6}, ...$, the maximum value of the slope is obtained $f'(x) = 6$.

Need only one equation of the tangent, so let $x = \frac{\pi}{2}$.

$f\left(\frac{\pi}{2}\right) = 2\cos\frac{3\pi}{2}$

$= 0$

Use the slope-point form of a line to find the equation of the tangent line.

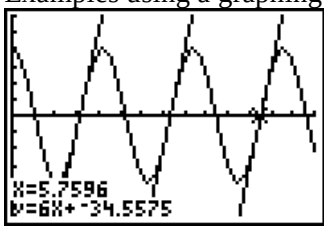
$y - 0 = 6\left(x - \frac{\pi}{2}\right)$

Therefore, the equation of the tangent is $y = 6x - 3\pi$.

b) No, there are an infinite number of tangent lines to the curve $y = 2\cos 3x$ whose slope is a maximum.

From part a), at $x = ..., -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{3\pi}{6}, \frac{7\pi}{6}, ...$, the maximum value of the slope is obtained $f'(x) = 6$. In more general terms, the set of all such x's can be expressed as $x = \frac{(4k+3)\pi}{6}, k \in \mathbb{Z}$. For each value of k, you will obtain a different equation for the tangent line with maximised slope.

Examples using a graphing calculator and the tangent function are shown below.



Chapter 4 Section Review **Question 9 Page 244**

a) $V(t) = 130\sin(5t) + 18$

$V'(t) = 650\cos(5t)$

The critical points are given by $V'(t) = 0$.

$0 = 650\cos(5t)$

$0 = \cos(5t)$

$5t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$t = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \ldots$

$V\left(\frac{\pi}{10}\right) = 130\sin\left(5\times\frac{\pi}{10}\right) + 18$

$= 130 + 18$

$= 148$

$V\left(\frac{3\pi}{10}\right) = 130\sin\left(5\times\frac{3\pi}{10}\right) + 18$

$= 130(-1) + 18$

$= -112$

Maximum voltage: 148 V at time, in seconds, $\left\{t \middle| t = \frac{(4k+1)\pi}{10},\ k \in \mathbb{Z},\ k \geq 0\right\}$.

Minimum voltage: –112 V at time, in seconds, $\left\{t \middle| t = \frac{(4k+3)\pi}{10},\ k \in \mathbb{Z},\ k \geq 0\right\}$.

b) Period: $T = \frac{2\pi}{5}$ s

$f = \frac{1}{T}$

$= \frac{5}{2\pi}$

Frequency: $\frac{5}{2\pi}$ Hz

$$A = \frac{1}{2}[148 - (-112)]$$
$$= 130$$

Amplitude: 130 V

Chapter 4 Section Review **Question 10 Page 245**

a) i) $\sin\theta = 1$

$$\theta = \frac{\pi}{2}$$

Maximum: The force $F = mg\sin\theta$ has a maximum value when $\theta = \frac{\pi}{2}$.

ii) $\sin\theta = 0$

$$\theta = 0$$

Minimum: The force $F = mg\sin\theta$ has a minimum value when $\theta = 0$.

b) Answers may vary. For example:

The formula for force is $F = mg\sin\theta$. The force will be a maximum at an angle where $\sin\theta$ is maximized i.e., $\frac{\pi}{2}$ = 90°, since sine has a maximum value at 90°. The force will be a minimum at an angle where $\sin\theta$ is minimized i.e., 0°, since sine has a minimum value at 0°.

Chapter 4 Review **Question 11 Page 245**

a) Answers may vary. For example:

Given:

$p = mv$ ① (p is the momentum of the body.)

Differentiate ① with respect to time

$\frac{dp}{dt} = m\frac{dv}{dt}$ ②

$\frac{dv}{dt} = a$ ③ (Acceleration is the rate of change of velocity.)

Therefore, $\frac{dp}{dt} = ma$.

Combined with Newton's second law of motion: $F = \frac{dp}{dt}$, this gives $F = ma$.

b) $F = ma$

$$= m\frac{dv}{dt}$$
$$= m\frac{d}{dt}(2\cos 3t)$$
$$= -6m\sin 3t$$

Therefore $F = 0$ when $\sin 3t = 0$.

$t = 0,\ \frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{3\pi}{3},\ \frac{4\pi}{3},\ \ldots$

$t = \frac{k\pi}{3},\ k \in \mathbb{Z},\ k \geq 0.$

c) $v(t) = 2\cos 3t$

$$v\left(\frac{k\pi}{3}\right) = 2\cos(k\pi)$$

$2\cos(k\pi)$ is 2 m/s when k is even and –2 m/s when k is odd.

Therefore the speed is $|v| = 2$ m/s.

Chapter 4 Practice Test

Chapter 4 Practice Test **Question 1 Page 246**

The correct answer is B.

Chapter 4 Practice Test **Question 2 Page 246**

The correct answer is C.

Chapter 4 Practice Test **Question 3 Page 246**

The correct answer is C.

Chapter 4 Practice Test **Question 4 Page 246**

The correct answer is D.

Chapter 4 Practice Test **Question 5 Page 246**

The correct answer is C.

Chapter 4 Practice Test **Question 6 Page 246**

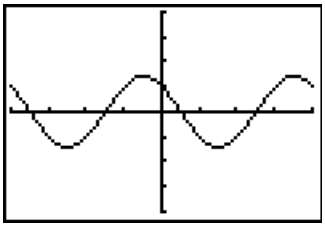
The correct answer is D.

$$\frac{dy}{dx} = 2\left(\frac{1}{2}\right)$$
$$= 1.$$

Chapter 4 Practice Test **Question 7 Page 246**

The correct answer is B.

Use a graphing calculator to graph $\frac{dy}{dx} = \cos x - \sin x$ for the given window.



Chapter 4 Practice Test **Question 8 Page 246**

The correct answer is B.

Chapter 4 Practice Test **Question 9 Page 247**

a) $\frac{dy}{dx} = -\sin x - \cos x$

b) $\frac{dy}{d\theta} = 6\cos 2\theta$

c) $f'(x) = \pi \sin x \cos x$

d) $f'(t) = 3t^2 \cos t + 6t \sin t$

Chapter 4 Practice Test **Question 10 Page 247**

a) $\frac{dy}{dx} = \cos\left(\theta + \frac{\pi}{4}\right)$

b) $\frac{dy}{d\theta} = -\sin\left(\theta - \frac{\pi}{4}\right)$

c) $\frac{dy}{d\theta} = 4\sin^3\theta\left(\cos\theta\right)$

d) $\frac{dy}{d\theta} = 4\theta^3 \cos\left(\theta^4\right)$

Chapter 4 Practice Test **Question 11 Page 247**

The slope of the tangent is the same as that of the curve, i.e., $\frac{dy}{dx}$ at $\frac{\pi}{4}$.

$\frac{dy}{dx} = 2\cos^2 x - 2\sin^2 x$

At $x = \frac{\pi}{4}$,

$$\begin{aligned}\frac{dy}{dx} &= 2\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right) \\ &= 0\end{aligned}$$

Therefore, the slope of the line tangent to the curve $y = 2\sin x \cos x$ at $x = \frac{\pi}{4}$ is 0.

Chapter 4 Practice Test **Question 12 Page 247**

$$\frac{dy}{dx} = -6\cos^2 x \sin x$$

At $x = \frac{\pi}{3}$,

$$\frac{dy}{dx} = -6\left(\frac{1}{4}\right)\left(\frac{\sqrt{3}}{2}\right)$$
$$= -\frac{3\sqrt{3}}{4}$$
$$= m$$

Also at $x = \frac{\pi}{3}$,

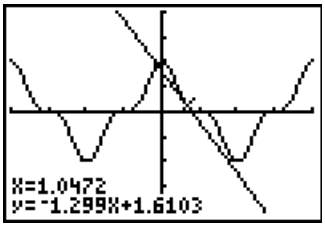
$$y = 2\left(\frac{1}{2}\right)^3$$
$$= \frac{1}{4}$$

Substituting for m, y, and x in $y = mx + b$ gives:

$$b = \frac{1}{4} + \frac{3\sqrt{3}}{4}\left(\frac{\pi}{3}\right)$$
$$= \frac{1}{4} + \frac{\sqrt{3}\pi}{4}$$

$y = -\frac{3\sqrt{3}}{4}x + \frac{\sqrt{3}\pi}{4} + \frac{1}{4}$ is the equation of the tangent line.

Use a graphing calculator and the tangent function to confirm this result.



Chapter 4 Practice Test **Question 13 Page 247**

a) $V(t) = 325\sin(100\pi t)$

$V'(t) = 325(100\pi)\cos(100\pi t)$

The critical points are given by $V'(t) = 0$.

$0 = 325(100\pi)\cos(100\pi t)$

$0 = \cos(100\pi t)$

$100\pi t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$t = \frac{1}{200}, \frac{3}{200}, \frac{5}{200}, \ldots$

$V\left(\frac{1}{200}\right) = 325\sin\left(100\pi\left(\frac{1}{200}\right)\right)$

$= 325$

$V\left(\frac{3}{200}\right) = 325\sin\left(100\pi\left(\frac{3}{200}\right)\right)$

$= -325$

Maximum voltage: 325 V at time, in seconds, $\left\{t \middle| t = \frac{4k+1}{200},\ k \in \mathbb{Z},\ k \geq 0\right\}$.

Minimum voltage: –325 V at time, in seconds, $\left\{t \middle| t = \frac{4k+3}{200},\ k \in \mathbb{Z},\ k \geq 0\right\}$.

b) i) $T = \frac{2\pi}{100\pi}$

$= \frac{1}{50}$

The period is $\frac{1}{50}$ s.

ii) $f = \frac{1}{T}$

$= 50$ Hz

The frequency is 50 Hz.

iii) $A = \frac{1}{2}[325 - (-325)]$

$= 325$

The amplitude is 325 V.

Chapter 4 Practice Test **Question 14 Page 247**

a) $V(t) = 170\sin(120\pi t)$

$V'(t) = 170(120\pi)\cos(120\pi t)$

The critical points are given by $V'(t) = 0$.

$0 = 170(120\pi)\cos(120\pi t)$

$0 = \cos(120\pi t)$

$120\pi t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$t = \frac{1}{240}, \frac{3}{240}, \frac{5}{240}, \ldots$

$V\left(\frac{1}{240}\right) = 170\sin\left(120\pi\left(\frac{1}{240}\right)\right)$

$= 170$

$V\left(\frac{3}{240}\right) = 170\sin\left(120\pi\left(\frac{3}{240}\right)\right)$

$= -170$

Maximum voltage: 170 V at time, in seconds, $\left\{t \middle| t = \frac{4k+1}{200},\ k \in \mathbb{Z},\ k \geq 0\right\}$.

Minimum voltage: –170 V at time, in seconds, $\left\{t \middle| t = \frac{4k+3}{200},\ k \in \mathbb{Z},\ k \geq 0\right\}$.

i) $T = \frac{2\partial}{120\partial}$

$= \frac{1}{60}$

The period is $\frac{1}{60}$ s.

ii) $f = \frac{1}{T}$

$= 60$ Hz

The frequency is 60 Hz.

iii) $A = \frac{1}{2}[170 - (-170)]$

$= 170$ V

The amplitude is 170 V.

b) Answers may vary. For example;

Similarities: Both functions are sinusoidal functions and both functions pass through the origin (0, 0).
Differences: The functions have different periods, frequencies, and amplitudes.

Chapter 4 Practice Test **Question 15 Page 247**

$$\text{L.S.} = \frac{d}{dx}(\sin 2x)$$
$$= 2\cos 2x$$

$$\text{R.S.} = \frac{d}{dx}(2\sin x \cos x)$$
$$= 2\cos^2 x - 2\sin^2 x$$

Recall:
$$\sin 2x = 2\sin x \cos x$$
$$2\cos 2x = 2\cos^2 x - 2\sin^2 x$$

Therefore, differentiating both sides of the given equation gives the identity $\cos 2x = \cos^2 x - \sin^2 x$.

Chapter 4 Practice Test **Question 16 Page 247**

a) $f'(x) = -\sin^2 x + \cos^2 x$
$f''(x) = -4\sin x \cos x$

b) $f'''(x) = -4\left[-\sin^2 x + \cos^2 x\right]$
$f^{(4)}(x) = 16\sin x \cos x$
$f^{(5)}(x) = 16\left[-\sin^2 x + \cos^2 x\right]$
$f^{(6)}(x) = -64\sin x \cos x$

Answers may vary. For example:

The first, third, and fifth derivatives all have the expression $-\sin^2 x + \cos^2 x$ in the derivative.
The second, fourth, and sixth derivative all have the expression $\sin x \cos x$ in the derivative.
The second derivative is the original function multiplied by −4.
The third derivative is the first derivative multiplied by −4.
This pattern continues for fourth to sixth derivatives so that the n derivative is the $(n - 2)$ derivative multiplied by –4.

c) Answers may vary. For example:

My prediction for the seventh derivative is $f^{(7)}(x) = -64\left[-\sin^2 x + \cos^2 x\right]$.

My prediction for the eighth derivative is $f^{(8)}(x) = 256\sin x \cos x$.

When the sixth derivative is differentiated, the seventh derivative is $f^{(7)}(x) = -64\left[-\sin^2 x + \cos^2 x\right]$ as predicted. When the seventh derivative is differentiated, the eighth derivative is found to be $f^{(8)}(x) = 256\sin x \cos x$ as predicted.

d) Answers may vary. For example:

i) $f^{(2n)}(x) = (-4)^n \sin x \cos x$

ii) $f^{(2n+1)}(x) = (-4)^n(-\sin^2 x + \cos^2 x)$

e) i) $f^{(12)}(x) = 4096 \sin x \cos x$

ii) $f^{(15)}(x) = -16384(-\sin^2 x + \cos^2 x)$

Chapter 4 Practice Test **Question 17 Page 247**

Answers will vary. For example:

a) $y = \sin x$

$$\frac{dy}{dx} = \cos x$$

$$\frac{d^2y}{dx^2} = -\sin x$$

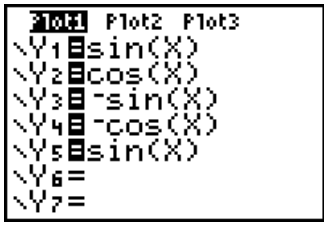
$$\frac{d^3y}{dx^3} = -\cos x$$

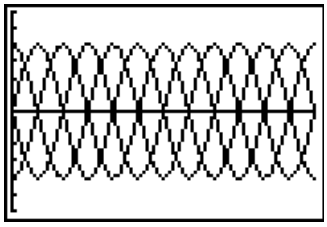
$$\frac{d^4y}{dx^4} = \sin x$$

$$\frac{d^5y}{dx^5} = \cos x$$

b) $y = \cos x$, $y = -\cos x$

c) There are four functions that satisfy this differential equation. The fourth function is $y = -\sin x$. The functions are sinusoidal and the derivatives of each of the functions are shifted horizontally to the left, or to the right, $\frac{\pi}{2}$ units. The graph of the fifth derivative of each of the functions will be the same as the graph of the first derivative of each of the functions.



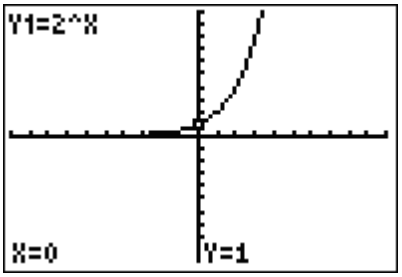


Chapter 5 **Exponential and Logarithmic Functions**

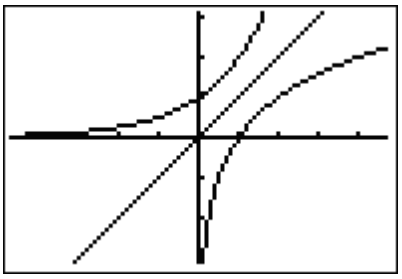
Chapter 5 Prerequisite Skills

Chapter 5 Prerequisite Skills **Question 1 Page 250**

a)



b)



c) Answers may vary. For example:
The equation of the inverse is $y = \log_2 x$ since $2^{\log_2 x} = x$.

Chapter 5 Prerequisite Skills **Question 2 Page 250**

a) $y = 2^x$: Domain: $(-\infty, \infty)$
Range: $(0, \infty)$

$y = \log_2 x$: Domain: $(0, \infty)$
Range: $(-\infty, \infty)$

b) $y = 2^x$: x-intercept: none
y-intercept: 1

$y = \log_2 x$: x-intercept: 1
y-intercept: none

c) $y = 2^x$: The function is increasing on the interval: $x \in (-\infty, \infty)$
$y = \log_2 x$: The function is increasing on the interval: $y \in (-\infty, \infty)$

d) $y = 2^x$: $y = 0$
$y = \log_2 x$: $x = 0$

Chapter 5 Prerequisite Skills **Question 3 Page 250**

a) $2^3 = 8$

b) $2^{3.5} \doteq 11.3$

c) $2^{1.5} \doteq 2.8$

d) $\log_2 10 \doteq 3.3$

e) $\log_2 7 \doteq 2.8$

f) $\log_2 4.5 \doteq 2.2$

Chapter 5 Prerequisite Skills **Question 4 Page 250**

a) $2^3 = 8$

b) $2^{3.5} \doteq 11.3$

c) $2^{1.5} \doteq 2.8$

d) $\log_2 10 \doteq 3.3$

e) $\log_2 7 \doteq 2.8$

f) $\log_2 4.5 \doteq 2.2$

Chapter 5 Prerequisite Skills **Question 5 Page 250**

a) $y = (2^3)^x$
$= 2^{3x}$

b) $y = (2^2)^{2x}$
$= 2^{4x}$

c) $y = (2^4)^{\frac{x}{2}}$
$= 2^{2x}$

d) $y = (2^{-2})^{-2x}$
$= 2^{4x}$

Chapter 5 Prerequisite Skills **Question 6 Page 250**

a) $\log_2 5 = \dfrac{\log_{10} 5}{\log_{10} 2}$

$\doteq 2.322$

b) $\log_4 66 = \dfrac{\log_{10} 66}{\log_{10} 4}$

$\doteq 3.022$

c) $\log_3 10 = \dfrac{\log_{10} 10}{\log_{10} 3}$

$\doteq 2.096$

d) $\log_2 7 = \dfrac{\log_{10} 7}{\log_{10} 2}$

$\doteq 2.807$

e) $\log_3 75 = \dfrac{\log_{10} 75}{\log_{10} 3}$

$\doteq 3.930$

f) $\log_5\left(\dfrac{1}{10}\right) = \dfrac{\log_{10} 0.1}{\log_{10} 5}$

$\doteq -1.431$

g) $\log_{\frac{1}{2}}\left(\dfrac{1}{4}\right) = \dfrac{\log_{10} 0.25}{\log_{10} 0.5}$

$= 2$

h) $\log_{0.5} 5 = \dfrac{\log_{10} 5}{\log_{10} 0.5}$

$\doteq -2.322$

Chapter 5 Prerequisite Skills **Question 7 Page 250**

a) $(h^2k^3)(hk^{-2}) = h^3k$

b) $(a^3)(ab^3)^2 = (a^3)(a^2b^6)$

$= a^5b^6$

c) $\dfrac{(x)(y^3)^{-2}}{(x^3y^3)^4} = \dfrac{(x)(y^{-6})}{x^{12}y^{12}}$

$= \dfrac{1}{x^{11}y^{18}}$

d) $\dfrac{8u^3v^{-2}}{4uv^{-1}} = \dfrac{2u^2}{v}$

e) $(g^2)(gh^3)^{-2} = (g^2)(g^{-2}h^{-6})$

$= \dfrac{1}{h^6}$

f) $x^2x^4 + (x^2)^3 = x^{2+4} + x^{2\times3}$

$= 2x^6$

g) $\dfrac{2^x4^x}{4^{-x}} = \dfrac{2^x(2^2)^x}{(2^2)^{-x}}$

$= 2^{5x}$

h) $\dfrac{a^xb^{2x}}{ab^x} = a^{x-1}b^x$

Chapter 5 Prerequisite Skills **Question 8 Page 250**

a) $\log 5 + \log 2 = \log(5\times2)$

$= \log 10$

$= 1$

b) $\log_2 24 - \log_2 3 = \log_2\left(\dfrac{24}{3}\right)$

$= \log_2 8$

$= 3$

c) $\log_5 50 - \log_5 0.08 = \log_5\left(\dfrac{50}{0.08}\right)$

$= \log_5 625$

$= 4$

d) $\log(0.01)^3 = 3(\log 0.01)$

$= -6$

e) $\log\sqrt{1000} + \log\sqrt[3]{100} = \frac{1}{2}(\log 1000) + \frac{1}{3}(\log 100)$

$= \frac{3}{2} + \frac{2}{3}$

$= \frac{13}{6}$

f) $2\log 2 + 2\log 5 = 2\log(2 \times 5)$

$= 2\log 10$

$= 2$

Chapter 5 Prerequisite Skills **Question 9 Page 250**

a) $\log a - \log 2a = \log\left(\frac{a}{2a}\right)$

$= \log\left(\frac{1}{2}\right)$

b) $\log ab + \log a - \log ab^2 = \log\left(\frac{aba}{ab^2}\right)$

$= \log\left(\frac{a}{b}\right)$

c) $4\log a^2 - 4\log a = \log\left(\frac{a^8}{a^4}\right)$

$= \log a^4$

$= 4\log a$

d) $3\log a^2 b + 3\log ab^2 = \log[(a^6 b^3)(a^3 b^6)]$

$= \log(a^9 b^9)$

$= 9\log ab$

e) $\log 2a^2 b + \log 2b^2 = \log[(2a^2 b)(2b^2)]$

$= \log(4a^2 b^3)$

Chapter 5 Prerequisite Skills **Question 10 Page 251**

a) $2^x = 4^{x+1}$

$2^x = (2^2)^{x+1}$

$x = 2x + 2$

$x = -2$

b) $4^{2x+1} = 64^x$

$4^{2x+1} = (4^3)^x$

$2x + 1 = 3x$

$x = 1$

c) $3^{2x-5} = \sqrt{27}$

$3^{2x-5} = (3^3)^{\frac{1}{2}}$

$2x - 5 = \frac{3}{2}$

$x = \frac{13}{4}$

d) $\log x - \log 2 = \log 5$

$\log x = \log(5 \times 2)$

$x = 10$

e) $\log 5 + \log x = 3$

$\log 5 + \log x = \log 1000$

$\log 5x = \log 1000$

$5x = 1000$

$x = 200$

f) $x - 3\log 5 = 3\log 2$

$x = 3\log(2 \times 5)$

$x = 3\log 10$

$x = 3$

Chapter 5 Prerequisite Skills **Question 11 Page 251**

a) $2 = 1.06^x$

$\log 2 = \log 1.06^x$

$x = \frac{\log 2}{\log 1.06}$

$x \doteq 11.9$

b) $50 = 5^{2x}$

$\log 50 = \log 5^{2x}$

$x = \dfrac{\log 50}{2\log 5}$

$x \doteq 1.2$

c) $10 = \left(\dfrac{1}{2}\right)^{x}$

$\log 10 = \log\left(\dfrac{1}{2}\right)^{x}$

$x = \dfrac{\log 10}{\log\left(\dfrac{1}{2}\right)}$

$x \doteq -3.3$

d) $75 = 225(2)^{-\frac{x}{4}}$

$\log\left(\dfrac{75}{225}\right) = \log(2)^{-\frac{x}{4}}$

$-\dfrac{x}{4} = \dfrac{\log\left(\dfrac{1}{3}\right)}{\log 2}$

$x \doteq 6.3$

Chapter 5 Prerequisite Skills **Question 12 Page 251**

a) i) 100 bacteria

ii) 200 bacteria

iii) 400 bacteria

b) C

c) Answers may vary. For example:
The formula for an exponential growth function is $P = P_0a^t$, where P is the bacteria population, P_0 is the initial bacteria population, a is the exponential base or growth rate, and t is the time for the population to grow, in this case, doubling time.

The initial population of bacteria is 50, so $P_0 = 50$.
The population doubles exponentially, so $a = 2$.
The population doubles after 3 days, so t = number of days ÷ 3.

Therefore, the correct equation is $P = 50(2)^{\frac{t}{3}}$.

Chapter 5 Prerequisite Skills **Question 13 Page 251**

a)

Time (min)	Amount Remaining (g)
0	100
5	50
10	25
15	12.5
20	6.25

b) $A(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{5}}$

c) i) $A(30) = 100\left(\frac{1}{2}\right)^{\frac{30}{5}}$

$= 100\left(\frac{1}{2}\right)^{6}$

$= 1.5625$

After half an hour, the amount remaining is 1.5625 g.

ii) Half a day is 12×60 min = 720 min.

$A(720) = 100\left(\frac{1}{2}\right)^{\frac{720}{5}}$

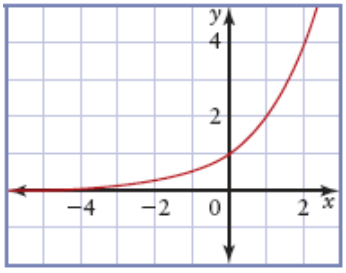
$\doteq 4.484 \times 10^{-42}$

After half a day, the amount remaining is 4.484×10^{-42} g.

Chapter 5 Section 1 **Rates of Change and the Number *e***

Chapter 5 Section 1 **Question 1 Page 256**

a)



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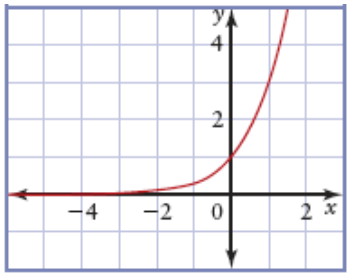
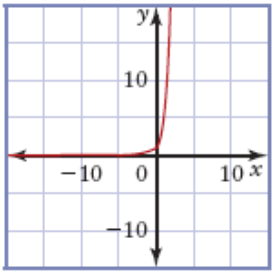
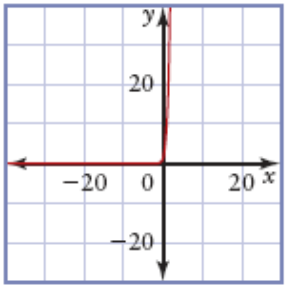


c)

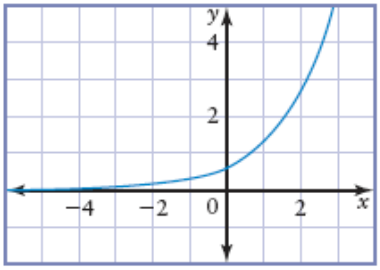


d)

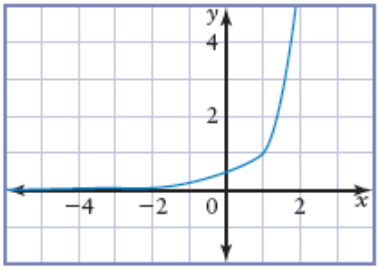


Chapter 5 Section 1 **Question 2 Page 256**

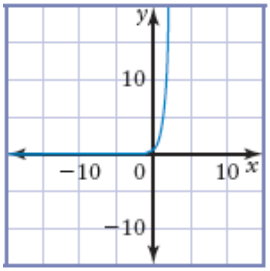
a)



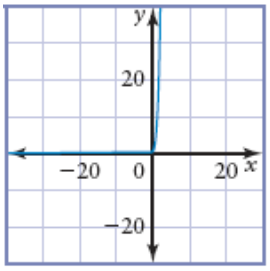
b)



c)



d)



Chapter 5 Section 1 **Question 3 Page 256**

a) $f(x) = 2^x : \{x \in \mathbb{R}\}$
$f(x) = e^x : \{x \in \mathbb{R}\}$

b) No

c) No

Chapter 5 Section 1 Question 4 Page 257

a) B. The graph of the derivative of a quadratic function is a straight line.

b) C. The graph of the derivative of a line is of the form $y = a$, where a is a constant.

c) D. The graph of the derivative of an exponential function is also an exponential function.

d) A. The graph of the derivative of a cubic function is a quadratic function.

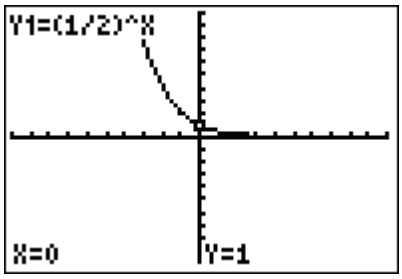
Chapter 5 Section 1 Question 5 Page 257

a) $b > e$

b) $0 \le b < e$

Chapter 5 Section 1 Question 6 Page 258

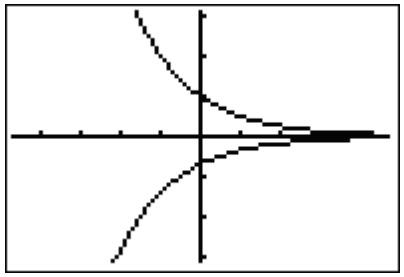
a)



b) Answers may vary. For example:

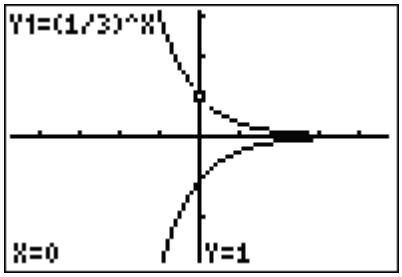
The graph of the rate of change of $y = \left(\frac{1}{2}\right)^x$ will be a compression and a reflection of the graph of y in the x-axis.

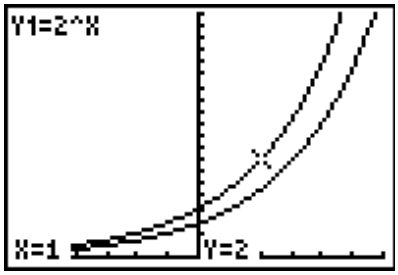
c)



Chapter 5 Section 1 **Question 7 Page 258**

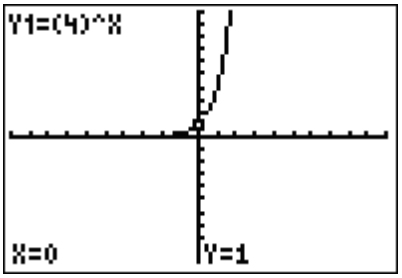
Answers may vary. For example:
If $0 < b < 1$, the graph of $y = b^x$ will be above the x-axis and the graph of the rate of change of this function will be below the x-axis. If $b > 1$, the graph of $y = b^x$ and the graph of the rate of change of this function will both be above the x-axis.



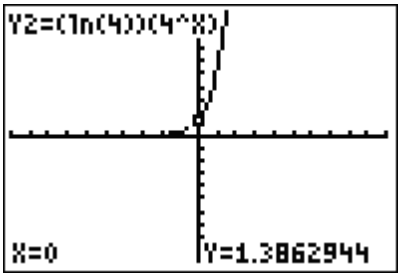


Chapter 5 Section 1 **Question 8 Page 258**

a)

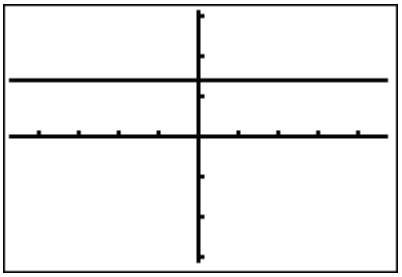


b)



c) Answers may vary. For example:
The graph of the combined function $g(x)$ will be a horizontal straight line.

d)



Answers may vary. For example:
The graph of $g(x) = \ln 4$ is a constant function. Therefore the graph is a horizontal straight line.

Chapter 5 Section 1 Question 9 Page 258

a) Answers may vary. For example:
No. The shape of the graph of g will not change. The shape of the graph of g will be a horizontal straight line. If the base is other than 4, the graph will be parallel to the graph of g and shifted up or down depending on the numerical value of the base.
If the value of the base is greater than 4, the graph will be shifted up. If the value of the base is greater than 1 and less than 4, the graph will be shifted down, but will still be above the x-axis.
If the value of the base is greater than 0 and less than 1, the graph will be shifted down and will be below the x-axis.

b) The graph is the line $g(x) = \ln e$, which is the horizontal straight line $g(x) = \dfrac{f'(x)}{f(x)}$ where $g(x) = 1$.

Chapter 5 Section 1 Question 10 Page 258

Solutions to the Achievement Checks are shown in the Teacher's Resource.

Chapter 5 Section 1 Question 11 Page 258

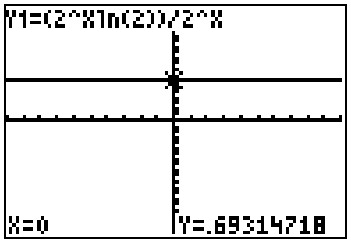
a) Answers may vary. For example:
The graph of the function $g'(x)$ will be the horizontal straight line, $y = 0$.

b) Answers may vary. For example:

If $f(x) = 2^x$, then $f'(x) = 2^x \ln 2$, and $g(x) = \dfrac{2^x \ln 2}{2^x}$.

$g(x) = \ln 2$, which is just a constant so $g'(x) = 0$. This is applicable for any base. Therefore, the graph of $g'(x)$ will be the graph of $y = 0$.

The graph of $g(x)$ is a constant function. The derivative of a constant function is 0.



c) Answers may vary. For example:
The function $g(x)$ will be a horizontal straight line for any value of b, $b > 0$ and the derivative function, $g'(x)$, will be $g'(x) = 0$, for any value of b, $b > 0$, since the derivative of any constant function is the horizontal straight line $y = 0$.

Chapter 5 Section 1 **Question 12 Page 258**

a) $\{x \in \mathbb{R}\}$

b) $\{y < 0 < 1, y \in \mathbb{R}\}$

c) As the value of c increases, the graph of the function is shifted to the right.

Chapter 5 Section 1 **Question 13 Page 258**

Answers may vary. For example:
Some answers include: Leonhard Euler; 1727; e is used in probability

Chapter 5 Section 1 **Question 14 Page 258**

B

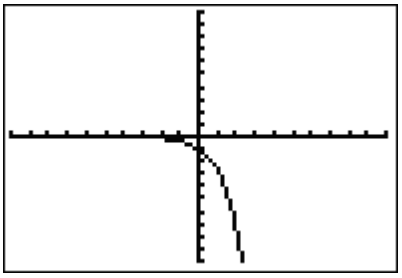
Chapter 5 Section 1 **Question 15 Page 258**

D

Chapter 5 Section 2 The Natural Logarithm

Chapter 5 Section 2 Question 1 Page 265

a)



b) i) Domain: $\{x \in \mathbb{R}\}$

ii) Range: $\{y < 0, y \in \mathbb{R}\}$

iii) x-intercepts: none; y-intercept: -1

iv) Horizontal asymptote: $y = 0$

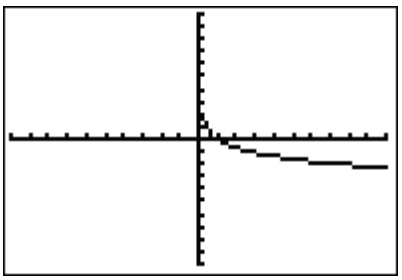
v) Decreasing on the interval $(-\infty, \infty)$

vi) Maximum or minimum points: none

vii) Points of inflection: none

Chapter 5 Section 2 Question 2 Page 265

a)



b) i) Domain: $\{x > 0, x \in \mathbb{R}\}$

ii) Range: $\{y \in \mathbb{R}\}$

iii) x-intercept: 1; y-intercepts: none

iv) Vertical asymptote: $x = 0$

v) Decreasing on the interval $(0, \infty)$

vi) Maximum or minimum points: none

vii) Points of inflection: none

Chapter 5 Section 2 Question 3 Page 265

Answers may vary. For example:
No. The function $f(x)$ and the function $g(x)$ are not inverse functions. They are not reflections of each other in the line $y = x$.

Chapter 5 Section 2 Question 4 Page 265

Answers may vary. For example:

a) $e^4 \doteq 55$

b) $e^5 \doteq 150$

c) $e^2 \doteq 7.5$

d) $e^{-2} \doteq 0.1$

Chapter 5 Section 2 Question 5 Page 265

a) $e^4 \doteq 54.598$

b) $e^5 \doteq 148.413$

c) $e^2 \doteq 7.389$

d) $e^{-2} \doteq 0.135$

Chapter 5 Section 2 Question 6 Page 265

a) $\ln 7 \doteq 1.946$

b) $\ln 200 \doteq 5.298$

c) $\ln\left(\frac{1}{4}\right) \doteq -1.386$

d) $\ln(-4)$ is undefined

Chapter 5 Section 2 Question 7 Page 265

Answers may vary. For example:
The value of $\ln 0$ is $-\infty$, which is undefined. Also, the domain of the function $y = \ln x$ is the interval $(0, \infty)$, so when $x = 0$ the function is undefined.

Chapter 5 Section 2 **Question 8 Page 265**

a) $\ln(e^{2x}) = 2x \ln e$
$= 2x$

b) $\ln(e^x) + \ln(e^x) = 2\ln(e^x)$
$= 2x \ln e$
$= 2x$

c) $e^{\ln(x+1)} = x + 1$

d) $e^{\ln(3x)}(\ln(e^{2x})) = 3x(2x \ln e)$
$= 6x^2$

Chapter 5 Section 2 **Question 9 Page 265**

a) $e^x = 5$
$\ln e^x = \ln 5$
$x \doteq 1.609$

b) $1000 = 20e^{\frac{x}{4}}$
$e^{\frac{x}{4}} = 50$
$\ln e^{\frac{x}{4}} = \ln 50$
$x = 4 \ln 50$
$x \doteq 15.648$

c) $\ln(e^x) = 0.442$
$x = 0.442$

d) $7.316 = e^{\ln(2x)}$
$2x = 7.316$
$x = 3.658$

Chapter 5 Section 2 **Question 10 Page 265**

a) $3^x = 15$
$\ln 3^x = \ln 15$
$x = \dfrac{\ln 15}{\ln 3}$
$x \doteq 2.465$

b) $3^x = 15$

$\log 3^x = \log 15$

$x = \dfrac{\log 15}{\log 3}$

$x \doteq 2.465$

c) Answers may vary. For example:
The value of x can be found by taking natural logarithms of both sides of the equation or by taking common logarithms of both sides of the equation.

Chapter 5 Section 2 **Question 11 Page 265**

a) $V(t) = V_{\max} e^{-\frac{t}{4}}$

$\dfrac{V_{\max}}{2} = V_{\max} e^{-\frac{t}{4}}$

$\ln\left(e^{-\frac{t}{4}}\right) = \ln\left(\dfrac{1}{2}\right)$

$t = -4\ln\left(\dfrac{1}{2}\right)$

$t \doteq 2.8$

It will take 2.8 s.

b) $V(t) = V_{\max} e^{-\frac{t}{4}}$

$\dfrac{V_{\max}}{10} = V_{\max} e^{-\frac{t}{4}}$

$\ln\left(e^{-\frac{t}{4}}\right) = \ln\left(\dfrac{1}{10}\right)$

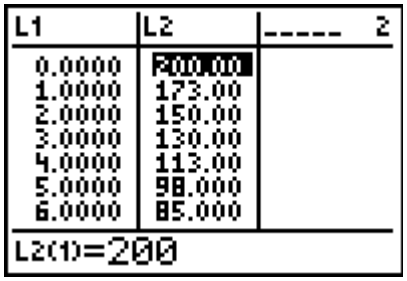
$t = -4\ln\left(\dfrac{1}{10}\right)$

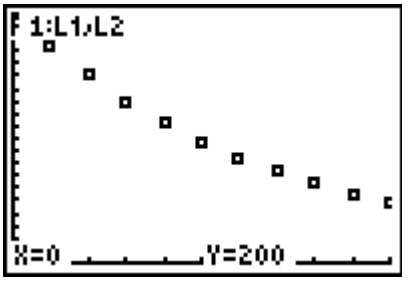
$t \doteq 9.2$

It will take 9.2 s.

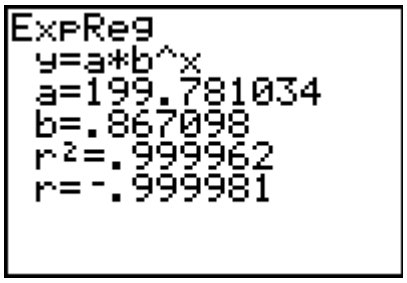
Chapter 5 Section 2 **Question 12 Page 266**

a)





Use the ExpReg function:



The equation of the function is:

$$T(t) = 200e^{-\frac{t}{k}}$$
$$= 200(0.867)^t$$

Taking the logarithm of both sides,

$$\ln e^{-\frac{t}{k}} = \ln(0.867)^t$$
$$-\frac{t}{k} = t\ln(0.867)$$
$$k = -\frac{1}{\ln(0.867)}$$
$$k = 7$$

b) $T(10) = 200e^{-\frac{10}{7}}$ $T(10) = 200(0.867)^{10}$

$\doteq 48$ $\doteq 48$

At 10 min, the temperature is 48ºC.

c) $T(15) = 200e^{-\frac{15}{7}}$

$\doteq 23$

At 15 min, the temperature is 23ºC.

Answers may vary. For example:
The pizza will reach room temperature (21ºC) after a long time.

Chapter 5 Section 2 Question 13 Page 266

a) $\ln 2 + \ln 3 \doteq 1.7918$

b) $\ln 6 \doteq 1.7918$

c) Answers may vary. For example:
The results seem to verify the Law of Logarithms for Multiplication. In terms of natural logarithms, the Law of Logarithms for Multiplication of natural logarithms is $\ln(a \times b) = \ln a + \ln b$, $a > 0$, $b > 0$.

Chapter 5 Section 2 Question 14 Page 266

a) i)
$$\frac{N_0}{10} = N_0 e^{-\frac{(\ln 2)t}{5700}}$$
$$\ln\left(e^{-\frac{(\ln 2)t}{5700}}\right) = \ln 0.1$$
$$t = -\frac{5700 \ln 0.1}{\ln 2}$$
$$t \doteq 18\,935$$
The age is 18 935 years.

ii)
$$\frac{N_0}{100} = N_0 e^{-\frac{(\ln 2)t}{5700}}$$
$$\ln\left(e^{-\frac{(\ln 2)t}{5700}}\right) = \ln 0.01$$
$$t = -\frac{5700 \ln 0.01}{\ln 2}$$
$$t \doteq 37\,870$$
The age is 37 870 years.

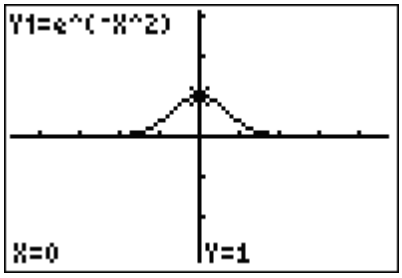
iii)
$$\frac{N_0}{2} = N_0 e^{-\frac{(\ln 2)t}{5700}}$$
$$\ln\left(e^{-\frac{(\ln 2)t}{5700}}\right) = \ln 0.5$$
$$t = -\frac{5700 \ln 0.5}{\ln 2}$$
$$t = 5700$$
The age is 5700 years.

b) Answers may vary. For example:
No. The half-life of C-14 is approximately 5700 years. It will take 5700 years for the sample to have a C-14 to C-12 ratio of half of today's level and it will take 11 400 years for the sample to have a C-14 to C-12 ratio of one quarter of today's level.

c) $t = -\dfrac{5700\left(\ln \dfrac{N(t)}{N_0}\right)}{\ln 2}$

Chapter 5 Section 2 **Question 15 Page 266**

a)



b) Answers may vary. For example:
The shape of the graph is similar to the shape of the normal distribution curve.

c) The maximum value is $y = 1$ and this occurs when $x = 0$.

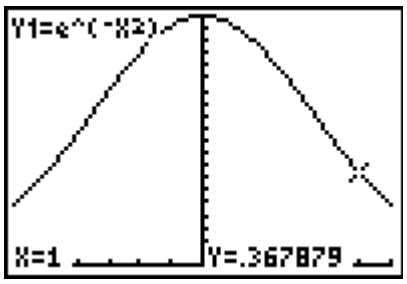
d) Answers may vary. For example:
From the graph, use a trapezoid to estimate the area with a base of 3 units, top 0.5 units, and height 1 unit.

$$A = h\left(\frac{a+b}{2}\right)$$
$$= 1\left(\frac{3+0.5}{2}\right)$$
$$\doteq 1.75$$

This gives an estimate of 1.75 units^2.

Note: The total area is given by the integral of the error function and is $\sqrt{\pi} = 1.77$ square units.

e)



From the graph it can be seen that an estimate of the area between $x = -1$ and $x = +1$ can be made by using the sum of the area of a rectangle and trapezoid.

$$\text{area of rectangle} = 2 \times 0.367$$
$$= 0.734$$
$$\text{area of trapezoid} = (1 - 0.376)\left(\frac{2 + 0.5}{2}\right)$$
$$= 0.78$$

Therefore, the estimated area between $x = -1$ and $x = 1$ is 0.734 units^2 + 0.78 units^2 $\doteq$ 1.5 units^2.

Note: The Empirical Rule for normal distributions states that this area should be 68% of the total area under the curve.

Chapter 5 Section 2 **Question 16 Page 266**

D

Chapter 5 Section 2 **Question 17 Page 266**

C

Chapter 5 Section 3 **Derivatives of Exponential Functions**

Chapter 5 Section 3 **Question 1 Page 274**

a) $g'(x) = 4^x \ln 4$

b) $f'(x) = 11^x \ln 11$

c) $\frac{dy}{dx} = \left(\frac{1}{2}\right)^x \ln \frac{1}{2}$

d) $N'(x) = -3e^x$

e) $h'(x) = e^x$

f) $\frac{dy}{dx} = \pi^x \ln \pi$

Chapter 5 Section 3 **Question 2 Page 274**

a) $f'(x) = e^x$; $f''(x) = e^x$; $f'''(x) = e^x$

b) $f^{(n)}(x) = e^x$

Chapter 5 Section 3 **Question 3 Page 274**

$$\frac{dy}{dx} = 5^x \ln 5$$
$$\left.\frac{dy}{dx}\right|_{x=2} = 5^2 \ln 5$$
$$\doteq 40.2$$

The instantaneous rate of change is 40.2.

Chapter 5 Section 3 **Question 4 Page 274**

$$\frac{dy}{dx} = \frac{1}{2}e^x$$
$$\left.\frac{dy}{dx}\right|_{x=4} = \frac{1}{2}e^4$$
$$\doteq 27.3$$

The slope is 27.3.

Chapter 5 Section 3 **Question 5 Page 274**

$$\frac{dy}{dx} = 8^x \ln 8$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{1}{2}} = 8^{\frac{1}{2}} \ln 8$$

$$= \left(2\sqrt{2}\right)3\ln 2$$

$$= 6\sqrt{2}\ln 2$$

When $x = \frac{1}{2}$, $y = 2\sqrt{2}$

Substitute $x_1 = \frac{1}{2}$, $y_1 = 2\sqrt{2}$, and $m = 6\sqrt{2}\ln 2$ in $y - y_1 = m(x - x_1)$.

Therefore,

$$y - 2\sqrt{2} = 6\sqrt{2}\ln 2\left(x - \frac{1}{2}\right)$$

$$y = \left(6\sqrt{2}\ln 2\right)x + \sqrt{2}(2 - 3\ln 2)$$

Chapter 5 Section 3 **Question 6 Page 274**

a) $N(t) = 10(2^t)$; t is the time in days; $N(t)$ is the number of fruit flies

b) $N(7) = 10(2^7)$

$= 1280$

After 7 days, there will be 1280 fruit flies.

c) Rate of increase $= N'(t)$

$= 10(2^t)\ln 2$

$N'(7) = 10(2^7)\ln 2$

$\doteq 887$

At 7 days, the rate is 887 fruit flies per day.

d) $500 = 10(2^t)$

$\ln 50 = t\ln 2$

$t = \frac{\ln 50}{\ln 2}$

$t \doteq 5.64$

It will take 5.64 days for the population to reach 500 flies.

e) $N'(5.64) = 10(2^{5.64})\ln 2$

$\doteq 346$

At 5.64 days, the rate is 346 fruit flies per day.

Chapter 5 Section 3 **Question 7 Page 275**

a) i) $20 = 10(2^t)\ln 2$

$$2^t = \frac{2}{\ln 2}$$

$$t = \frac{\ln\left(\frac{2}{\ln 2}\right)}{\ln 2}$$

$$t \doteq 1.53$$

The time is 1.53 days.

ii) $2000 = 10(2^t)\ln 2$

$$2^t = \frac{200}{\ln 2}$$

$$t = \frac{\ln\left(\frac{200}{\ln 2}\right)}{\ln 2}$$

$$t \doteq 8.17$$

The time is 8.17 days.

b) Answers may vary. For example:
Since the growth rate increases exponentially, it is most desirable to begin an extermination program very soon after 2 days. At 8 days, the population becomes out of control.

Chapter 5 Section 3 **Question 8 Page 275**

$$f'(x) = \frac{1}{2}e^x$$

$$f'(\ln 3) = \frac{1}{2}e^{\ln 3}$$

$$= \frac{3}{2}$$

Therefore, the slope of the perpendicular line is $-\frac{2}{3}$.

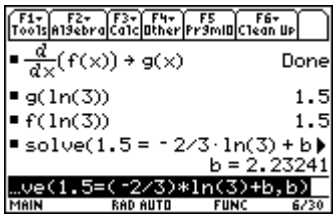
When $x = \ln 3$, $y = \frac{3}{2}$.

Substitute $x_1 = \ln 3$, $y_1 = \frac{3}{2}$, and $m = -\frac{2}{3}$ in $y - y_1 = m(x - x_1)$.

$$y - \frac{3}{2} = -\frac{2}{3}(x - \ln 3)$$

$$y = -\frac{2}{3}x + \frac{2}{3}\ln 3 + \frac{3}{2}$$

Chapter 5 Section 3 **Question 9 Page 275**



$$y = -\frac{2}{3}x + \frac{2}{3}\ln 3 + \frac{3}{2}$$

$$y \doteq -\frac{2}{3}x + 2.2341$$

Chapter 5 Section 3 **Question 10 Page 275**

a) Answers may vary. For example:
The shape of the graph of $g(x)$ is a horizontal straight line.

b) $f'(x) = kb^x \ln b$; $g(x) = \frac{kb^x \ln b}{kb^x}$

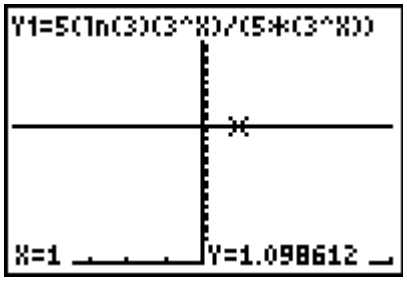
$$= \ln b$$

c) The simplified form of the function is the graph of a horizontal straight line: $g(x) = \ln b$.

Chapter 5 Section 3 **Question 11 Page 275**

Answers may vary. For example:
Use a graphing calculator. Let $k = 5$ and $b = 3$. Then $g(x) = \ln 3$

$\doteq 1.0986$



Chapter 5 Section 3 **Question 12 Page 275**

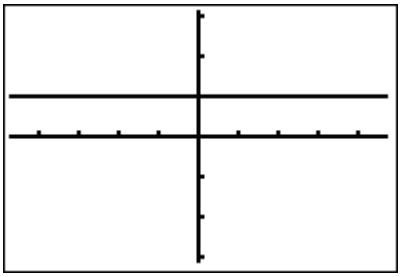
a) $g(x) = 1$; Answers may vary. For example:

The derivative of the exponential function of the form $f(x) = ke^x$ is $f'(x) = ke^x$.

The simplified form of the function $g(x)$ is the function

$$g(x) = \frac{f'(x)}{f(x)}$$
$$= \frac{ke^x}{ke^x}$$
$$= 1$$

b) $g(x) = 1$



Chapter 5 Section 3 **Question 13 Page 275**

a) $f^{(n)}(x) = b^x (\ln b)^n$

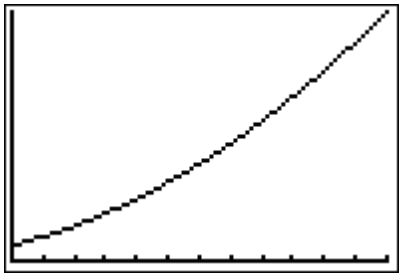
b) Answers may vary. For example:

$f'(x) = b^x \ln b,\ f''(x) = b^x (\ln b)^2,\ f'''(x) = b^x (\ln b)^3, \ldots, f^{(n)}(x) = b^x (\ln b)^n$

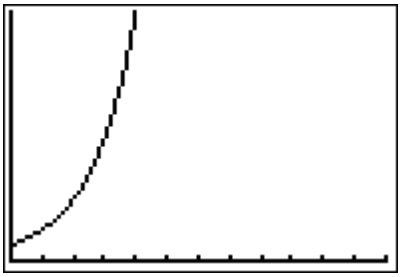
Chapter 5 Section 3 Question 14 Page 275

a) Answers may vary. For example:
Both functions have the same *y*-value of 16, when the *x*-value is 4. Both of the functions are increasing functions that do not have a local maximum or minimum point, or a point of inflection. The function $g(x) = 2^x$ is increasing more rapidly than the function $f(x) = x^2$ over the given interval, $4 \le x \le 16$.

$f(x) = x^2, 4 \le x \le 16$

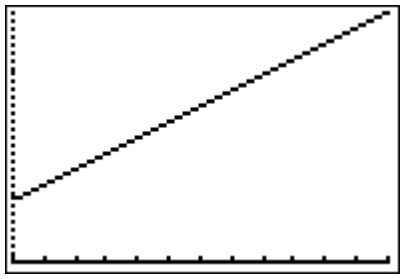


$g(x) = 2^x, 4 \le x \le 16$

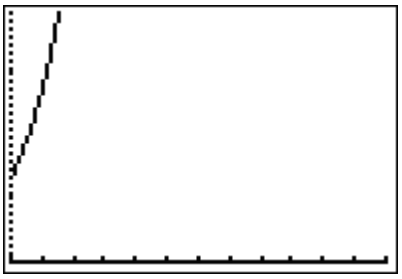


b) Answers may vary. For example:
No. The derivatives of the two functions will not be similar. The derivative of the quadratic function $f(x) = x^2$ is $f'(x) = 2x$. When graphed, the derivative function is a linear function with a slope of 2. The derivative of the exponential function $g(x) = 2^x$ is $g'(x) = 2^x \ln 2$. When graphed, the derivative function is also an exponential function.

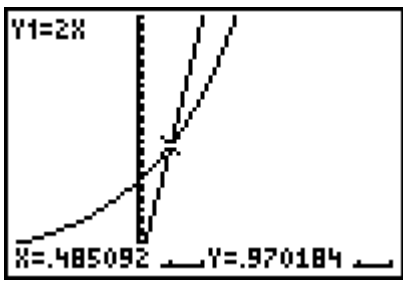
c) $f'(x) = 2x, 4 \le x \le 16$



$g'(x) = 2^x \ln 2, 4 \le x \le 16$



d) Answers may vary. For example:
Yes. There are two *x*-values for which the slope of $f(x)$ will be approximately the same as the slope of $g(x)$ when rounded to five decimal places. When the *x*-value is 0.485 09, the slope of $f(x)$ and the slope of $g(x)$ is 0.970 18. When the *x*-value is 3.212 43, the slope of $f(x)$ and the slope of $g(x)$ is 6.424 87. The *x*-values can be found using a graphing calculator.



Chapter 5 Section 3 **Question 15 Page 275**

a) $P(h) = 101.3e^{-kh}$

$95.6 = 101.3e^{-1000k}$

$\ln\left(\frac{95.6}{101.3}\right) = (-1000k)\ln e$

$k = -0.001\ln\left(\frac{95.6}{101.3}\right)$

$k \doteq 0.000\ 057\ 9$

b) $P(2000) = 101.3e^{-(0.000\ 057\ 9)(2000)}$

$\doteq 90.2$

The pressure is 90.2 kPa.

c) $P'(h) = 101.3(-0.000\ 057\ 9)e^{-0.000\ 057\ 9h}$

$\doteq -0.005\ 87e^{-0.000\ 057\ 9h}$

d) $P'(1500) = -0.005\ 87e^{-0.000\ 057\ 9(1500)}$

$\doteq -0.005\ 38$

The rate is −0.005 38 kPa/m.

Chapter 5 Section 3 **Question 16 Page 276**

a) $N(t) = 50(2^t)$ where N is the number of visitors and t is the time in weeks.

b) i) $N(4) = 50(2^4)$

$= 800$

After 4 weeks, there will be 800 visitors.

ii) $N(12) = 50(2^{12})$

After 12 weeks, there will be 204 800 visitors.

c) $N'(t) = 50(2^t)\ln 2$

i) $N'(4) = 50(2^4)\ln 2$

$\doteq 555$

The rate is 555 visitors per week.

ii) $N'(12) = 50(2^{12})\ln 2$

$\doteq 141\,957$

The rate is 141 957 visitors per week.

d) Answers may vary. For example:
No. This trend will not continue indefinitely. The number of people visiting the site will eventually level off.

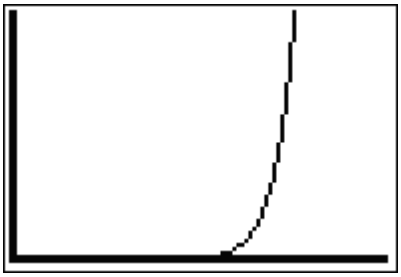
Chapter 5 Section 3 Question 17 Page 276

a) Answers may vary. For example:
One Internet site claims the current population is growing at a rate of 205 000 per day, or 8500 per hour, or 140 per minute, or 2.3 people per second.

b) Answers may vary. For example:

i) Equation form: $P(t) = 4e^{0.019t}$, where P is the population, in billions, and t is the time, in years, since 1975.

ii) Graphical form:



c) i) $P(50) = 4e^{0.019(50)}$

$= 10.342\,838\,64$

The population would be 10.342 838 64 billion.

ii) $P(525) = 4e^{0.019(525)}$

$\doteq 85\,930.521\,67$

The population would be 85 930.521 67 billion.

iii) $P(1025) = 4e^{0.019(1025)}$

$\doteq 1\,148\,008\,296$

The population would be 1 148 008 296 billion.

d) Answers may vary. For example:
No. This model is not sustainable over the long term. Other factors that could affect this trend are the amount of resources available to sustain the population and the available areas on the earth that could sustain this number of people.

e) Answers may vary. For example:
If the resources, such as food, start to diminish then the population increase would slow down, since the death rate would increase relative to birth. Poor nutrition is one contributing factor to low birth rates. The factor 0.019 would be reduced.

Chapter 5 Section 3 Question 18 Page 276

a) Answers may vary. For example:

i) $10^{-6} = 4e^{0.019t}$

$\ln(2.5\times10^{-7}) = 0.019t$

$t \doteq -800$

Since $t = 0$ in 1975, the year when the population would have been 1000 is predicted as 1975 – 800 = 1175.

ii) $10^{-7} = 4e^{0.019t}$

$\ln(2.5\times10^{-8}) = 0.019t$

$t \doteq -921$

Since $t = 0$ in 1975, the year when the population would have been 100 is predicted as 1975 – 921 = 1054.

iii) $2\times10^{-9} = 4e^{0.019t}$

$\ln(5\times10^{-10}) = 0.019t$

$t \doteq -1127$

Since $t = 0$ in 1975, the year when the population would have been 2 is predicted as 1975 – 1127 = 848.

b) Answers may vary. For example:
No. The answers in part a) do not seem reasonable.

Chapter 5 Section 3 Question 19 Page 276

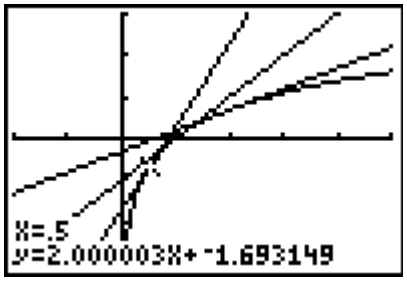
Answers may vary. For example:
Students may use a graphing calculator to graph $y = \ln x$ and use the tangent function for some values of x.

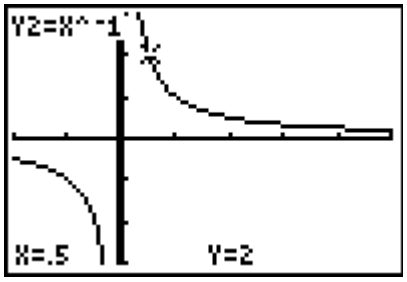
$x = 1$: tangent is $y = x - 1$ where the slope is 1

$x = 2$: tangent is $y = 0.5x - 0.31$ where the slope is 0.5

$x = 0.5$: tangent is $y = 2x - 1.69$ where the slope is 2

Graph the derivative function of $y = \ln x$ (i.e., $f(x) = x^{-1}$) and compare the ordered pairs with the (x, slope) values for $y = \ln x$. They are the same:





Chapter 5 Section 3 **Question 20 Page 276**

D

Chapter 5 Section 3 **Question 21 Page 276**

E

Chapter 5 Section 4 **Differentiation Rules for Exponential Functions**

Chapter 5 Section 4 **Question 1 Page 282**

a) $y = e^{x \ln b}$

b) $\dfrac{dy}{dx} = e^{x \ln b} \ln b$

Chapter 5 Section 4 **Question 2 Page 282**

a) $\dfrac{dy}{dx} = -3e^{-3x}$

b) $f'(x) = 4e^{4x-5}$

c) $\dfrac{dy}{dx} = 2e^{2x} - (-2)e^{-2x}$

$= 2e^{2x} + 2e^{-2x}$

d) $\dfrac{dy}{dx} = 2^x \ln 2 + 3^x \ln 3$

e) $f(x) = 3e^{2x} - (2^x)^3$

$f'(x) = (2)3e^{2x} - 3(2^x)^2 2^x \ln 2$

$= 6e^{2x} - 3(2^{3x}) \ln 2$

f) $\dfrac{dy}{dx} = 4xe^x + 4e^x$

$= 4e^x(x+1)$

g) $\dfrac{dy}{dx} = 5^x e^{-x} \ln 5 - 5^x(e^{-x})$

$= -5^x e^{-x}(1 - \ln 5)$

h) $f'(x) = 2xe^{2x} + e^{2x} - 6e^{-3x}$

Chapter 5 Section 4 **Question 3 Page 282**

a) $\frac{dy}{dx} = -e^{-x}\sin x + e^{-x}\cos x$

$= e^{-x}(\cos x - \sin x)$

b) $\frac{dy}{dx} = -\sin x(e^{\cos x})$

c) $f'(x) = 2e^{2x}(x^2 - 3x + 2) + e^{2x}(2x - 3)$

$= e^{2x}(2x^2 - 4x + 1)$

d) $g'(x) = 4xe^{\cos 2x} + 2x^2e^{\cos 2x}(-2\sin x)$

$= -4xe^{\cos 2x}(x\sin 2x - 1)$

Chapter 5 Section 4 **Question 4 Page 282**

$f'(x) = e^x - 2e^{2x}$

If $f'(x) = 0$ then $e^x - 2e^{2x} = 0.$

Therefore,

$e^x(1 - 2e^x) = 0$

$e^x = \frac{1}{2}$ since $e^x > 0$

$x = \ln 0.5$

$f(\ln 0.5) = e^{\ln 0.5} - e^{2\ln 0.5}$

$= 0.5 - 0.5^2$

$= 0.25$

Therefore, by using derivative tests, there is a local maximum of $y = 0.25$ when $x = \ln(0.5)$.

Chapter 5 Section 4 **Question 5 Page 282**

Adding two exponential functions gives an exponential function.

$f'(x) = e^x + 2e^{2x}$

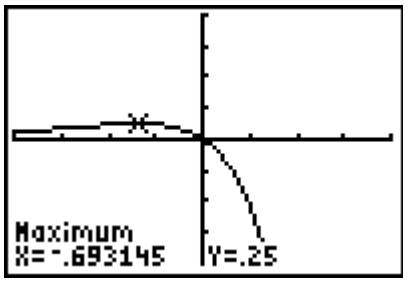
Set $f'(x) = 0.$

$e^x(1 + 2e^x) = 0$

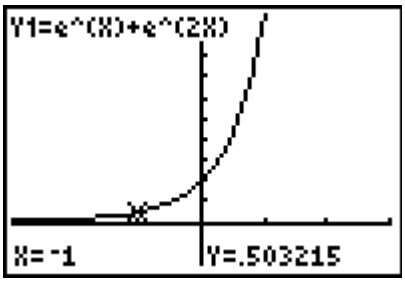
e^x must equal $-\frac{1}{2}$ or 0 but since $e^x > 0,$ the function has no local extrema.

Chapter 5 Section 4 **Question 6 Page 282**

Graph of $y = e^x - e^{2x}$



Graph of $y = e^x + e^{2x}$



Chapter 5 Section 4 **Question 7 Page 283**

a) $P(3) = 50e^{0.5(3)}$

$\doteq 224$

After 3 days, there will be 224 bacteria.

b) Initial population = 50 (i.e., $t = 0$)

$10(50) = 50e^{0.5t}$

$10 = e^{0.5t}$

$\ln 10 = 0.5t$

$t = \dfrac{\ln 10}{0.5}$

$t \doteq 4.6$

The time is 4.6 days.

c) $e^{0.5t} = 10^x$

$0.5t = x \ln 10$

$x = \dfrac{t}{2 \ln 10}$

$x \doteq \dfrac{t}{4.6}$

$P(t) = 50e^{0.5t}$

$\doteq 50(10)^{\frac{t}{4.6}}$

d) $P(5) = 50(10)^{\frac{5}{4.6}}$

$\doteq 611$

After 5 days, there will be 611 bacteria.

e) $P(5) = 50e^{0.5(5)}$

$\doteq 609$

After 5 days, there will be 609 bacteria.

Answers may vary. For example:
The function from part c) approximates the relationship between e and t from the initial function.

Chapter 5 Section 4 Question 8 Page 283

a) i) $A(2) = 3000e^{0.065(2)}$

$= 3416.49$

After 2 years, the amount will be $3416.49.

ii) $A(5) = 3000e^{0.065(5)}$

$= 4152.09$

After 5 years, the amount will be $4152.09.

iii) $A(25) = 3000e^{0.065(25)}$

$= 15\,235.26$

After 25 years, the amount will be $15 235.26.

b) $6000 = 3000e^{0.065t}$

$e^{0.065t} = 2$

$0.065t = \ln 2$

$$t = \frac{\ln 2}{0.065}$$

$t \doteq 10.7$

It will take 10.7 years.

c) $A'(t) = 3000(0.065)e^{0.065t}$

$A'(t) = 195e^{0.065t}$

$A'(10.7) = 195e^{0.065(10.7)}$

$= 390.92$

The investment is growing at a rate of $390.92 per year.

Chapter 5 Section 4 Question 9 Page 283

a) $f'(x) = e^x(\cos x + \sin x)$

$f''(x) = 2e^x(\cos x)$

$f'''(x) = 2e^x(\cos x - \sin x)$

$f^{(4)}(x) = -4e^x(\sin x)$

$f^{(5)}(x) = -4e^x(\cos x + \sin x)$

$f^{(6)}(x) = -8e^x(\cos x)$

b) Answers may vary. For example:
The first and fifth derivatives have the expression cos x + sin x. The third derivative has the expression cos x − sin x. The second and third derivatives have the same coefficient $2e^x$. The fourth and fifth derivatives have the same coefficient $-4e^x$ The second and sixth derivatives have the expression cos x. The fourth derivative has the expression sin x. The derivatives all have the expression e^x in them.

c) i) $f^{(7)}(x) = -8e^x(\cos x - \sin x)$

ii) $f^{(8)}(x) = 16e^x(\sin x)$

d) Answers may vary. For example:

$f^{(n)}(x) = (-4)^{\frac{n-1}{4}} e^x(\cos x + \sin x)$, for $n \in \{1,5,9,13,...\}$

$f^{(n)}(x) = 2(-4)^{\frac{n-2}{4}} e^x \cos x$, for $n \in \{2,6,10,14,...\}$

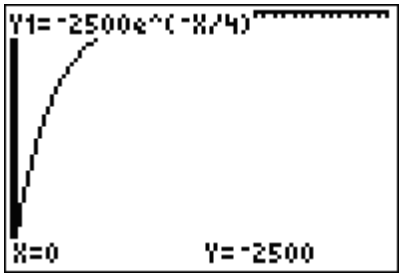
$f^{(n)}(x) = 2(-4)^{\frac{n-3}{4}} e^x(\cos x - \sin x)$, for $n \in \{3,7,11,15,...\}$

$f^{(n)}(x) = (-4)^{\frac{n}{4}} e^x(\sin x)$, for $n \in \{4,8,12,16,...\}$

Chapter 5 Section 4 Question 10 Page 283

Answers may vary. For example:
Laura's motorcycle depreciates in value the fastest when she first drives it off the lot. The rate of depreciation at time $t = 0$ is calculated as −\$2500 per year. Therefore, her motorcycle is depreciating at the rate of \$2500 per year when $t = 0$.

Graph of $V'(t)$



Chapter 5 Section 4 **Question 11 Page 283**

a) $P_0 = 2000$

$$4000 = 2000\left(a^{\frac{1}{4}}\right)$$

$$a^{\frac{1}{4}} = 2$$

$$a = 16$$

b) $$P\left(\frac{1}{6}\right) = 2000\left(16^{\frac{1}{6}}\right)$$

$$\doteq 3175$$

After 10 min, there will be approximately 3175 algae.

c) i) $P'(t) = P_0(a^t)\ln a$

$$P'(1) = 2000(16^1)\ln 16$$

$$\doteq 88\,723$$

After 1 h, the rate of change of the population will be approximately 88 723 algae per hour.

ii) $P'(3) = 2000(16^3)\ln 16$

$$\doteq 22\,713\,047$$

After 3 h, the rate of change of the population will be approximately 22 713 047 algae per hour.

Chapter 5 Section 4 **Question 12 Page 283**

a) Answers may vary. For example:
Cheryl has tried to differentiate the exponential function using the power rule. The power rule cannot be used to differentiate an exponential function, since the exponent is a variable.

b) Answers may vary. For example:
Cheryl saw a term that was in exponent form and thought that she could use the power rule.

c) The derivative of an exponential function $y = a^x$ is $\frac{dy}{dx} = a^x \ln a$, so the correct answer is

$$\frac{dy}{dx} = 10^x \ln 10.$$

Chapter 5 Section 4 **Question 13 Page 283**

$$\frac{dy}{dx} = -2xe^{-x^2}$$

$$\frac{d^2y}{dx^2} = 4x^2e^{-x^2} - 2e^{-x^2}$$

Points of inflection occur for x-values that satisfy $\frac{d^2y}{dx^2} = 0$.

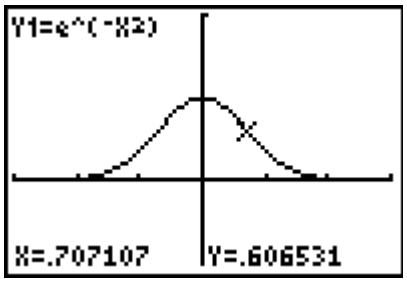
$$0 = 4x^2e^{-x^2} - 2e^{-x^2}$$

$4x^2 = 2$ since $e^{-x^2} > 0$

$$x = \pm\frac{1}{\sqrt{2}}$$

If $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{e}}$; If $x = -\frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{e}}$.

Therefore, the points of inflection are $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}\right)$.



Chapter 5 Section 4 **Question 14 Page 284**

a) $\frac{dy}{dx} = e^x\cos x - e^x\sin x$

Let $\frac{dy}{dx} = 0$ to find the x-values of the local extrema.

$$0 = e^x(\cos x - \sin x)$$

$\cos x = \sin x$ since $e^x > 0$

$x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ in the interval $0 \le x \le 2\pi$.

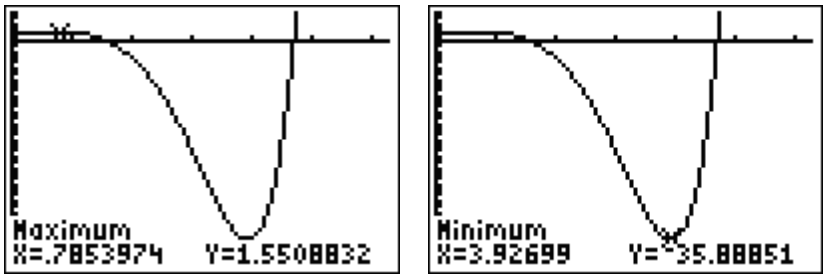
$$\frac{d^2y}{dx^2} = e^x(-\sin x) + e^x \cos x - e^x \cos x - e^x \sin x$$
$$= -2e^x \sin x$$

At $x = \frac{\pi}{4}$, $\frac{d^2y}{dx^2}$ is negative, so there is a local maximum.

At $x = \frac{5\pi}{4}$, $\frac{d^2y}{dx^2}$ is positive, so there is a local minimum.

Using a graphing calculator:

Two local extrema occur on the interval: a local maximum at (0.785, 1.551) or $\left(\frac{\pi}{4}, 1.551\right)$; and a local minimum at (3.927, −35.889) or $\left(\frac{5\pi}{4}, -35.889\right)$ over the interval $[0, 2\pi]$.



b) The local maximum for $f(x)$ occurs $\frac{\pi}{4}$ rad to the right and $\frac{7\pi}{4}$ rad to the left of where the local maximums (0, 1) and (2π, 1) occur for the function $y = \cos x$ over the interval $[0, 2\pi]$.

The local minimum for the function $y = e^x \cos x$ occurs $\frac{\pi}{4}$ rad to the right of where the local minimum (π, −1) occurs for the function $y = \cos x$ over the interval $[0, 2\pi]$

Chapter 5 Section 4 **Question 15 Page 284**

a) $0.75(V_{\max}) = V_{\max}\left(1 - e^{-\frac{t}{8}}\right)$

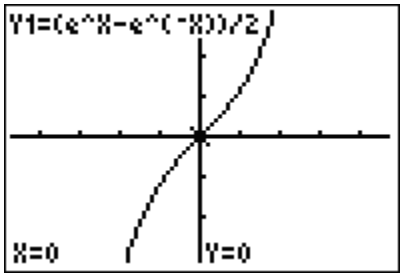
$$e^{-\frac{t}{8}} = 0.25$$
$$-\frac{t}{8} = \ln(0.25)$$
$$t = -8\ln(0.25)$$
$$t \doteq 11.1$$

The time required is 11.1 h.

b) $V'(t) = \frac{1}{8}V_{\max}e^{-\frac{t}{8}}$

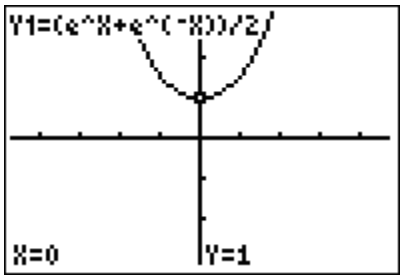
Chapter 5 Section 4 **Question 16 Page 284**

a)



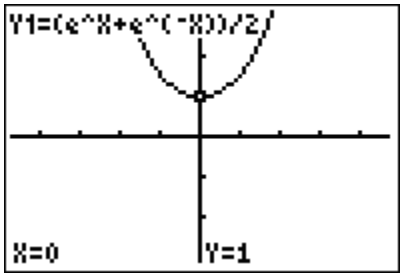
Answers may vary. For example:
The function is an increasing function on the interval $(-\infty, \infty)$. The function is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, \infty)$.

b) Answers may vary. For example:
The shape of the derivative of the function will be concave up on the interval $(-\infty, \infty)$ with a local minimum value when $x = 0$.



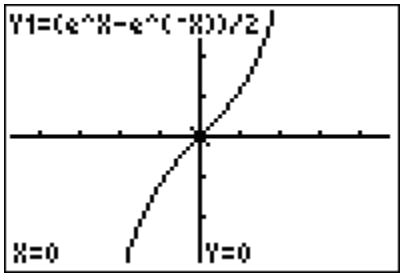
Chapter 5 Section 4 **Question 17 Page 284**

a)



Answers may vary. For example:
The shape of the function will be concave up on the interval $(-\infty, \infty)$ with a local minimum value when $x = 0$.

b)



Answers may vary. For example:
The derivative of the function is an increasing function on the interval $(-\infty, \infty)$. The derivative function is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, \infty)$.

Chapter 5 Section 4 **Question 18 Page 284**

a) i) $\dfrac{d}{dx}\left(\sinh x\right) = \dfrac{e^x - (-1)e^{-x}}{2}$

$= \dfrac{e^x + e^{-x}}{2}$

$= \cosh x$

ii) $\dfrac{d}{dx}\left(\cosh x\right) = \dfrac{e^x + (-1)e^{-x}}{2}$

$= \dfrac{e^x - e^{-x}}{2}$

$= \sinh x$

b) Answers may vary. For example:
The predictions in part a) were correct.

Chapter 5 Section 4 **Question 19 Page 284**

a) Take the derivative with respect to x of both sides of the equation $x = e^y$.

b) Since $x = e^y$ by taking the derivatives of both sides,

$$1 = e^y\left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

$$\frac{dy}{dx} = \frac{1}{x}$$

Chapter 5 Section 4 **Question 20 Page 284**

A

Chapter 5 Section 4 **Question 21 Page 284**

E

Chapter 5 Section 5 **Making Connections: Exponential Models**

Chapter 5 Section 5 **Question 1 Page 289**

a) $N(10) = 100e^{-\lambda(10)}$

$73 = 100e^{-\lambda(10)}$

$\ln(0.73) = -10\lambda$

$\lambda = -\dfrac{\ln(0.73)}{10}$

$\lambda \doteq 0.031$

The disintegration constant is 0.031/min.

b) $\dfrac{100}{2} = 100e^{-0.031t}$

$\ln 0.5 = -0.031t$

$t = -\dfrac{\ln 0.5}{0.031}$

$t \doteq 22$

The half-life is 22 min.

c) $N(t) = 100\left(\left(\dfrac{1}{2}\right)^{\log_{\frac{1}{2}} e}\right)^{-0.031t}$

$= 100\left(\left(\dfrac{1}{2}\right)^{\frac{\log e}{\log 0.5}}\right)^{-0.031t}$

$\doteq 100\left(\dfrac{1}{2}\right)^{\frac{t}{22}}$

d) $N'(t) = -\lambda N_0 e^{-\lambda t}$

$N'(5) = -(0.031)(100)e^{-(0.031)(5)}$

$\doteq -2.65$

The sample is decaying at 2.65 mg/min after 5 min.

Chapter 5 Section 5 **Question 2 Page 290**

a) i) $M_{Rn}(1) = 100\left(\frac{1}{2}\right)^{\frac{1}{3.8}}$

$\doteq 83.3$

After 1 day, there will be 83.3 mg of radon.

ii) $M_{Rn}(7) = 100\left(\frac{1}{2}\right)^{\frac{7}{3.8}}$

$\doteq 27.9$

After 1 week, there will be 27.9 mg of radon.

b) $0.25(100) = 100\left(\frac{1}{2}\right)^{\frac{t}{3.8}}$

$\left(\frac{1}{2}\right)^{\frac{t}{3.8}} = \frac{1}{4}$

$\left(\frac{1}{2}\right)^{\frac{t}{3.8}} = \left(\frac{1}{2}\right)^{2}$

$\frac{t}{3.8} = 2$

$t = 7.6$

It will take 7.6 days.

c) $M_{Rn}(t) = 100\left(\left(\frac{1}{2}\right)^{t}\right)^{\frac{1}{3.8}}$

$M'_{Rn}(t) = \frac{100}{3.8}\left(\left(\frac{1}{2}\right)^{t}\right)^{-\frac{2.8}{3.8}}\left(\frac{1}{2}\right)^{t}\ln\frac{1}{2}$

$M'_{Rn}(t) = \frac{100}{3.8}\left(\frac{1}{2}\right)^{\frac{t}{3.8}}\ln\frac{1}{2}$

i) $M'_{Rn}(1) = \frac{100}{3.8}\left(\frac{1}{2}\right)^{\frac{1}{3.8}}\ln\frac{1}{2}$

$\doteq -15.2$

The rate of decay is –15.2 mg/day.

ii) $M'_{Rn}(7) = \frac{100}{3.8}\left(\frac{1}{2}\right)^{\frac{7}{3.8}} \ln\frac{1}{2}$

$\doteq -5.1$

The rate of decay is –5.1 mg/day.

$M'_{Rn}(7.6) = \frac{100}{3.8}\left(\frac{1}{2}\right)^{\frac{7.6}{3.8}} \ln\frac{1}{2}$

$\doteq -4.6$

The rate of decay is –4.6 mg/day.

Chapter 5 Section 5 Question 3 Page 290

a) i) Since initially there is 100 mg of radon, there is 0 mg of polonium.

ii) $M_{Po}(1) = 100\left[1-\left(\frac{1}{2}\right)^{\frac{1}{3.8}}\right]$

$\doteq 16.7$

There will be 16.7 mg of polonium.

b) $M_{Po}(t) = 100\left[1-\left(\left(\frac{1}{2}\right)^{t}\right)^{\frac{1}{3.8}}\right]$

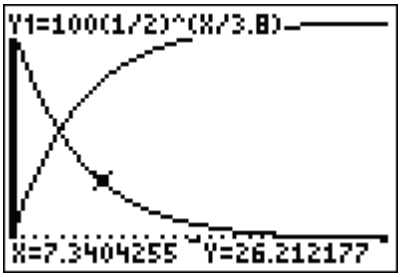
$M'_{Po}(t) = 100\left[-\frac{1}{3.8}\left(\left(\frac{1}{2}\right)^{t}\right)^{-\frac{2.8}{3.8}}\left(\frac{1}{2}\right)^{t}\ln\frac{1}{2}\right]$

$M'_{Po}(t) = -\frac{100}{3.8}\left(\frac{1}{2}\right)^{\frac{t}{3.8}}\ln\frac{1}{2}$

Answers may vary. For example:
The first derivative of the function is the rate of change of the amount of polonium in milligrams per day.

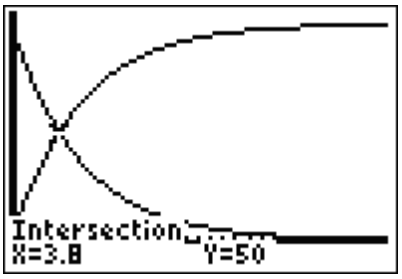
Chapter 5 Section 5 Question 4 Page 290

a)



Answers may vary. For example
No. The two functions are not inverses of each other. They are not a reflection of each other in the line $y = x$.

b)

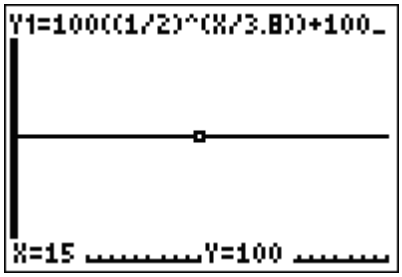


The coordinates are (3.8, 50). The point of intersection is the half-life of radon.

c) Answers may vary. For example:
At the point of intersection, which is the half-life of radon, the derivatives of each function are equal in value, but opposite in sign (± 9.12 mg/day).

The rate of change of radon is negative, since the amount of radon is decreasing, and the rate of change of polonium is positive, since the amount of polonium is increasing. This makes sense from a physical perspective since the radon is being converted into polonium, so the rate of decay of radon must equal the rate of growth of polonium.

d)



Answers may vary. For example:
The shape of this graph is the horizontal straight line $y = 100$. This makes sense from a physical perspective since the sum of the amount of radon and the amount of polonium will always be equal to 100 as the radon decays.

Chapter 5 Section 5 Question 5 Page 290

a) Answers may vary. For example:
Yes. The function in the graph is an example of damped harmonic motion. The curve is sinusoidal with diminishing amplitude as the time is increasing.

b) i) 2.27 ms

ii) 0.002 27 s

c) $f = \dfrac{1}{0.002\,27}$

$\doteq 440$

The frequency is 440 Hz.

d) $I(t) = 4\cos(2\pi(440)t)e^{-kt}$

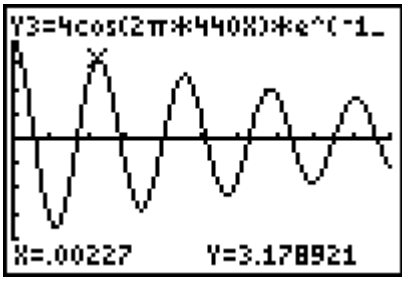
Chapter 5 Section 5 Question 6 Page 291

a) k = 101.2/s

$I(t) = 4\cos[2\pi(440t)]e^{-101.2t}$

b) Answers may vary. For example: I found the value by substituting the $I(t)$ value of 2 that occurs when t = 0.006 804 5 ms into the equation in question 5 part d).

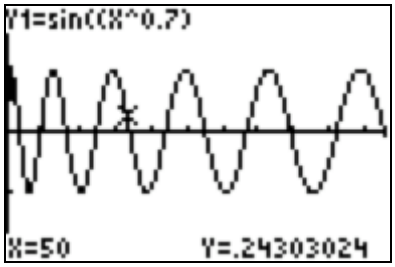
c)



Chapter 5 Section 5 Question 7 Page 291

a) The frequency of the sound is not a function of time. Therefore, it does not diminish over time.

b) Pitch decay could look like the following graph. As the frequency diminishes, the period will increase leading to a "stretched" sinusoidal curve.

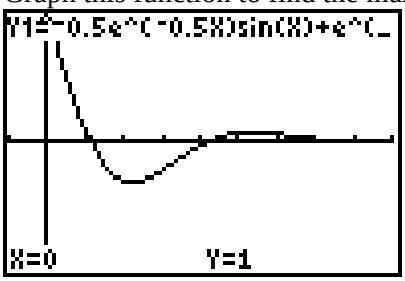


Chapter 5 Section 5 Question 8 Page 291

a) $v(t) = h'(t)$

$= -0.5e^{-0.5t} \sin t + e^{-0.5t} \cos t$

Graph this function to find the maximum velocity.



The maximum value is at $t = 0$ s and is $v_{max} = 1$ m/s.

b) $v(t) = -0.5e^{-0.5t}\sin t + e^{-0.5t}\cos t$

$a(t) = v'(t)$

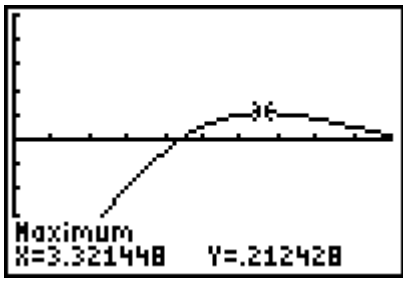
$$= \frac{d}{dt}\left(e^{-0.5t}(-0.5\sin t + \cos t)\right)$$

$$= e^{-0.5t}(-0.5\cos t - \sin t) - 0.5e^{-0.5t}(-0.5\sin t + \cos t)$$

$$= e^{-0.5t}[\sin t(-1 + 0.25) + \cos t(-0.5 - 0.5)]$$

$$= e^{-0.5t}[-0.75\sin t - \cos t]$$

Graph $a(t) = e^{-0.5t}[-0.75\sin t - \cos t]$ and find the maximum.



$F = ma$

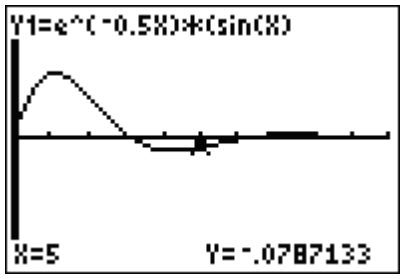
$\doteq 60(0.212\ 4)$

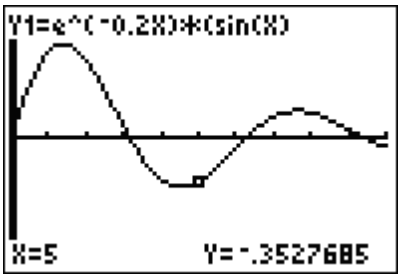
$\doteq 12.7$

The greatest force is 12.7 N.

Chapter 5 Section 5 **Question 9 Page 291**

a)





b) Answers may vary. For example:
As the shock absorbers wear out with time, the vertical displacement of the shock absorber will increase since the amplitude of the function is larger.

c) Answers will vary. For example:
As the vertical displacement of the shock absorber increases with wear, the modelling equation will change to reflect the increases.

Chapter 5 Section 5 **Question 10 Page 291**

a) Answers may vary. For example:
Yes. Rocco's motion is an example of damped harmonic motion. The curve is sinusoidal with diminishing amplitude as the time is increasing.

b) Answers may vary. For example:
No. Biff will not be able to rescue Rocco. Rocco will not swing back to within 1 m from where he started falling.

Rocco's initial horizontal position, in metres: $x(0) = 5\cos\left(\frac{\pi t}{2}\right)e^{-0.1(0)}$

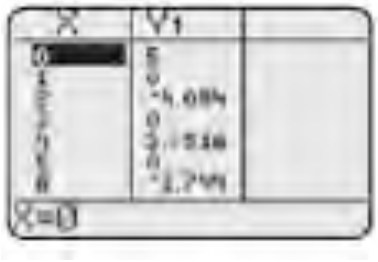
$= 5$

Rocco will swing back towards his horizontal position when $\frac{\pi t}{2} = 2\pi$ so when t = 4 s.

Rocco's horizontal position at t = 4 s, in metres: $x(4) = 5\cos\left(\frac{\pi t}{2}\right)e^{-0.1(4)}$

$\doteq 3.35$

Rocco will be 5 m – 3.35 m = 1.65 m away from his initial position so not within 1 m.



c) Find the horizontal velocity and find when the speed is less than 2 m/s.

$$x'(t) = -\frac{5\pi}{2}\sin\left(\frac{\pi t}{2}\right)e^{-0.1t} + 5(-0.1)\cos\left(\frac{\pi t}{2}\right)e^{-0.1t}$$

$$= -e^{-0.1t}\left[\frac{5\pi}{2}\sin\left(\frac{\pi t}{2}\right) + 0.5\cos\left(\frac{\pi t}{2}\right)\right]$$

Rocco is at the bottom of the swing at t = 1 s, 3 s, 5 s, …

$$x'(1) = -e^{-0.1(1)}\left(\frac{5\pi}{2}\right) \qquad x'(3) = -e^{-0.1(3)}\left(-\frac{5\pi}{2}\right)$$

$$= -7.11 \qquad = 5.82$$

$$x'(5) = -e^{-0.1(5)}\left(\frac{5\pi}{2}\right) \qquad x'(7) = -e^{-0.1(7)}\left(-\frac{5\pi}{2}\right)$$

$$= -4.76 \qquad = 3.90$$

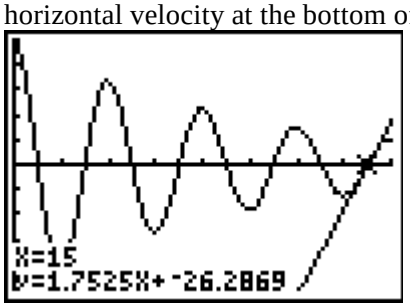
$$x'(9) = -e^{-0.1(9)}\left(\frac{5\pi}{2}\right) \qquad x'(11) = -e^{-0.1(11)}\left(-\frac{5\pi}{2}\right)$$

$$= -3.19 \qquad = 2.61$$

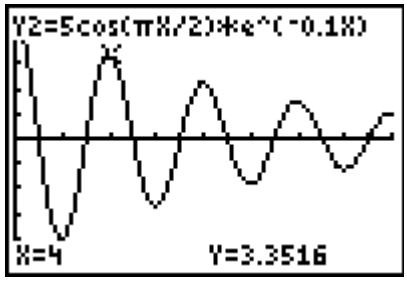
$$x'(13) = -e^{-0.1(13)}\left(\frac{5\pi}{2}\right) \qquad x'(15) = -e^{-0.1(15)}\left(-\frac{5\pi}{2}\right)$$

$$= -2.14 \qquad = 1.75$$

Therefore, Rocco must swing back and forth 3.75 times before he can safely drop to the ground. The graph below shows that the slope of the tangent at t = 15 is 1.7525 m/s. i.e., $x'(t)$, the horizontal velocity at the bottom of the swing is less than 2 m/s after 15 s.



d)



Chapter 5 Section 5 **Question 11 Page 292**

a) The period of Rocco's vine is 4 s.

Therefore, use $T = 2\pi\sqrt{\frac{l}{g}}$ with $T = 4$ s and $g = 9.8$ m/s^2 to find l.

$$4 = 2\pi\sqrt{\frac{l}{9.8}}$$

$$l = 9.8\left(\frac{2}{\pi}\right)^2$$

$$\doteq 4.0$$

The vine is about 4.0 m long.

b) i) Answers may vary. For example:
If the vine were shorter, Rocco's position graph would have a shorter period and a smaller amplitude.

ii) Answers may vary. For example:
If the vine were longer, Rocco's position graph would have a longer period and a larger amplitude.

c) Answers may vary. For example:
If the vine were longer, Rocco could swing back to within 1 m of Biff. A shorter vine would slow to 2 m/s in fewer swings.

Chapter 5 Section 5 **Question 12 Page 292**

$$I(t) = I_{pk}\left[1 - e^{-\left(\frac{1000}{200}\right)t}\right]$$

$$I(t) = I_{pk}\left[1 - e^{-5t}\right]$$

a) i) $0.50I_{pk} = I_{pk}\left[1 - e^{-5t}\right]$

$$\ln(0.5) = -5t$$

$$t = \frac{\ln(0.5)}{-5}$$

$$t \doteq 0.14$$

It will take 0.14 s.

ii) $0.90I_{pk} = I_{pk}\left[1 - e^{-5t}\right]$

$-0.10 = -e^{-5t}$

$\ln(0.1) = -5t$

$t = \dfrac{\ln(0.1)}{-5}$

$t \doteq 0.46$

It will take 0.46 s.

b) $I'(t) = I_{pk}\left[-e^{-\left(\frac{R}{L}\right)t}\left(-\dfrac{R}{L}\right)\right]$

$= I_{pk}[-e^{-5t}(-5)]$

$= 5I_{pk}e^{-5t}$

i) $I'(0.14) = 5I_{pk}e^{-5(0.14)}$

$\doteq 2.5I_{pk}$

The rate is $2.5I_{pk}$ A/s.

ii) $I'(0.46) = 5I_{pk}e^{-5(0.46)}$

$\doteq 0.5I_{pk}$

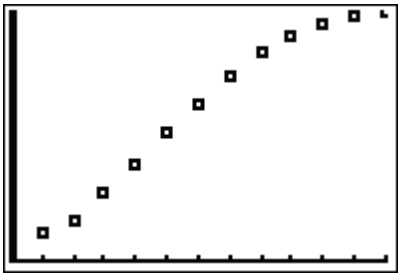
The rate is $0.5I_{pk}$ A/s.

Chapter 5 Section 5 **Question 13 Page 292**

Solutions to the Achievement Checks are shown in the Teacher's Resource.

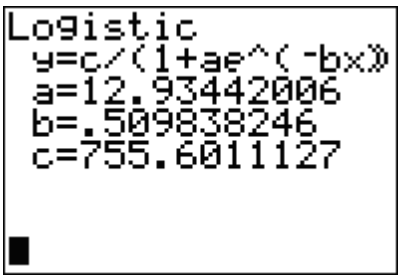
Chapter 5 Section 5 **Question 14 Page 293**

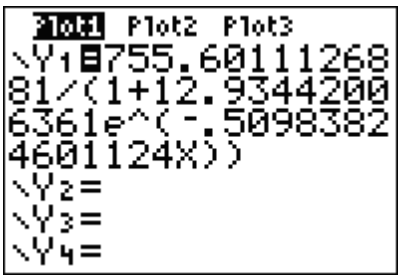
a)



The graph has the shape of a logistic function.

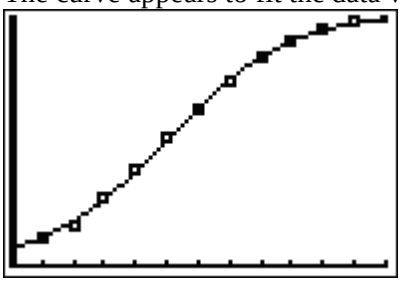
b)



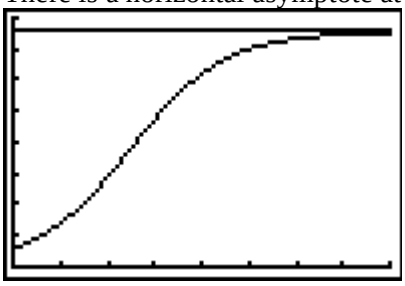


$$P(t) \doteq \frac{755.6}{1+12.9e^{-0.5t}}$$

c) Answers may vary. For example:
The curve appears to fit the data very well as shown in the graph.



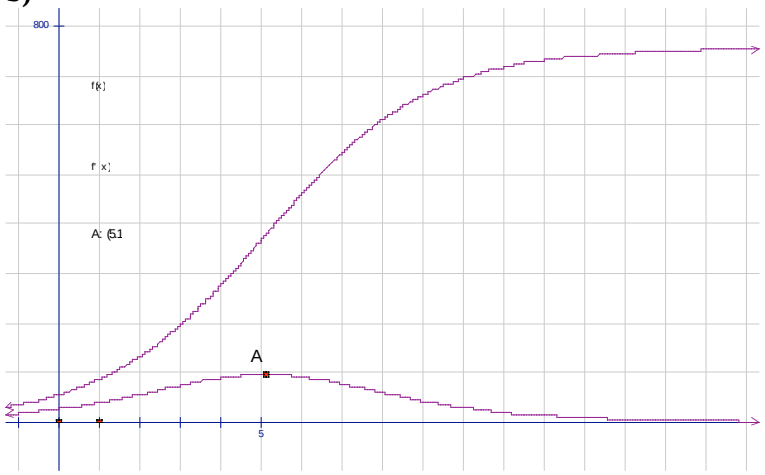
d) There is a horizontal asymptote at $y = 756$.



e) Answers may vary. For example:
The rabbit population will not reach 756.

Chapter 5 Section 5 **Question 15 Page 293**

a), b)



Answers may vary. For example:
The graph has a maximum value at (5.1, 94.4). The growth rate of the rabbits will increase to a maximum and then decrease to zero.

c) The rabbit population was growing the fastest at the fifth year.

d) The rabbit population was growing at the rate of 94.4 rabbits per year.

Chapter 5 Section 5 **Question 16 Page 293**

$$\begin{aligned}
P(t) &= \frac{755.6}{1+12.9e^{-0.5t}} \\
&= 755.6(1+12.9e^{-0.5t})^{-1} \\
P'(t) &= -755.6(1+12.9e^{-0.5t})^{-2}(12.9)(-0.5)e^{-0.5t} \\
&= \frac{4873.62e^{-0.5t}}{(1+12.9e^{-0.5t})^2} \\
&= \frac{4873.62}{e^{0.5t}+25.8+166.41e^{-0.5t}} \\
&= \frac{4873.62}{\sqrt{e^t}+25.8+\frac{166.41}{\sqrt{e^t}}}
\end{aligned}$$

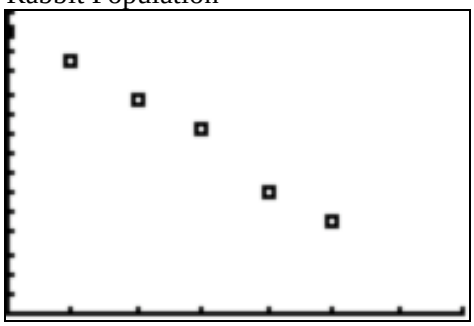
Chapter 5 Section 5 Question 17 Page 293

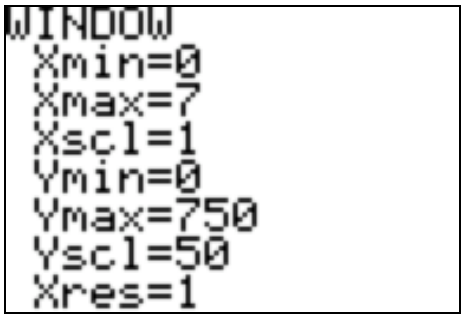
Answers may vary. For example:

a) Assume that in the pack of 5 wolves, only the dominant male and female breed. Each year, the female gives birth to a litter of 5 pups. According to question 14, the maximum population for the rabbits is 755. Assume that 50 rabbits are consumed per year by 10 wolves and the rate of consumption is proportional with the wolf population.

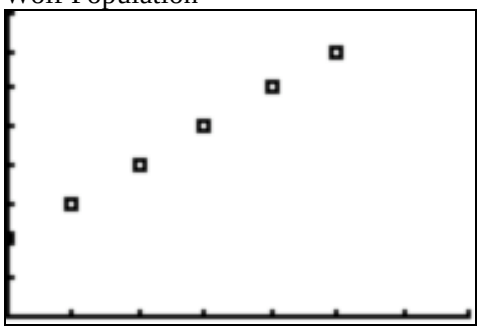
Year	Rabbit Population	Wolf Population
15	705	10
16	630	15
17	530	20
18	455	25
19	305	30
20	230	35

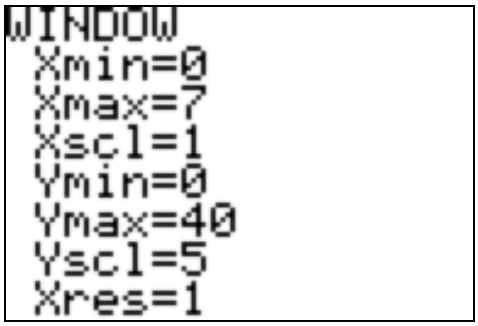
Rabbit Population





Wolf Population





b) The rabbit population is steadily declining as the wolf population is steadily increasing.

c) Answers will vary.
Rabbit: $y = -97.9x + 720.5$
Wolf: $y = 5x + 10$

Both models are linear. The wolf population in increasing as the rabbit population declines. The rate of decline is greater in the rabbit population than the rate of increase in the wolf population.

Chapter 5 Section 5 **Question 18 Page 293**

B

Chapter 5 Review

Chapter 5 Review Question 1 Page 294

a) $y = \dfrac{3^x}{\ln 3}$

b) $y = \dfrac{3x^2}{2}$

c) $y = x^3$

d) $y = 2^x$

Chapter 5 Review Question 2 Page 294

Answers may vary. For example:

a) Choose the values of n to be $n \in \mathbb{N}$. Evaluate the limit for values of n that are larger and larger. As the values of n become larger, the value of the limit will approach the value e.

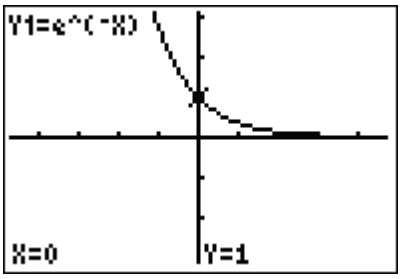
b) Choose n = 100 000.

$$\lim_{n \to 100\,000} \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{100\,000}\right)^{100\,000}$$

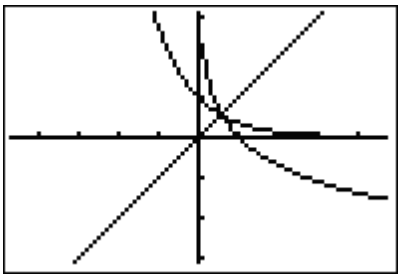
$$e = 2.72$$

Chapter 5 Review Question 3 Page 294

a)



b)



c) $y = -\ln x$

Answers may vary. For example:

$-\ln(e^{-x}) = x$

$= e^{-(-\ln x)}$

Chapter 5 Review **Question 4 Page 294**

a) $e^{-3} = 0.050$

b) $\ln(6.2) = 1.825$

c) $\ln\left(e^{\frac{3}{4}}\right) = 0.75$

d) $e^{\ln(0.61)} = 0.61$

Chapter 5 Review **Question 5 Page 294**

a) $x = 1.10$

b) $x = 0.01$

c) $x = 2.23$

d) $x = 9.21$

Chapter 5 Review **Question 6 Page 294**

a) 50 bacteria

b) $P(4) = 50e^{0.12(4)}$

$\doteq 81$

After 4 days, there will be 81 bacteria.

c) $100 = 50e^{0.12t}$

$e^{0.12t} = 2$

$0.12t = \ln 2$

$t = \dfrac{\ln 2}{0.12}$

$t \doteq 5.78$

It will take about 6 days for the population to double.

d) $P(t) = 50\left(2^{\log_2 e}\right)^{0.12t}$

$$= 50\left(2^{\frac{\log e(0.12t)}{\log 2}}\right)$$

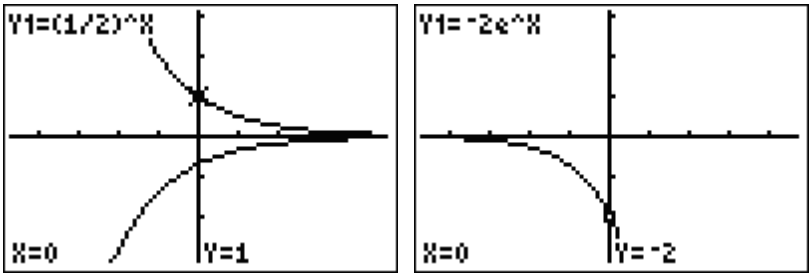
$$= (50)2^{\frac{t}{5.8}}$$

Chapter 5 Review **Question 7 Page 295**

a) i) $f'(x) = \left(\frac{1}{2}\right)^x \ln\frac{1}{2}$

ii) $g'(x) = -2e^x$

b)



Chapter 5 Review **Question 8 Page 295**

$$\frac{dy}{dx} = 2\ln 3(3^x)$$

$$\text{slope} = \left.\frac{dy}{dx}\right|_{x=1}$$

$$= 2\ln 3(3^1)$$

$$= 6\ln 3$$

At $x = 1$, $m = 6\ln 3$, and $y = 6$.

Substitute these values into $y = mx + b$.
$b = 6 - 6\ln 3$

Therefore, the equation of the tangent is $y = 6x\ln 3 + 6 - 6\ln 3$.

Chapter 5 Review **Question 9 Page 295**

$$\frac{dy}{dx} = -3e^{x}$$

$$\text{slope} = \left.\frac{dy}{dx}\right|_{x=\ln 2}$$
$$= -3e^{\ln 2}$$
$$= -6$$

At $x = \ln 2$, $m = -6$, and $y = -6$.

Substitute these values into $y = mx + b$.
$b = -6 + 6\ln 2$

Therefore, the equation of the tangent is $y = -6x - 6 + 6\ln 2$.

Chapter 5 Review **Question 10 Page 295**

a) $A(5) = 1000(2)^{\frac{5}{9}}$
$= 1469.73$
The value after 5 years is \$1469.73.

b) i) $2000 = 1000(2)^{\frac{t}{9}}$
$\frac{t}{9} = 1$
$t = 9$
It will take 9 years to double in value.

ii) $3000 = 1000(2)^{\frac{t}{9}}$
$3 = (2)^{\frac{t}{9}}$
$\ln 3 = \frac{t}{9}\ln 2$
$t = \frac{9\ln 3}{\ln 2}$
$t \doteq 14.26$
It will take 14.26 years to triple in value.

c) $A(t) = 1000(2^t)^{\frac{1}{9}}$

$$A'(t) = \frac{1000}{9}(2^t)^{-\frac{8}{9}}(2^t \ln 2)$$

$$= \frac{1000}{9}(2^{\frac{t}{9}})\ln 2$$

i) $A'(9) = \frac{1000}{9}(2^{\frac{9}{9}})\ln 2$

$= 154.03$

The rate is \$154.03 per year.

ii) $A'(14.26) = \frac{1000}{9}(2^{\frac{14.26}{9}})\ln 2$

$= 230.97$

The rate is \$230.97 per year.

Chapter 5 Review **Question 11 Page 295**

a) $\frac{dy}{dx} = (6x-2)e^{3x^2-2x+1}$

b) $f'(x) = e^{2x}(2x-1)$

c) $\frac{dy}{dx} = 3 - e^{-x}$

d) $\frac{dy}{dx} = e^x(\cos(2x) - 2\sin(2x))$

e) $g(x) = \left(\left(\frac{1}{3}\right)^x\right)^4 - 2e^{\sin x}$

$$g'(x) = 4\left(\left(\frac{1}{3}\right)^x\right)^3\left(\frac{1}{3}\right)^x\left(\ln\frac{1}{3}\right) - 2e^{\sin x}(\cos x)$$

$$g'(x) = 4\left(\frac{1}{3}\right)^{4x}\left(\ln\frac{1}{3}\right) - 2e^{\sin x}(\cos x)$$

Chapter 5 Review **Question 12 Page 295**

Local extrema occur when $\frac{dy}{dx} = 0$.

$$\frac{dy}{dx} = 2xe^{x^2}$$
$$0 = 2xe^{x^2}$$
$$x = 0 \qquad \text{since } e^{x^2} > 0$$

When $x = 0$, $y = 1$.

$\frac{d^2y}{dx^2} = 2e^{x^2} + 4x^2e^{x^2}$ which is positive for $x = 0$.

Therefore, there is a local minimum at (0, 1).

Chapter 5 Review **Question 13 Page 295**

Local extrema occur when $\frac{dy}{dx} = 0$.

$$\frac{dy}{dx} = 2e^x$$
$$0 = 2e^x$$

But $2e^x > 0$ for all values of x.
Therefore, there are no local extrema.

Chapter 5 Review **Question 14 Page 295**

a) \$900

b) $V(1) = 900e^{-\frac{1}{3}}$
$= 644.88$
The value after one year is \$644.88.

c) $450 = 900e^{-\frac{t}{3}}$
$e^{-\frac{t}{3}} = 0.5$
$-\frac{t}{3} = \ln(0.5)$
$t = -3\ln(0.5)$
$t \doteq 2.1$
It will take 2.1 years for the computer to be worth half its original value.

d) $V'(t) = -\left(\frac{1}{3}\right)900e^{-\frac{t}{3}}$

$= -300e^{-\frac{t}{3}}$

$V'(2.1) = -300e^{-\frac{2.1}{3}}$

$\doteq -149.98$

The rate of depreciation is \$149.98 per year.

Chapter 5 Review **Question 15 Page 295**

a) $70 = 80\left(\frac{1}{2}\right)^{\frac{5}{h}}$

$\left(\frac{1}{2}\right)^{\frac{5}{h}} = \frac{7}{8}$

$\frac{5}{h}\ln\left(\frac{1}{2}\right) = \ln\left(\frac{7}{8}\right)$

$h = \frac{5\ln(0.5)}{\ln(0.875)}$

$h \doteq 26$

The half-life is 26 days.

b) $N(t) = 80\left(\frac{1}{2}\right)^{\frac{t}{26}}$

c) $N(t) = 80\left(\left(\frac{1}{2}\right)^{t}\right)^{\frac{1}{26}}$

$N'(t) = \frac{80}{26}\left(\left(\frac{1}{2}\right)^{t}\right)^{-\frac{25}{26}}\left(\frac{1}{2}\right)^{t}\ln\left(\frac{1}{2}\right)$

$N'(t) = \frac{80}{26}\left(\frac{1}{2}\right)^{\frac{t}{26}}\ln\left(\frac{1}{2}\right)$

$N'(5) = \frac{80}{26}\left(\frac{1}{2}\right)^{\frac{5}{26}}\ln\left(\frac{1}{2}\right)$

$\doteq -1.9$

The rate of decay is –1.9 mg/day.

Chapter 5 Review **Question 16 Page 295**

a) $x(0) = 3\cos(0)e^{-0.05(0)}$

$= 3$

The horizontal distance is 3 m.

b) $x'(t) = 3(-0.05)\cos(t)e^{-0.05t} - 3\sin(t)e^{-0.05t}$

$= -0.15\cos(t)e^{-0.05t} - 3\sin(t)e^{-0.05t}$

The maximum value occurs at $t = \frac{\pi}{2}$, where $x'\left(\frac{\pi}{2}\right) \doteq -2.8$.

The greatest speed is 2.8 m/s.

c) Kara's maximum horizontal distance occurs at $t = 0, \pi, 2\pi, \ldots, n\pi$ for $n \in \mathbb{N}$.
At $t = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi$, $x(t) > 1$.

$x(6\pi) = 3\cos(6\pi)e^{-0.05(6\pi)}$

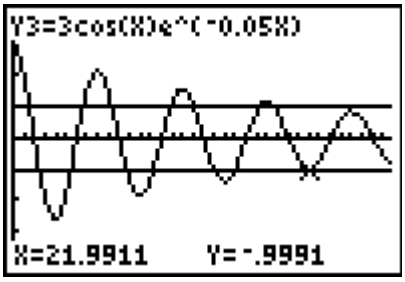
$\doteq 1.169$

$x(7\pi) = 3\cos(7\pi)e^{-0.05(7\pi)}$

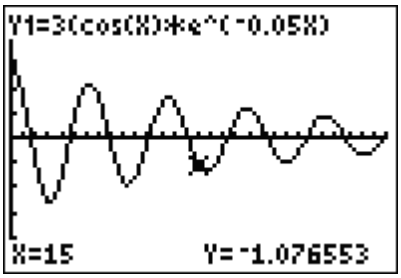
$\doteq -0.999$

Therefore Kara will be within a distance of 1 m in her maximum displacement at $t \doteq 22$ s.
It will take her 3.5 swings.

Graph $y = x(t)$, $y = 1$, and $y = -1$.



d)



Chapter 5 Practice Test

Chapter 5 Practice Test Question 1 Page 296

A

Chapter 5 Practice Test Question 2 Page 296

C

Chapter 5 Practice Test Question 3 Page 296

A

Chapter 5 Practice Test Question 4 Page 296

D

Chapter 5 Practice Test Question 5 Page 296

a) $\frac{dy}{dx} = e^{-\frac{1}{2}x}$

b) $f'(x) = 2x^3e^{2x} + 3x^2e^{2x} + 2x^2e^{-2x} - 2xe^{-2x}$

Chapter 5 Practice Test Question 6 Page 296

Local extrema occur when $\frac{dy}{dx} = 0$.

$$\frac{dy}{dx} = 2xe^{-2x} - 2x^2e^{-2x}$$
$$0 = 2xe^{-2x}(1-x)$$

$x = 0,\ x = 1$

If $x = 0$, then $y = 0$.
If $x = 1$, then $y = e^{-2}$.

$$\frac{d^2y}{dx^2} = 2e^{-2x} - 4xe^{-2x} - 4xe^{-2x} + 4x^2e^{-2x}$$

At $x = 0$, $\frac{d^2y}{dx^2}$ is positive.

At $x = 1$, $\frac{d^2y}{dx^2}$ is negative.

There is a local minimum at (0, 0).
There is a local maximum at (1, e^{-2}) or (1, 0.135).

Chapter 5 Practice Test Question 7 Page 296

a) $P(0) = 50$

Initially, 50 people had the virus.

b) $P(7) = 50(2)^{\frac{7}{2}}$

$\doteq 566$

After 1 week, 566 people will be infected.

c) $P'(t) = 50\left(\frac{1}{2}\right)(2)^{\frac{t}{2}} \ln 2$

$= 25(2)^{\frac{t}{2}} \ln 2$

$P'(7) = 25(2)^{\frac{7}{2}} \ln 2$

$\doteq 196$

After 1 week, the virus will be spreading to 196 people/day.

d) $1000 = 50(2)^{\frac{t}{2}}$

$(2)^{\frac{t}{2}} = 20$

$\frac{t}{2} \ln 2 = \ln 20$

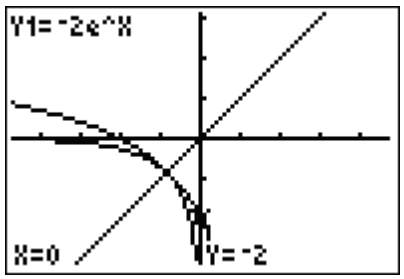
$t = 2\left(\frac{\ln 20}{\ln 2}\right)$

$t \doteq 8.64$

It will take 8.64 days for 1000 people to be infected.

Chapter 5 Practice Test Question 8 Page 296

a)



b) Answers may vary. For example:

Both graphs are decreasing functions. The y-intercept of the function $y = -2e^x$ is (0, –2). The x-intercept of the function $y = \ln\left(-\frac{1}{2}x\right)$ is (−2, 0).

Chapter 5 Practice Test Question 9 Page 296

$f'(x) = -2e^x$

slope $= f'(\ln 2)$

$= -2e^{\ln 2}$

$= -4$

At $x = \ln 2$, $m = -4$, and $y = -4$.

Substitute these values in $y = mx + b$.

$b = -4 + 4\ln 2$

Therefore, the equation of the tangent is $y = -4x - 4 + 4\ln 2$.

Chapter 5 Practice Test Question 10 Page 296

a) $N(t) = N_0 e^{-\lambda t}$

$64 = 100e^{-10\lambda}$

$-10\lambda = \ln(0.64)$

$\lambda = -\dfrac{\ln(0.64)}{10}$

$\lambda \doteq 0.045$

The disintegration constant is 0.045/min.

b) $\dfrac{N_0}{2} = N_0 e^{-\lambda t}$

$-\lambda t = \ln 0.5$

$t = -\dfrac{\ln 0.5}{0.045}$

$t \doteq 15.4$

The half-life is 15.4 min.

c) $N(t) = N_0\left(\dfrac{1}{2}\right)^{\frac{t}{15.4}}$

d) $N'(t) = -\lambda N_0 e^{-\lambda t}$

$N'(15) = -(0.045)(25)e^{-(0.045)(15)}$

$\doteq -0.57$

After 15 min, the sample is decaying at –0.57 mg/min.

Chapter 5 Practice Test Question 11 Page 296

a) $V(10) = 1000(1.05)^{10}$

$= 1628.89$

The value is \$1628.89 after 10 years.

b) $2000 = 1000(1.05)^t$

$t = \dfrac{\ln 2}{\ln 1.05}$

$t \doteq 14.2$

It will take 14.2 years to double in value.

c) $V(t) = 1000(1.05)^t$

$V'(t) = 1000(1.05)^t \ln 1.05$

d) $V'(10) = 1000(1.05)^{10} \ln 1.05$

$= 79.47$

The rate is \$79.47 per year.

Chapter 5 Practice Test Question 12 Page 297

a) $\dfrac{dy}{dx} = e^x(2\sin x \cos x + \sin^2 x)$

b) $\dfrac{dy}{dx} = e^{-x}(-x^2 + 2x - 1)$

c) $\dfrac{dy}{dx} = e^{\sin x}(x^2 \cos x + 2x)$

Chapter 5 Practice Test Question 13 Page 297

a) $15 = 20e^{-\frac{t}{16}}$

$-\dfrac{t}{16} = \ln(0.75)$

$t = -16\ln(0.75)$

$t \doteq 4.6$

It will take 4.6 h.

b) $V'(t) = 20\left(-\dfrac{1}{16}\right)e^{-\frac{t}{16}}$

$= -\dfrac{5}{4}e^{-\frac{t}{16}}$

c) $V'(1) = -\frac{5}{4}e^{-\frac{1}{16}}$

$\doteq -1.174$

The rate of change of the voltage is –1.174 V/h after 1 h.

d) Answers may vary. For example:

The voltage is dropping at a slower rate since the slope of the function $y = e^{-\frac{t}{16}}$ becomes less negative with time.

e) Answers may vary. For example:

$V'(2) = -\frac{5}{4}e^{-\frac{2}{16}}$

$\doteq -1.103$

The rate of change of the voltage is –1.103 V/h after 2 h.

Chapter 5 Practice Test Question 14 Page 297

a) $N_0 = 1000$

$N(t) = N_0 e^{kt}$

$1500 = 1000e^{k(1)}$

$k = \ln 1.5$

b) $2000 = 1000e^{(\ln 1.5)t}$

$2 = 1.5^t$

$t = \frac{\ln 2}{\ln 1.5}$

$t \doteq 1.7$

It will take 1.7 days.

c) $N'(t) = 1000e^{(\ln 1.5)t}(\ln 1.5)$

d) $N'(5) = 1000e^{(\ln 1.5)5}(\ln 1.5)$

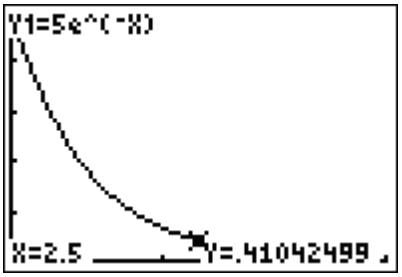
$\doteq 3079$

The growth rate is 3079 bacteria per day.

Chapter 5 Practice Test **Question 15 Page 297**

a) $d(t) = 5e^{-t}$

b)



c) At $t = 1$,

$d(1) = 5e^{-(1)}$

$\doteq 1.84$

At $t = 2$,

$d(2) = 5e^{-(2)}$

$\doteq 0.68$

At 1 s, the displacement is 1.84 m.
At 2 s, the displacement is 0.68 m.

d) $d'(t) = -5e^{-t}$

e) $d'(1) = -5e^{-(1)}$

$\doteq -1.84$

The rate is –1.84 m/s.

Chapter 5 Practice Test **Question 16 Page 297**

$A'(x) = -30xe^{-x^2}$

$A'(1) = -30e^{-1^2}$

$\doteq -11.04$

The amplitude is changing at –11.04 cm/cm.

Chapters 4 and 5 Review

Chapters 4 and 5 Review Question 1 Page 298

Answers may vary. For example:

a) $y = -2\cos(2x)$

b) The instantaneous rate of change is zero at the point $\left(\frac{\pi}{2}, 2\right)$.

This point is a local maximum point on the graph.

c) The instantaneous rate of change is a maximum at the point $\left(\frac{5\pi}{4}, 0\right)$. This point is a zero of the second derivative of f on a section of the graph where the function is increasing.

d) The instantaneous rate of change is a minimum at the point $\left(\frac{3\pi}{4}, 0\right)$. This point is a zero of the second derivative of f on a section of the graph where the function is decreasing.

e) A point on the graph that is a point of inflection is the point $\left(\frac{3\pi}{4}, 0\right)$. The function is decreasing at this point and the curve changes from concave down to concave up at this point.

Chapters 4 and 5 Review Question 2 Page 298

a) $\frac{dy}{dx} = 1 + \cos x$

b) Since the slope m is given by $\left.\frac{dy}{dx}\right|_{x=\pi}$

$$\left.\frac{dy}{dx}\right|_{x=\pi} = 1 + \cos(\pi)$$
$$= 0$$
$$f(\pi) = \pi$$

Substitute these values in the equation $y = mx + b$.

$b = \pi$

So the equation of the tangent is $y = \pi$.

c) Answers may vary. For example:
The tangent to $f(x)$ at $x = 3\pi$ will not have the same equation. The tangent line will be parallel to the tangent equation in part b), and the equation will be $y = 3\pi$.

Chapters 4 and 5 Review Question 3 Page 298

a) $f'(x) = 2\cos 2x$

b) $g'(x) = 2(\cos^2 x - \sin^2 x)$

c) Use the double angle identity $\cos 2x = \cos^2 x - \sin^2 x$.

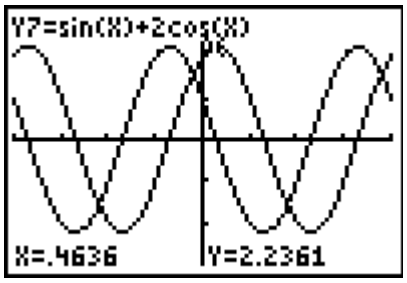
$$\begin{aligned} f'(x) &= 2\cos 2x \\ &= 2(\cos^2 x - \sin^2 x) \\ &= g'(x) \end{aligned}$$

d) Answers may vary. For example:
The original functions are equal. The expansion for the double angle identity for $f(x)$ is $\sin 2x = 2\sin x \cos x$.

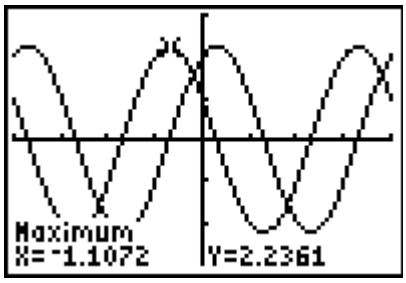
Chapters 4 and 5 Review Question 4 Page 298

Answers may vary. For example:
$A(t) = \sin t + 2\cos t$ and $A'(t) = \cos t - 2\sin t$. When the functions are graphed in the same viewing screen of a graphing calculator the rate of change of the amplitude never exceeds the maximum value of the amplitude itself.

$A(t) = \sin t + 2\cos t$



$A'(t) = \cos t - 2\sin t$



Chapters 4 and 5 Review Question 5 Page 298

a) $$\begin{aligned} s(t) &= 8\sin\left(\frac{2\pi}{30}t\right) \\ &= 8\sin\left(\frac{\pi}{15}t\right) \end{aligned}$$

t is the time in s, $s(t)$ is the north-south position of the rider in m

b) $s'(t) = \frac{8\pi}{15}\cos\left(\frac{\pi}{15}t\right)$

c) The maximum speed occurs when $s''(t) = 0$.

$$s''(t) = -\frac{8\pi^2}{15^2}\sin\left(\frac{\pi}{15}t\right)$$

$$0 = \sin\left(\frac{\pi}{15}t\right)$$

$$\frac{\pi}{15}t = k\pi,\ k \in \mathbb{N}$$

$$t = 15k$$

Choose any value of k and substitute t into $s'(t)$ to find the maximum value. Take the absolute value since speed is always positive.

$$\left|s'(15)\right| = \left|\frac{8\pi}{15}\cos\left(\frac{\pi}{15}(15)\right)\right|$$

$$= \frac{8\pi}{15}$$

$$\doteq 1.68$$

The maximum speed is 1.68 m/s.

d) From part c), the maximum speed occurs for $t = 15k,\ k \in \mathbb{N}$.

The positions at which the maximum speed occurs are:

$s(0) = 0$

$s(15) = 0$

$s(30) = 0$

Therefore, the position of the rider is 0 m each time the maximum speed is reached.

Chapters 4 and 5 Review Question 6 Page 298

a) $D(t) = 2\cos\left(\frac{\pi}{6}t\right) + 9$ assuming t = 0 at 9:00 A.M., t is in hours, and D is the depth in feet.

b) $D'(t) = -\frac{\pi}{3}\sin\left(\frac{\pi}{6}t\right)$

c) The maximum speed occurs when $\sin\left(\frac{\pi}{6}t\right) = -1$.

$$\frac{\pi}{6}t = \frac{3\pi}{2} + 2k\pi,\ k \in \mathbb{N}$$

$$t = 9 + 12k$$

The depth is rising fastest at t = 9 h, t = 21 h, t = 33 h, …

d) At $t = 9$,

$$D'(9) = -\frac{\pi}{3}\sin\left(\frac{(9)\pi}{6}\right)$$
$$= \frac{\pi}{3}$$
$$\doteq 1.047$$

The water is rising at 1.047 ft/h.

Chapters 4 and 5 Review Question 7 Page 299

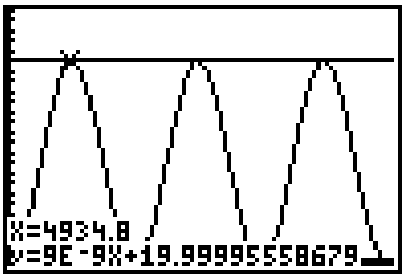
a) $$A'(t) = 20(2)\left(\sin\left(\frac{t}{1000\pi}\right)\right)\cos\left(\frac{t}{1000\pi}\right)\frac{1}{1000\pi}$$
$$= \frac{1}{25\pi}\sin\left(\frac{t}{1000\pi}\right)\cos\left(\frac{t}{1000\pi}\right)$$

b) $$A'(t) = \frac{1}{25\pi}\sin\left(\frac{t}{1000\pi}\right)\cos\left(\frac{t}{1000\pi}\right)$$
$$= 0$$

$$\sin\left(\frac{t}{1000\pi}\right) = 0 \qquad \cos\left(\frac{t}{1000\pi}\right) = 0$$
$$\frac{t}{1000\pi} = \pi \qquad \frac{t}{1000\pi} = \frac{\pi}{2}$$
$$t = 1000\pi^2 \qquad t = 500\pi^2$$
$$t \doteq 9869.60 \qquad t \doteq 4934.8$$

The first value of t is 4934.8 s.

c)



Chapters 4 and 5 Review Question 8 Page 299

Answers may vary. For example:

a) $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{2^{(x+h)} - 2^x}{h}$

$= \lim\limits_{h \to 0} \dfrac{2^x(2^h) - 2^x}{h}$

$= \lim\limits_{h \to 0} \dfrac{2^x(2^h - 1)}{h}$

$= 2^x \left[\lim\limits_{h \to 0} \dfrac{(2^h - 1)}{h} \right]$

$= 2^x \ln 2$

$f'(0) = \ln 2$

$\doteq 0.693$

b) The result will be ln 3 = 1.099, which is greater than the previous result.

c) $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{3^{(x+h)} - 3^x}{h}$

$= \lim\limits_{h \to 0} \dfrac{3^x(3^h) - 3^x}{h}$

$= \lim\limits_{h \to 0} \dfrac{3^x(3^h - 1)}{h}$

$= 3^x \left[\lim\limits_{h \to 0} \dfrac{(3^h - 1)}{h} \right]$

$= 3^x \ln 3$

$f'(0) = \ln 3$

$= 1.0986$

Chapters 4 and 5 Review Question 9 Page 299

a) \$2.00

b) $A = 1(1 + 0.5)^2$

$= 2.25$

After 1 year, it will be worth \$2.25.

c) $A = 1\left(1 + \frac{1}{n}\right)^n$ where n is the number of compounding periods.

If $n = 3$, $A = \$2.37$.

If $n = 4$, $A = \$2.44$.

If $n = 5$, $A = \$2.49$.

If $n = 100$, $A = \$2.70$.

d) Since $\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = e$, which is about 2.718, the amount A is approaching \$2.72.

Chapters 4 and 5 Review Question 10 Page 299

a) $\ln(e^{\sin^2 x}) + \ln(e^{\cos^2 x}) = \sin^2 x + \cos^2 x$

$= 1$

b) $\ln(e^{5x})\left(e^{\ln\left(\frac{1}{x}\right)}\right) = 5x\left(\frac{1}{x}\right)$

$= 5$

Chapters 4 and 5 Review Question 11 Page 299

a) Answers may vary. For example:
No. The smoke detector is not likely to fail while I own it.
The half-life of americium-241 is 432.2 years.

b) After 50 years,

$m = 0.2(0.5)^{\frac{50}{432.2}}$

$\doteq 0.1846$

There will be 0.1846 mg.

c) $0.05 = 0.2(0.5)^{\frac{t}{432.2}}$

$0.25 = (0.5)^{\frac{t}{432.2}}$

$\frac{t}{432.2} = \frac{\ln 0.25}{\ln 0.5}$

$t = 432.2\left(\frac{\ln 0.25}{\ln 0.5}\right)$

$t \doteq 864.4$

It will take 864.4 years.

Chapters 4 and 5 Review **Question 12 Page 299**

a) $f'(x) = 12^x \ln 12$

b) $g'(x) = \left(\frac{3}{4}\right)^x \ln\left(\frac{3}{4}\right)$

c) $h'(x) = -5e^x$

d) $i'(x) = \theta^x \ln \theta$

Chapter 6 **Geometric Vectors**

Chapter 6 Prerequisite Skills

Chapter 6 Prerequisite Skills **Question 1 Page 302**

Starting with the base and moving counterclockwise, the measured side lengths are 5.5 cm, 2.4 cm, 2.9 cm, 2.5 cm, 1.3 cm, and 2.7 cm.

Using the scale of 3 cm represents 1 cm, the actual lengths are 1.8 cm, 0.8 cm, 1.0 cm, 0.8 cm, 0.4 cm, and 0.9 cm.

The angle measures are 106°, 92°, 139°, 229°, 63°, and 91°.

Chapter 6 Prerequisite Skills **Question 2 Page 302**

Answers may vary. For example:
a) 1 cm represents 50 km; Answers may vary.

$$\frac{\text{Scale: 1 cm : 50 km}}{\text{200 km}}$$

b) 1 cm represents 10 m; Answers may vary.

$$\frac{\text{Scale: 1 cm : 10 m}}{\text{50 m}}$$

c) 1 cm represents 20 cm; Answers may vary.

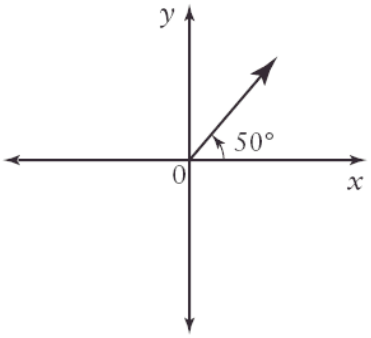
$$\frac{\text{Scale: 1 cm : 20 cm}}{\text{120 cm}}$$

d) 1 cm represents 1000 km; Answers may vary.

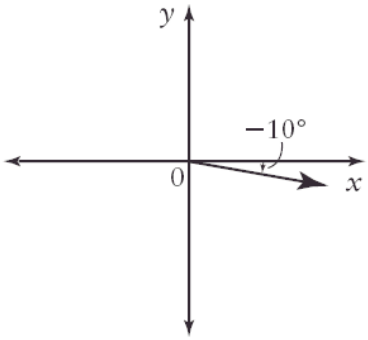
$$\frac{\text{Scale 1 cm : 1000 km}}{\text{4000 km}}$$

Chapter 6 Prerequisite Skills **Question 3 Page 302**

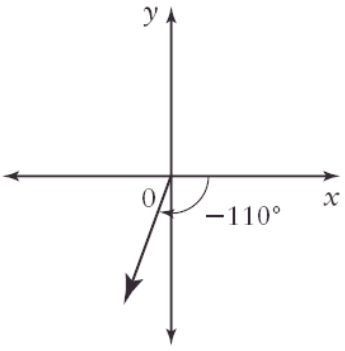
a) The measure of the angle between the positive y-axis and the terminal arm is 40°.



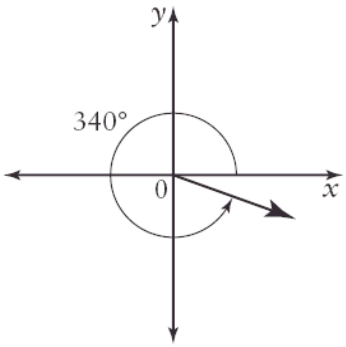
b) The measure of the angle between the positive y-axis and the terminal arm is 100°.



c) The measure of the angle between the positive y-axis and the terminal arm is 160°.



d) The measure of the angle between the positive y-axis and the terminal arm is 110°.



Chapter 6 Prerequisite Skills **Question 4 Page 302**

a) measure of the angle between the positive y-axis and the reflected terminal arm: 120°

b) measure of the angle between the positive y-axis and the reflected terminal arm: 10°

c) measure of the angle between the positive y-axis and the reflected terminal arm: 25°

d) measure of the angle between the positive y-axis and the reflected terminal arm: 120°

Chapter 6 Prerequisite Skills **Question 5 Page 302**

a) $\dfrac{c}{\sin 40^\circ} = \dfrac{3}{\sin 65^\circ}$

$$c = \frac{3\sin 40^\circ}{\sin 65^\circ}$$

$$c \doteq 2.1$$

The length of side c is approximately 2.1 cm.

b) $\dfrac{\sin P}{7} = \dfrac{\sin 110^\circ}{15}$

$$\sin P = \frac{7\sin 110^\circ}{15}$$

$$\doteq 0.4385$$

$$P \doteq \sin^{-1}(0.4385)$$

$$\angle P \doteq 26$$

The measure of $\angle P$ is approximately 26.0°.

c) $5^2 = 8^2 + 8^2 - 2(8)(8)\cos E$

$$\cos E = \frac{64 + 64 - 25}{128}$$
$$\doteq 0.8047$$
$$\angle E \doteq 36.4$$

The measure of $\angle E$ is approximately 36.4°.

Chapter 6 Prerequisite Skills **Question 6 Page 303**

a) $a^2 = 12^2 + 10^2 - 2(12)(10)\cos 25^\circ$

$$\doteq 244 - 240(0.9063)$$
$$= 26.488$$
$$a \doteq 5.1$$

The length of side a is approximately 5.1 cm.

b) $7^2 = 9^2 + 6^2 - 2(9)(6)\cos P$

$$\cos P = \frac{81 + 36 - 49}{108}$$
$$\doteq 0.6296$$
$$\angle P \doteq 51.0$$

The measure of $\angle P$ is approximately 51.0°.

c) $\frac{\sin F}{4.0} = \frac{\sin 120^\circ}{6.9}$

$$\sin F = \frac{4.0 \sin 120^\circ}{6.9}$$
$$\doteq 0.0502$$
$$\angle F \doteq \sin^{-1}(0.0502)$$
$$\angle F \doteq 30.1$$

The measure of $\angle F$ is 30.1°.

Chapter 6 Prerequisite Skills **Question 7 Page 303**

Use the cosine law to find b.

$$b^2 = 12^2 + 8^2 - 2(12)(8)\cos 40^\circ$$
$$\doteq 208 - 192(0.7660)$$
$$= 60.928$$
$$b \doteq 7.8$$

The length of side b is 7.8 mm.

Use the sine law to find $\angle C$.

$$\frac{\sin C}{8} = \frac{\sin 40^\circ}{7.8}$$

$$\sin C = \frac{8 \sin 40^\circ}{7.8}$$

$$\doteq 0.6593$$

$$\angle C \doteq \sin^{-1}(0.6593)$$

$$\angle C \doteq 41.2$$

The measure of $\angle C$ is 41.2°.

Use the sum of the angles in a triangle to find the measure of $\angle A$.

$$\angle A + 41.2 + 40 = 180$$

$$\angle A = 98.8$$

The measure of $\angle A$ is 98.8°.

Chapter 6 Prerequisite Skills **Question 8 Page 303**

Label the lower right base angle as $\angle X$.

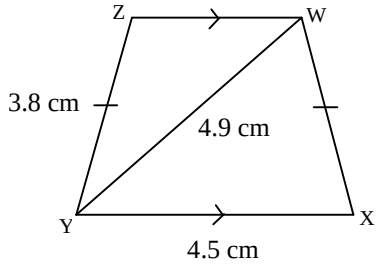
$$4.9^2 = 4.5^2 + 3.8^2 - 2(4.5)(3.8)\cos X$$

$$\cos X = \frac{20.25 + 14.44 - 24.01}{34.2}$$

$$\doteq 0.3123$$

$$\angle X \doteq 71.8$$

The measure of $\angle X$ is 71.8°.



Using the properties of parallel lines,

$$\angle ZWX + \angle X = 180^\circ$$

$$\angle ZWX + 71.8^\circ = 180^\circ$$

$$\angle ZWX = 108.2^\circ$$

The four interior angles are 71.8°, 71.8°, 108.2°, and 108.2°.

Chapter 6 Prerequisite Skills **Question 9 Page 303**

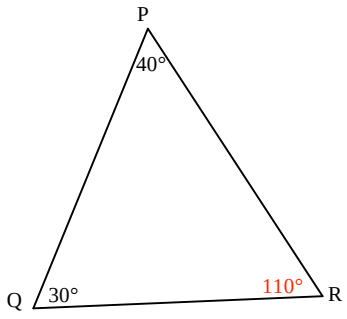
Using the sum of the angles in a triangle, $\angle R = 110^\circ$.

Use the sine law to find *c*.

$$\frac{p}{\sin 40^\circ} = \frac{10}{\sin 30^\circ}$$

$$p = \frac{10 \sin 40^\circ}{\sin 30^\circ}$$

$$p \doteq 12.86$$



Use the sine law to find r.

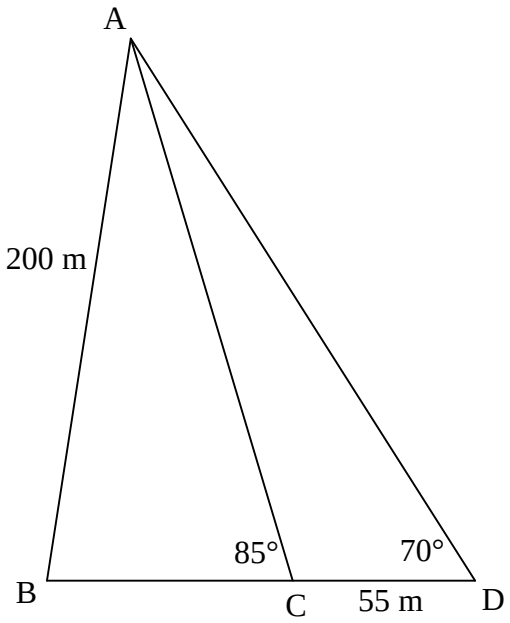
$$\frac{r}{\sin 110^\circ} = \frac{10}{\sin 30^\circ}$$

$$r = \frac{10 \sin 110^\circ}{\sin 30^\circ}$$

$$r \doteq 18.79$$

The missing dimensions are $\angle R = 110^\circ$, p = 12.9 cm, and r = 18.8 cm.

Chapter 6 Prerequisite Skills **Question 10 Page 303**



Use the sine law in ΔACD to find AC.

$\angle BCA = 70^\circ + \angle CAD$ (exterior angle to a triangle)

$85^\circ = 70^\circ + \angle CAD$

$\angle CAD = 15^\circ$

$$\frac{AC}{\sin 70^\circ} = \frac{55}{\sin 15^\circ}$$

$$AC = \frac{55 \sin 70^\circ}{\sin 15^\circ}$$

$$AC \doteq 199.7$$

Now use the sine law in ΔABC.

$$\frac{\sin B}{199.7} = \frac{\sin 85^\circ}{200}$$

$$\sin B = \frac{199.7 \sin 85^\circ}{200}$$

$$\doteq 0.9947$$

$$\angle B \doteq \sin^{-1}(0.9947)$$

$$\angle B \doteq 84.1$$

Therefore, the acute angle from the horizontal through which the tower is leaning is 84.1°.

Chapter 6 Prerequisite Skills **Question 11 Page 303**

$a + b = b + a; \qquad a \times b = b \times a$

When you add or multiply two numbers, the order of performing the operation does not matter; the answer is the same in both cases. This property is called the commutative property of addition (or multiplication).

$(a + b) + c = a + (b + c), \qquad (ab)c = a(bc)$

When you add (or multiply) three numbers, you have to perform the operation in stages since you only know how to add (or multiply) two numbers at a time. The associative property for addition (or multiplication) says that it does not matter how you group the operations; the answer will be the same in both cases.

$a(b+c) = ab + ac$

When you multiply the sum of two numbers inside parentheses by a factor, you can multiply each of the terms in the parentheses by the factor and then add the resulting products. This is the distributive property for multiplication over addition.

Chapter 6 Prerequisite Skills **Question 12 Page 303**

Answers may vary. For example:

a) $a + 4 + (-a)$

$= a + (-a) + 4$ commutative property of addition

$= (a + (-a)) + 4$ associative property of addition

$= 4$

b) $3(b + 10)$

$= 3(b) + 3(10)$ distributive property of multiplication over addition

$= 3b + 30$

c) $c \times (-4) \times a$

$= (-4) \times c \times a$ commutative property of multiplication

$= (-4) \times a \times c$ commutative property of multiplication

$= -4ac$

d) $(a + 2)(a - 2)$

$= (a + 2)a - (a + 2)2$ distributive property

$= a(a + 2) - 2(a + 2)$ commutative property of multiplication

$= a^2 + 2a - 2a - 4$ distributive property

$= a^2 - 4$

Chapter 6 Prerequisite Skills **Question 13 Page 303**

Answers may vary. For example:

$5(3 + (x + 4))$

$= 5(3) + 5(x + 4)$ distributive property

$= 15 + 5x + 20$ distributive property

$= 5x + 15 + 20$ commutative property of addition

$= 5x + 35$ associative property of addition

Chapter 6 Section 1: Introduction to Vectors

Chapter 6 Section 1 Question 1 Page 310

a) A vector would be a suitable model. The magnitude is 35 km/h and the direction is east.

b) A vector would not be a suitable model. The magnitude is 10 knots, but no direction is given. This is a scalar quantity.

c) A vector could be a suitable model. The magnitude is 6 cm, and the direction is 30° to the horizontal. However, the direction is only partially specified.

d) A vector would not be a suitable model. The magnitude is 220 km/h, but no direction is given. This is a scalar quantity.

e) A vector would not be a suitable model. The magnitude is 2.9 kg, and there is no direction. This is a scalar quantity.

f) A vector would not be a suitable model. The magnitude is 10 m, but the direction is not clearly specified. This is a scalar quantity.

g) A vector would be a suitable model. The magnitude is 50 N, and the direction is down towards the centre of the earth, the direction of gravity.

h) A vector would not be a suitable model. The magnitude is 90°C, and there is no direction. This is a scalar quantity.

i) A vector would be a suitable model. The magnitude is 1000 N, and the direction is upwards.

Chapter 6 Section 1 Question 2 Page 310

Answers may vary. For example:

vectors: the path from your desk to the classroom door; a car is travelling at 100 km/h in a north easterly direction; a dog is pulling a sled with a force of 150 N at 45° to the horizontal.

scalars: a woman's age is 60 years; a table has a mass of 30 kg; the number of players on the soccer team is 11.

Chapter 6 Section 1 Question 3 Page 310

Quantity	**Vector or Scalar?**
$\vec{v}$	Vector. The arrow above the letter indicates it is a vector.
$\lvert\vec{v}\rvert$	Scalar. The absolute value brackets designate the magnitude or size of vector $\vec{v}$. There is no direction.
6	Scalar. The number 6 has magnitude only. There is no direction.
$-\overrightarrow{\text{CD}}$	Vector. The vector has the same magnitude and opposite direction as vector $\overrightarrow{\text{CD}}$. It also has the same magnitude and direction as $\overrightarrow{\text{DC}}$.

$-\left\|\overrightarrow{AB}\right\|$	Scalar. The absolute value brackets designate the magnitude or size of the vector $\overrightarrow{AB}$. What is indicated is the opposite of that magnitude.
n	Scalar. This variable represents a number that has no direction.
$-\sqrt{7}$	Scalar. This is a number. No direction is implied.

Chapter 6 Section 1 **Question 4 Page 311**

a) The line segment is 4 cm in length. Using the scale, the magnitude is 4×2 or 8 km. The direction is left.

b) The line segment is 5 cm in length. Using the scale, the magnitude is 5×5 or 25 km/h. The direction is 45° counterclockwise to the horizontal.

c) The line segment is 3 cm in length. Using the scale, the magnitude is 3×10 or 30 N. The direction is down.

Chapter 6 Section 1 **Question 5 Page 311**

a) N70°E

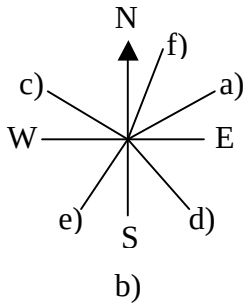
b) due south

c) N60°W

d) S40°E

e) S30°W

f) N24°E



b)

Chapter 6 Section 1 **Question 6 Page 311**

a) 035°

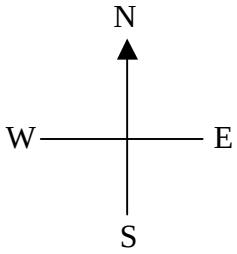
b) 290°

c) 190°

d) 128°

e) 162°

f) 273°



Chapter 6 Section 1 **Question 7 Page 311**

a) Vectors parallel to vector $\overrightarrow{AB}$ are $\overrightarrow{EF}$, $\overrightarrow{IJ}$, $\overrightarrow{KL}$, and $\overrightarrow{GH}$.

b) There is only one vector equivalent to vector $\overrightarrow{AB}$, that is, $\overrightarrow{EF}$.

c) There is only one vector opposite to vector $\overrightarrow{AB}$, that is, $\overrightarrow{GH}$.

Chapter 6 Section 1 **Question 8 Page 311**

a) $\overrightarrow{QP} = \overrightarrow{TU} = \overrightarrow{RS}$,
$\overrightarrow{PU} = \overrightarrow{US} = \overrightarrow{QT} = \overrightarrow{TR}$,
$\overrightarrow{UP} = \overrightarrow{SU} = \overrightarrow{TQ} = \overrightarrow{RT}$,
$\overrightarrow{PS} = \overrightarrow{QR}$,
$\overrightarrow{SP} = \overrightarrow{RQ}$

b) $\overrightarrow{AE} = \overrightarrow{BD}$,
$\overrightarrow{EA} = \overrightarrow{DB}$,
$\overrightarrow{AF} = \overrightarrow{BC}$,
$\overrightarrow{FA} = \overrightarrow{CB}$,
$\overrightarrow{FE} = \overrightarrow{CD}$,
$\overrightarrow{EF} = \overrightarrow{DC}$,
$\overrightarrow{AB} = \overrightarrow{FC} = \overrightarrow{ED}$,
$\overrightarrow{BA} = \overrightarrow{CF} = \overrightarrow{DE}$

c) $\overrightarrow{JK} = \overrightarrow{NL}$,
$\overrightarrow{KJ} = \overrightarrow{LN}$,
$\overrightarrow{JN} = \overrightarrow{KL}$,
$\overrightarrow{NJ} = \overrightarrow{LK}$

Chapter 6 Section 1 **Question 9 Page 311**

a) 200 km west

b) 500 N downward

c) 25 km/h on a bearing of 240°

d) 150 km/h on a quadrant bearing of N50°E

e) $-\overrightarrow{AB}$ or $\overrightarrow{BA}$

f) $-\vec{v}$

Chapter 6 Section 1 Question 10 Page 311

Answers may vary. For example:

a) 300 km east

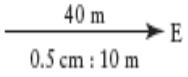
b) 700 N upward

c) 35 km/h on a bearing of 060°

d) 200 km/h on a quadrant bearing of S50°W

Chapter 6 Section 1 Question 11 Page 311

a)



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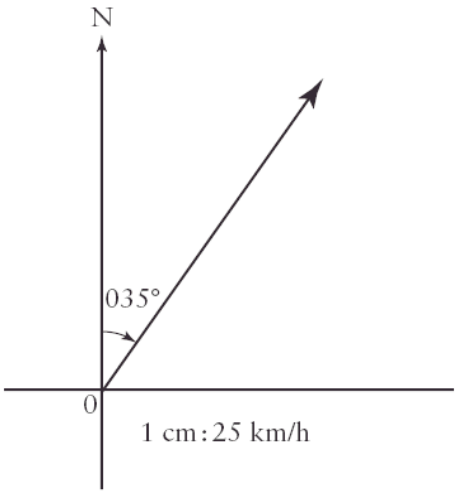

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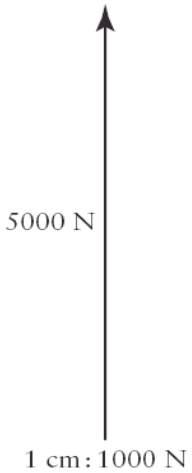
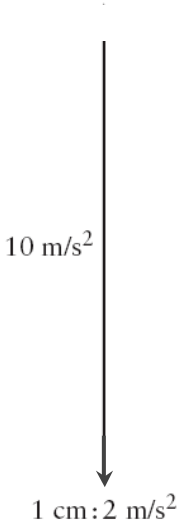


d)



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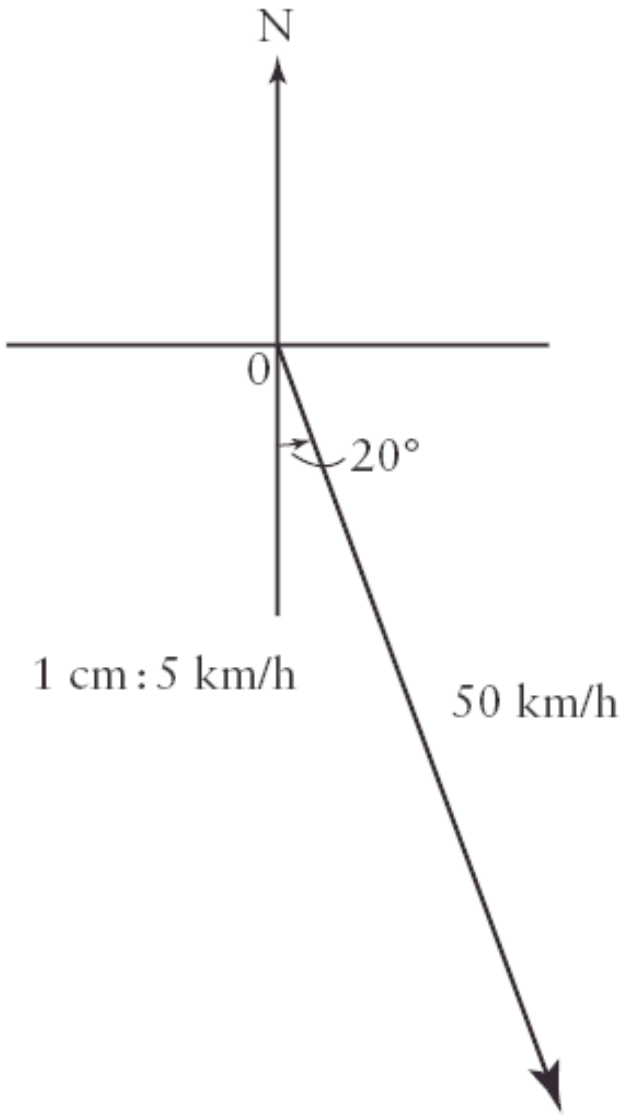

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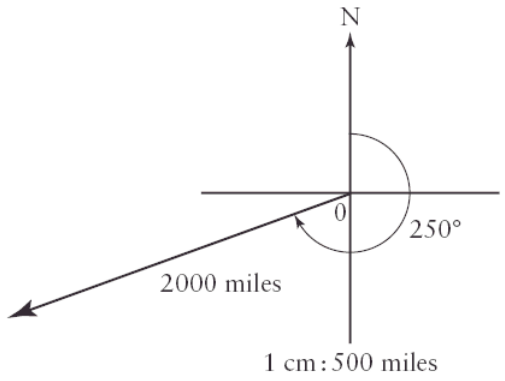
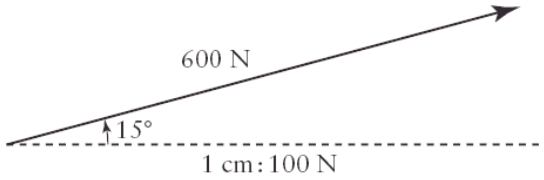
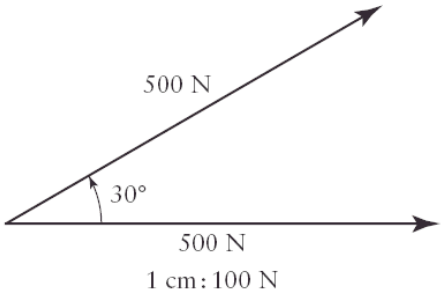


g)



h) Answers may vary. For example:



Chapter 6 Section 1 **Question 12 Page 312**

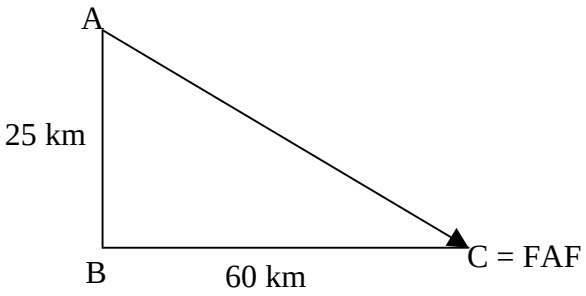
Answers may vary. For example: Distance is the most likely cause of the tread on the car's tires being worn down. The greater the distance the car is driven, the greater the wear on the tread on the car's tires.

Chapter 6 Section 1 **Question 13 Page 312**

a) $\overrightarrow{AB} = \overrightarrow{DC}$. Since ABCD is a parallelogram, therefore AB = DC and AB || DC. Vectors $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are equal in length and direction.

b) $\overrightarrow{BC} = -\overrightarrow{DA}$. Since ABCD is a parallelogram, therefore BC = DA and BC || DA. Vectors $\overrightarrow{BC}$ and $\overrightarrow{DA}$ are equal in length and opposite in direction.

Chapter 6 Section 1 **Question 14 Page 312**



To find the approach vector, two quantities must be determined: speed and direction.

Use the Pythagorean theorem to find AC.

$$\begin{aligned} AC^2 &= 25^2 + 60^2 \\ &= 4225 \\ AC &= 65 \end{aligned}$$

Since the 65 km need to be covered in 10 min, the speed necessary is $\frac{65}{10} \times 60$ or 390 km/h.

Use trigonometric ratios to find $\angle A$.

$$\begin{aligned} \tan A &= \frac{60}{25} \\ &= 2.4 \\ \angle A &= \tan^{-1}(2.4) \\ \angle A &\doteq 67.4 \end{aligned}$$

The approach vector is a speed of 390 km/h on a quadrant bearing of S67.4°E.

Chapter 6 Section 1 **Question 15 Page 312**

a) $70 \times 9.8 = 686$
The person's weight on earth: is 686 N.
$70 \times 1.63 = 114.1$
The person's weight on the moon is 114.1 N.

b) $2000 \times 9.8 = 19\,600$
The truck's weight on earth is 19 600 N.
$2000 \times 1.63 = 3260$
The truck's weight on the moon is 3260 N.

c) A partially submerged object displaces a volume of water that has the same weight as the object**.** On the moon, the object will weigh less but so too will the water weigh less. Therefore, the object will also be submersed 75% on the moon.

Chapter 6 Section 1 **Question 16 Page 312**

a) The statement is true. If $\vec{a} = \vec{b}$, then the vectors are equal. If the vectors are equal, they have the same magnitude and the same direction. Since the magnitudes of the vectors are equal, then $|\vec{a}| = |\vec{b}|$.

b) The statement is not true. If $|\vec{a}| = |\vec{b}|$, then the magnitudes of the vectors are equal. It is possible to choose two vectors that are equal in length, but have different directions. For example, the vectors could be opposite to each other. Clearly $\vec{a} \neq \vec{b}$ in this case.

Chapter 6 Section 1 **Question 17 Page 312**

Answers may vary. For example: A parallelepiped has six faces that are all parallelograms. Opposite faces are congruent.

a) i) $\overrightarrow{AB} = \overrightarrow{DC} = \overrightarrow{EF} = \overrightarrow{HG}$. Vectors opposite to $\overrightarrow{AB}$ are $\overrightarrow{BA} = \overrightarrow{CD} = \overrightarrow{FE} = \overrightarrow{GH}$.

ii) $\overrightarrow{ED} = \overrightarrow{FC}$. A vector opposite to $\overrightarrow{ED}$ is $\overrightarrow{CF}$.

iii) $\overrightarrow{BD} = \overrightarrow{FH}$ A vector opposite to $\overrightarrow{BD}$ is $\overrightarrow{HF}$.

iv) $\overrightarrow{FB} = \overrightarrow{GC} = \overrightarrow{HD} = \overrightarrow{EA}$. Vectors opposite to $\overrightarrow{FB}$ are $\overrightarrow{BF} = \overrightarrow{CG} = \overrightarrow{AE} = \overrightarrow{DH}$.

b) No. Vectors $\overrightarrow{AG}$ and $\overrightarrow{CE}$ have different directions, for example. $\overrightarrow{AG}$ goes from the front of the parallelepiped to the back, while $\overrightarrow{CE}$ goes from the back to the front, in clearly differing directions. They also have different lengths if the parallelepiped is not rectangular i.e., a cuboid.

Chapter 6 Section 1 **Question 18 Page 312**

C is the correct answer.

$$\begin{aligned}\overrightarrow{EU} - \overrightarrow{EP} - \overrightarrow{PU} &= \overrightarrow{EU} + \overrightarrow{PE} + \overrightarrow{UP} \\ &= \overrightarrow{PE} + \overrightarrow{EU} + \overrightarrow{UP} \\ &= \vec{0}\end{aligned}$$

Chapter 6 Section 1 **Question 19 Page 312**

(2, 8)

A variety of solutions are possible. For example:

Lemma 1*:* If X is the midpoint of PQ, then $\overrightarrow{OX} = \frac{1}{2}\overrightarrow{OP} + \frac{1}{2}\overrightarrow{OQ}$.

Proof:

$$\begin{aligned}\overrightarrow{OX} &= \overrightarrow{OP} + \frac{1}{2}\overrightarrow{PQ} \\ &= \overrightarrow{OP} + \frac{1}{2}\left(\overrightarrow{OQ} - \overrightarrow{OP}\right) \\ &= \overrightarrow{OP} + \frac{1}{2}\overrightarrow{OQ} - \frac{1}{2}\overrightarrow{OP} \\ &= \frac{1}{2}\overrightarrow{OP} + \frac{1}{2}\overrightarrow{OQ}\end{aligned}$$

Lemma 2: If F is the centroid of ΔABC, then $\overrightarrow{OF} = \frac{1}{3}\left(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}\right)$.

Proof: not given but uses only techniques similar to those in the proof of Lemma 1.

Using Lemma 2, let $F\left(x, y\right)$ be the third vertex of the triangle.

$$[3,4] = \frac{1}{3}([x,y] + [1,3] + [6,1])$$
$$= \frac{1}{3}[x+1+6, y+3+1]$$
$$= \left[\frac{x+7}{3}, \frac{y+4}{3}\right]$$

Therefore,

$3 = \frac{x+7}{3}$ and $4 = \frac{y+4}{3}$

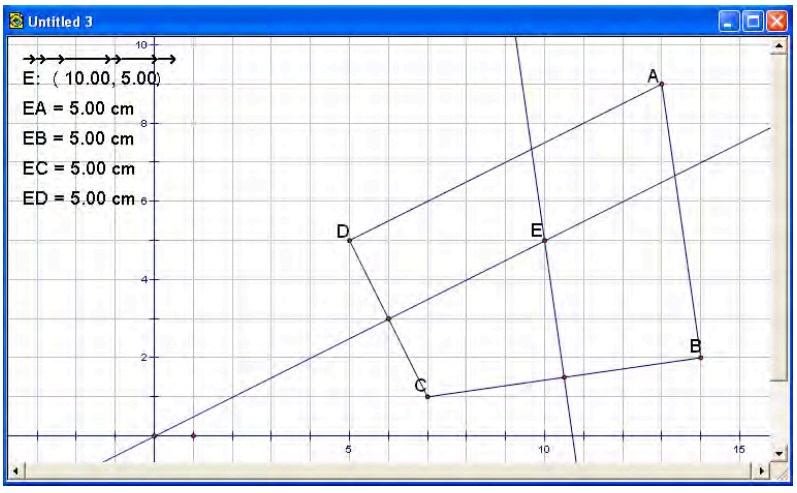
$x = 2$ $y = 8$

The third vertex is F(2, 8).

Chapter 6 Section 1 **Question 20 Page 312**

Method 1: Use *The Geometer's Sketchpad*®.

Plot the points.
Construct the midpoints of 2 sides.
Construct lines perpendicular to these sides and passing through the midpoints.
Construct the intersection point of these 2 lines. Label it E, a potential centre of the circle.
Measure EA, EB, EC, and ED.
If these four distances are equal, then A, B, C, and D lie on a circle with centre E.



Method 2: Use the property of cyclic quadrilaterals.

If a pair of opposite angles of a quadrilateral is supplementary, then the quadrilateral is cyclic.

Use the distance formula to find the lengths of all four sides of the quadrilateral and the two diagonals. Then use the cosine law to prove that any one pair of opposite angles is supplementary. (Note: if one pair is supplementary, then the other pair will be supplementary as well).

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\begin{aligned} |AD| &= \sqrt{(13-5)^2 + (9-5)^2} \\ &= \sqrt{64+16} \\ &= \sqrt{80} \end{aligned} \qquad \begin{aligned} |AB| &= \sqrt{(14-13)^2 + (2-9)^2} \\ &= \sqrt{1+49} \\ &= \sqrt{50} \end{aligned} \qquad \begin{aligned} |DB| &= \sqrt{(14-5)^2 + (2-5)^2} \\ &= \sqrt{81+9} \\ &= \sqrt{90} \end{aligned}$$

$$\begin{aligned} |CD| &= \sqrt{(5-7)^2 + (5-1)^2} \\ &= \sqrt{4+16} \\ &= \sqrt{20} \end{aligned} \qquad \begin{aligned} |CB| &= \sqrt{(14-7)^2 + (2-1)^2} \\ &= \sqrt{49+1} \\ &= \sqrt{50} \end{aligned}$$

$$\angle \mathrm{A} = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right)$$

$$\angle A = \cos^{-1}\left(\frac{80 + 50 - 90}{2\left(\sqrt{80}\right)\left(\sqrt{50}\right)}\right)$$

$$\doteq 71.6°$$

$$\angle C = \cos^{-1}\left(\frac{20 + 50 - 90}{2\left(\sqrt{20}\right)\left(\sqrt{50}\right)}\right)$$

$$\doteq 108.4°$$

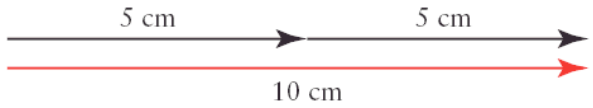
$$\begin{aligned} \angle \mathrm{A} + \angle \mathrm{C} &= 71.6° + 108.4° \\ &= 180° \end{aligned}$$

Therefore, one pair of opposite angles is supplementary, then the second pair of opposite angles is supplementary as well. Therefore, the quadrilateral is cyclic.
Using a similar approach to find ∠B and ∠D, these two angles are supplementary as well.

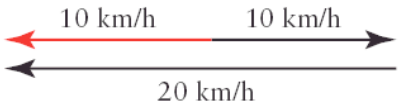
Chapter 6 Section 2: **Addition and Subtraction of Vectors**

Chapter 6 Section 2 **Question 1 Page 325**

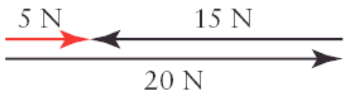
a)



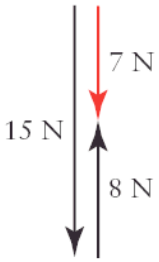
b)



c)



d)



Chapter 6 Section 2 **Question 2 Page 325**

a) First measure the length of the path.

$$d = 1+1+1.5+2+1.5+1+1+2.5+1.5+1.5+1+2.5+1.5+1+1+1.5$$
$$= 23$$

Since the scale is 1 cm : 3 m, the actual distance is 23×3 or 69 m.

The displacement is represented by a vector from the start (tail) of the first vector to the end (head) of the last vector. This measures 5 cm in length. Therefore, the actual displacement is 15 m to the right.

b) The distance and the displacement are different. The distance represents the total distance travelled, 69 m. The displacement represents the straight line distance between the initial and final positions, 15 m. Also, the displacement, a vector quantity, has a direction implied while the distance has no direction since it is a scalar quantity.

Chapter 6 Section 2 **Question 3 Page 325**

Answers may vary. For example:

a) The shortest vector is $\vec{w}$; $\vec{w} = \vec{u} - \vec{v}$

b) The shortest vector is $\overrightarrow{AB}$; $\overrightarrow{AB} = \overrightarrow{AC} - \overrightarrow{BC}$

c) The shortest vector is $\overrightarrow{RQ}$; $\overrightarrow{RQ} = \overrightarrow{PQ} - \overrightarrow{PR}$

d) The shortest vector is $\vec{e}$; $\vec{e} = \vec{v} - \vec{f}$

Chapter 6 Section 2 **Question 4 Page 325**

a) $\overrightarrow{AB}$

b) $\overrightarrow{BD}$ (parallelogram method)

c) $\overrightarrow{AC}$

d) $\overrightarrow{AC}$ (parallelogram method)

e) $\overrightarrow{BC}$

f) $\overrightarrow{AB} - \overrightarrow{DB} = \overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$

g) $\begin{aligned}\overrightarrow{AB} - \overrightarrow{CB} - \overrightarrow{DC} &= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} \\ &= \overrightarrow{AD}\end{aligned}$

h) The vectors form a triangle.

$\begin{aligned}\overrightarrow{AE} - \overrightarrow{EB} - \overrightarrow{BC} &= \overrightarrow{AE} + \overrightarrow{BE} + \overrightarrow{CB} \\ &= \overrightarrow{AE} + \overrightarrow{ED} + \overrightarrow{DA} \\ &= \vec{0}\end{aligned}$

Chapter 6 Section 2 **Question 5 Page 326**

Since the hexagon is regular, all sides are equal and all interior angles are equal. Consequently opposite sides are equal and parallel. Also, the interior triangles formed with the centre are equilateral triangles.

$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= \vec{b} - \vec{a}\end{aligned}$

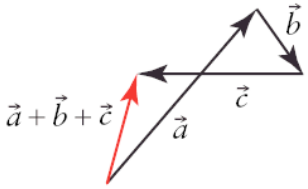
$$\overrightarrow{OC} = \overrightarrow{AB}$$
$$= \vec{b} - \vec{a}$$

$$\overrightarrow{CO} = -\overrightarrow{OC}$$
$$= -(\vec{b} - \vec{a})$$
$$= -\vec{b} + \vec{a}$$
$$= \vec{a} - \vec{b}$$

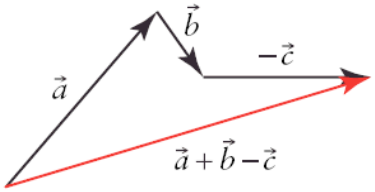
$$\overrightarrow{AE} = \overrightarrow{AO} + \overrightarrow{OE}$$
$$= -\overrightarrow{OA} - \overrightarrow{OB}$$
$$= -\vec{a} - \vec{b}$$

Chapter 6 Section 2 **Question 6 Page 326**

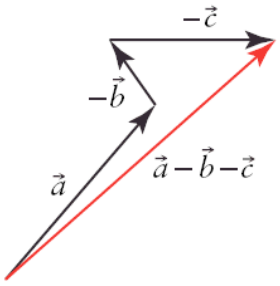
a)



b)



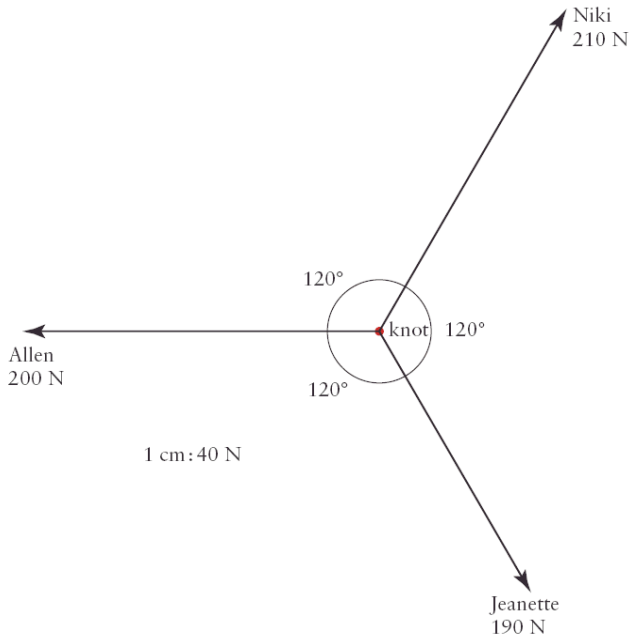
c)



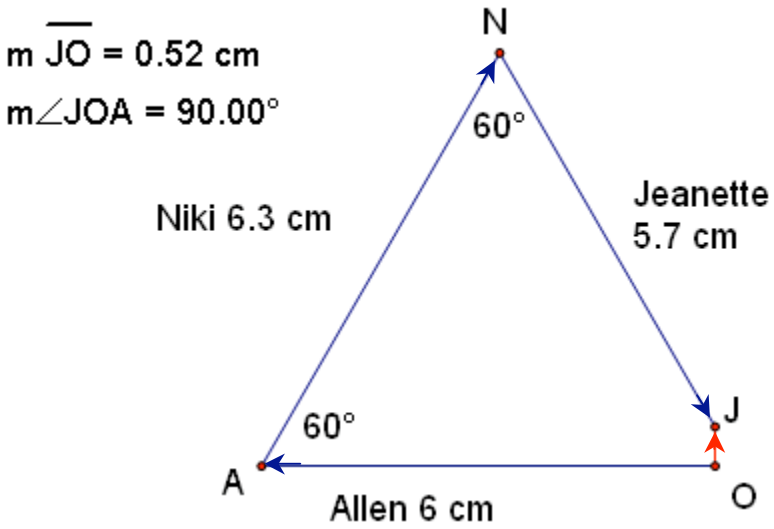
Chapter 6 Section 2 **Question 7 Page 326**

Answers may vary. For example:

a)



b)

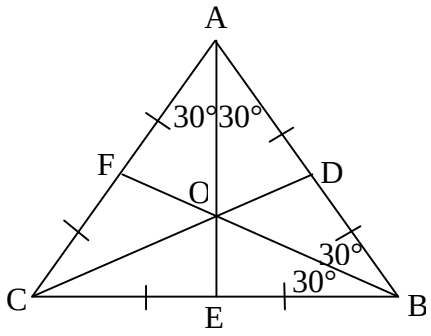


Use *The Geometer's Sketchpad*® to draw a scale diagram. Use a scale of 1 cm : 33.3 N. Let O represent the knot. The resultant vector is $\overrightarrow{OJ}$. The magnitude of the resultant force on the knot is 17.3 N. The direction of the resultant force on the knot is upward if Allen's force is directly to the left (000°).

It is also possible to solve this problem using trigonometry involving the sine law, solving triangle OAN first and then triangle NOJ.

Chapter 6 Section 2 **Question 8 Page 326**

a)



The centroid is the point where the three medians of a triangle meet.
In the equilateral triangle, ΔABC, with centroid, *O*, the medians are also the angle bisectors.
Therefore, in ΔABO, $\angle OBA = \angle OAB = 30°$. This is an isosceles triangle with $OA = OB$.
It is easy to show, in a similar way, that $OC = OB$. Therefore $OA = OB = OC$ and they meet each other at 120° angles.
Construct an equilateral triangle, ΔPQR, with $\overrightarrow{PQ} = \overrightarrow{OA}$, $\overrightarrow{QR} = \overrightarrow{OB}$, and $\overrightarrow{RP} = \overrightarrow{OC}$.
Since $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP} = \vec{0}$, therefore $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \vec{0}$.

b) Yes. $\overrightarrow{AO} + \overrightarrow{BO} + \overrightarrow{CO} = \vec{0}$.
From part a), $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \vec{0}$.

$$\begin{aligned}\overrightarrow{AO} + \overrightarrow{BO} + \overrightarrow{CO} &= -\overrightarrow{OA} - \overrightarrow{OB} - \overrightarrow{OC} \\ &= -\left(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}\right) \\ &= -\vec{0} \\ &= \vec{0}\end{aligned}$$

Chapter 6 Section 2 **Question 9 Page 326**

$$\begin{aligned}\overrightarrow{AH} &= \overrightarrow{AE} + \overrightarrow{AD} \\ &= \vec{v} + \vec{w}\end{aligned}$$

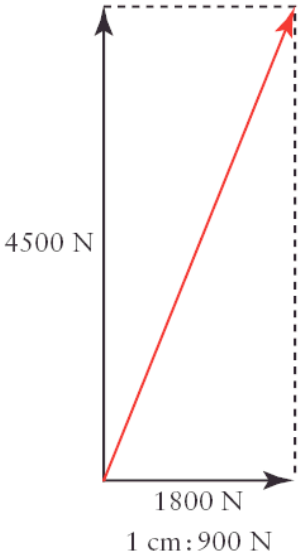
$$\begin{aligned}\overrightarrow{DG} &= \overrightarrow{DC} + \overrightarrow{DH} \\ &= \overrightarrow{AB} + \overrightarrow{AE} \\ &= \vec{w} + \vec{u}\end{aligned}$$

$$\begin{aligned}\overrightarrow{AG} &= \overrightarrow{AD} + \overrightarrow{DC} + \overrightarrow{CG} \\ &= \vec{v} + \vec{u} + \vec{w}\end{aligned}$$

$$\begin{aligned}\overrightarrow{CE} &= \overrightarrow{CB} + \overrightarrow{BA} + \overrightarrow{AE} \\ &= -\vec{u} - \vec{v} + \vec{w}\end{aligned}$$

$$\begin{aligned}\overrightarrow{BH} &= \overrightarrow{BA} + \overrightarrow{AD} + \overrightarrow{DH} \\ &= -\vec{u} + \vec{v} + \vec{w}\end{aligned}$$

Chapter 6 Section 2 **Question 10 Page 326**



Draw a scale diagram as shown. The resultant vector measures 5.4 cm and makes an angle of 68.2° with the horizontal.
Therefore, the force acting on the airplane wing: 4846.6 N, 68.2° above the horizontal.

Chapter 6 Section 2 **Question 11 Page 327**

a) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$
$= \overrightarrow{OB} + \overrightarrow{AO}$
$= \overrightarrow{OB} - \overrightarrow{OA}$

b) The proof is similar to part a).
$\overrightarrow{AB} = \overrightarrow{AX} + \overrightarrow{XB}$
$= \overrightarrow{XB} + \overrightarrow{AX}$
$= \overrightarrow{XB} - \overrightarrow{XA}$

Chapter 6 Section 2 **Question 12 Page 327**

a) Show that $\overrightarrow{AO}+\overrightarrow{BO}+\overrightarrow{CO}+\overrightarrow{DO}+\overrightarrow{EO}+\overrightarrow{FO}+\overrightarrow{GO}+\overrightarrow{HO}=\vec{0}$.

Since the hexagon is regular, all sides are equal and all segments to the centre are equal. In addition, all interior angles of the hexagon are equal (135°) as are the angles formed at the centre of the hexagon (45°).

Therefore, EOA is a straight line and $\overrightarrow{AO}=\overrightarrow{OE}$ and also $\overrightarrow{AO}+\overrightarrow{EO}=\vec{0}$.

$$\begin{aligned}&\overrightarrow{AO}+\overrightarrow{BO}+\overrightarrow{CO}+\overrightarrow{DO}+\overrightarrow{EO}+\overrightarrow{FO}+\overrightarrow{GO}+\overrightarrow{HO}\\&=\overrightarrow{AO}+\overrightarrow{EO}+\overrightarrow{BO}+\overrightarrow{FO}+\overrightarrow{CO}+\overrightarrow{GO}+\overrightarrow{DO}+\overrightarrow{HO}\\&=\vec{0}\end{aligned}$$

b) Yes. However, the above argument only works for regular polygons with an even number of sides. If the number of sides is odd, a different proof is needed. Suppose the polygon has *n* sides, where *n* is odd.

The segments AO, BO, CO, and so on, are all equal and make equal angles of $\frac{360°}{n}$ with each other.

If you create a vector sum of the segments, $\overrightarrow{AO}+\overrightarrow{BO}+\overrightarrow{CO}+\overrightarrow{DO}+\overrightarrow{EO}+\overrightarrow{FO}+\overrightarrow{GO}+\overrightarrow{HO}+\ldots$, placing the vectors head to tail, you will create a polygonal path.

You can see that the path is closed, since the vectors will form a regular polygon where each interior angle is $180°-\frac{360°}{n}$. Note that *n* of these angles add to $180°(n-2)$ which is the total of the interior angles of any *n*-gon. But this is a regular *n*-gon since all the segments are equal in length.

Therefore, $\overrightarrow{AO}+\overrightarrow{BO}+\overrightarrow{CO}+\overrightarrow{DO}+\overrightarrow{EO}+\overrightarrow{FO}+\overrightarrow{GO}+\overrightarrow{HO}+\ldots$ forms a complete new polygon and the vector sum is $\vec{0}$.

Chapter 6 Section 2 **Question 13 Page 327**

$$\begin{aligned}\overrightarrow{AG}&=\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DE}+\overrightarrow{EF}+\overrightarrow{FG}\\&=\vec{p}+\vec{q}+\vec{r}+\vec{s}+\vec{t}+\vec{u}\\&=\vec{r}+\vec{u}+\vec{t}+\vec{p}+\vec{q}+\vec{s}\qquad\text{commutative and associative properties of addition}\\&=\left(\vec{r}+\vec{u}\right)+\left(\vec{t}+\vec{p}\right)+\left(\vec{q}+\vec{s}\right)\end{aligned}$$

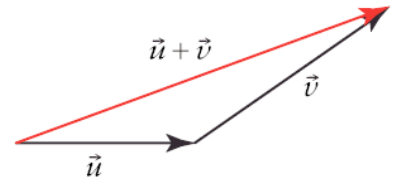
Chapter 6 Section 2 **Question 14 Page 327**

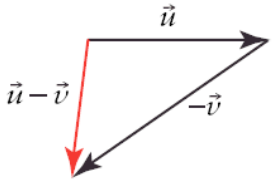
Solutions for Achievement Checks are shown in the Teacher Resource.

Chapter 6 Section 2 **Question 15 Page 327**

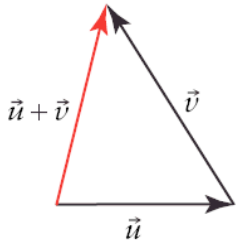
For all three parts, assume that $\vec{u}$ and $\vec{v}$ are not zero vectors.

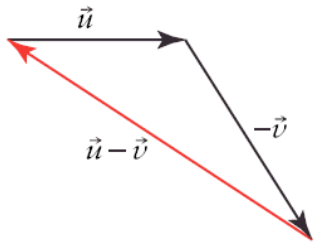
a) $|\vec{u}+\vec{v}| > |\vec{u}-\vec{v}|$ if the angle formed between the two vectors when they are placed head to tail is greater than 90°. If the angle between any two vectors is 180°, they lie on a straight line. The given inequality is satisfied in this case, as well. Therefore, the inequality holds true for $90° < \theta$.



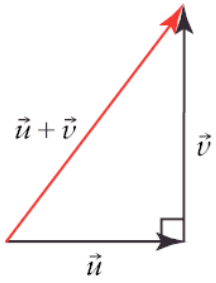


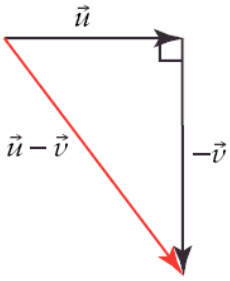
b) $|\vec{u}+\vec{v}| < |\vec{u}-\vec{v}|$ if the angle formed between the two vectors when they are placed head to tail is greater then 0°, but less than 90°, that is, $0° \leq \theta \leq 90°$.





c) $|\vec{u}+\vec{v}| = |\vec{u}-\vec{v}|$ if the angle formed between the two vectors when they are placed head to tail is equal to 90°, that is, $\theta = 90°$.

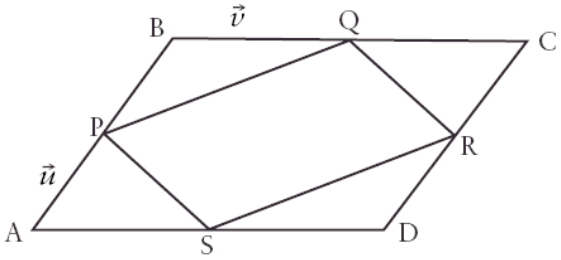




Chapter 6 Section 2 **Question 16 Page 327**

Answers may vary. For example:

In parallelogram ABCD ,



$\vec{u} = \overrightarrow{AP} = \overrightarrow{PB} = \overrightarrow{DR} = \overrightarrow{RC}$ and $\vec{v} = \overrightarrow{BQ} = \overrightarrow{QC} = \overrightarrow{AS} = \overrightarrow{SD}$.

A quadrilateral is a parallelogram if both pairs of opposite sides are parallel.
It is easy to show that a quadrilateral is a parallelogram if one pair of sides is equal and parallel.
Follow this approach.

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PB} + \overrightarrow{BQ} \\ &= \vec{u} + \vec{v} \\ &= \overrightarrow{DR} + \overrightarrow{SD} \\ &= \overrightarrow{SD} + \overrightarrow{DR} \\ &= \overrightarrow{SR}\end{aligned}$$

Since one pair of sides is equal and parallel, PQRS is a parallelogram. Note the efficiency of using vectors in that the proof proves equality of length and direction at the same time.

Chapter 6 Section 2 **Question 17 Page 327**

Vectors $\vec{u}, \vec{v}$, and $\vec{u}+\vec{v}$ form a triangle when $\vec{u}$ and $\vec{v}$ are placed head to tail.
In every triangle, the sum of the lengths of any two sides of the triangle is always greater than the length of the third side. This property is known as the triangle inequality.
Note that equality holds only when the two vectors have the same direction.

Chapter 6 Section 2 **Question 18 Page 327**

The correct answer is D.
A $\vec{a}=\vec{b}+\vec{c}$ means that $\vec{a}, \vec{b}$, and $\vec{c}$ form a triangle, but it is not necessary that $\vec{a}$ and $\vec{b}$ or $\vec{a}$ and $\vec{c}$ be collinear.
B The triangle formed by $\vec{a}, \vec{b}$, and $\vec{c}$ could be equilateral, which contradicts this statement.
C The triangle formed by $\vec{a}, \vec{b}$, and $\vec{c}$ could be equilateral, which contradicts this statement.
D True.

Chapter 6 Section 2 **Question 19 Page 327**

Two cubic functions can intersect in up to three points.

$$x^3+24x=-x^3+12x^2+16$$
$$2x^3-12x^2+24x-16=0$$
$$x^3-6x^2+12x-8=0$$

Use the factor theorem. Let $f(x)=x^3-6x^2+12x-8$.

$$f(1)=-1$$
$$f(-1)=-27$$
$$f(2)=0$$

Therefore, $(x-2)$ is a factor of $f(x)$.

Using long division, synthetic division, or trial and error,

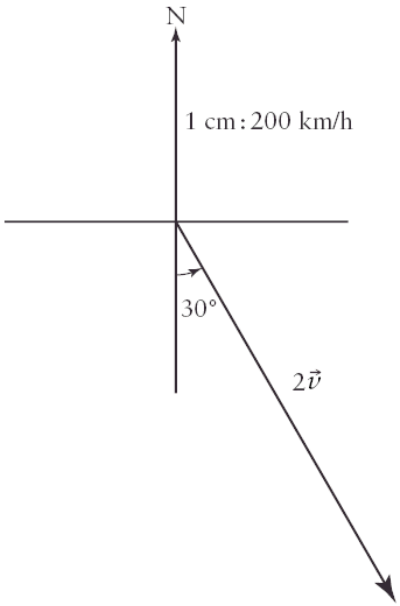
$$f(x)=x^3-6x^2+12x-8$$
$$=(x-2)(x^2-4x+4)$$

Factoring the trinomial, you get
$f(x)=(x-2)(x-2)(x-2)$
There is only one value for x, $x = 2$.
The intersection point is $(2, f(2))=(2,56)$.

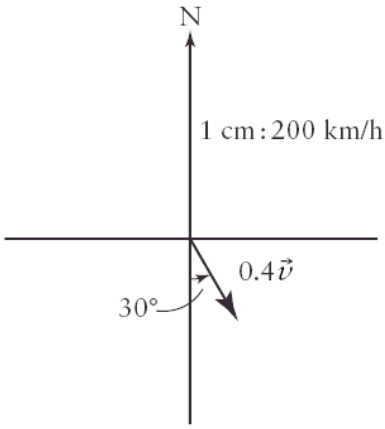
Chapter 6 Section 3 **Multiplying a Vector by a Scalar**

Chapter 6 Section 3 **Question 1 Page 334**

a)



b)



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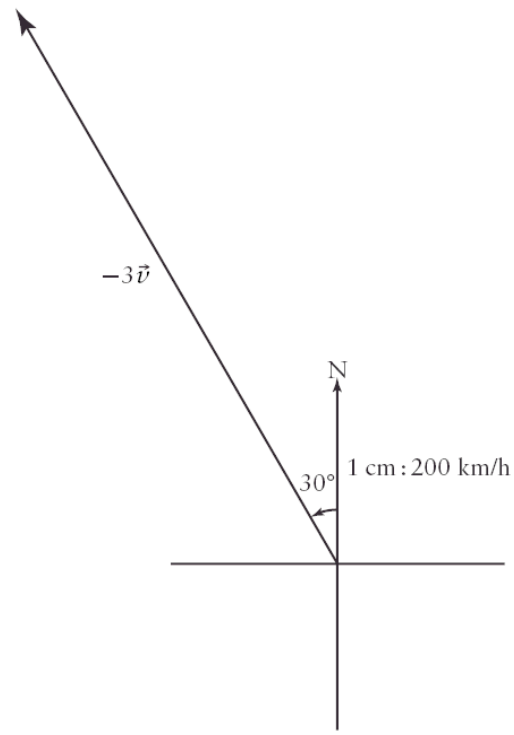
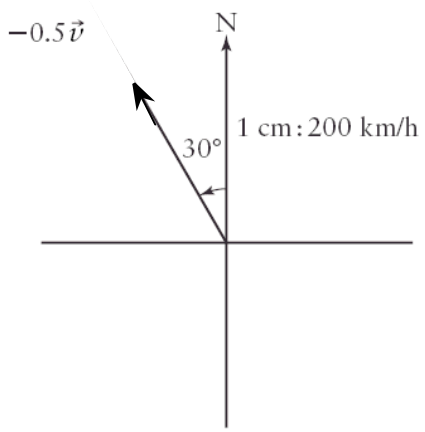


d)



Chapter 6 Section 3 **Question 2 Page 334**

a) $3\vec{u}$

b) $2\vec{u} - 3\vec{v} - 3\vec{u} + \vec{v}$
$= 2\vec{u} - 3\vec{u} - 3\vec{v} + \vec{v}$
$= -\vec{u} - 2\vec{v}$

c) $3\left(\vec{u} + \vec{v}\right) - 3\left(\vec{u} - \vec{v}\right)$
$= 3\vec{u} + 3\vec{v} - 3\vec{u} + 3\vec{v}$
$= 6\vec{v}$

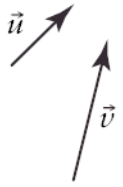
d) $3\vec{u} + 2\vec{v} - 2\left(\vec{v} - \vec{u}\right) + \left(-3\vec{v}\right)$
$= 3\vec{u} + 2\vec{v} - 2\vec{v} + 2\vec{u} - 3\vec{v}$
$= 5\vec{u} - 3\vec{v}$

e) $-\left(\vec{u} + \vec{v}\right) - 4\left(\vec{u} - 2\vec{v}\right)$
$= -\vec{u} - \vec{v} - 4\vec{u} + 8\vec{v}$
$= -5\vec{u} + 7\vec{v}$

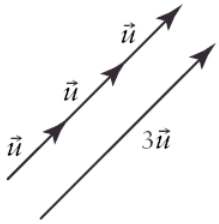
f) $2\left(\vec{u} + \vec{v}\right) - 2\left(\vec{u} + \vec{v}\right)$
$= 2\vec{u} + 2\vec{v} - 2\vec{u} - 2\vec{v}$
$= \vec{0}$

Chapter 6 Section 3 **Question 3 Page 334**

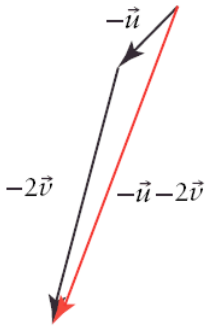
Answers may vary. For example if $\vec{u}$ and $\vec{v}$ are drawn as shown:



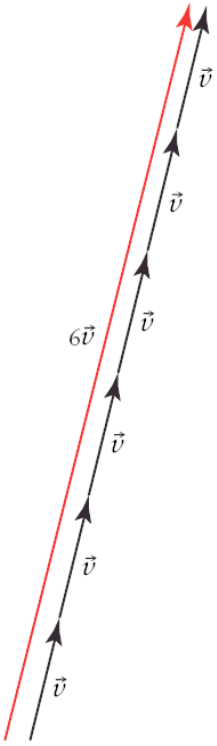
a)



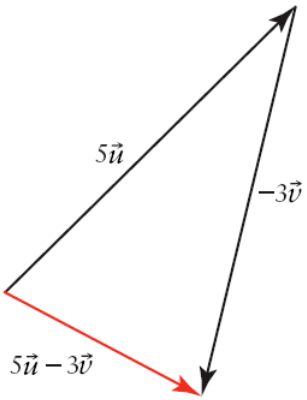
b)


$-\vec{u}$
$-2\vec{v}$
$-\vec{u}-2\vec{v}$

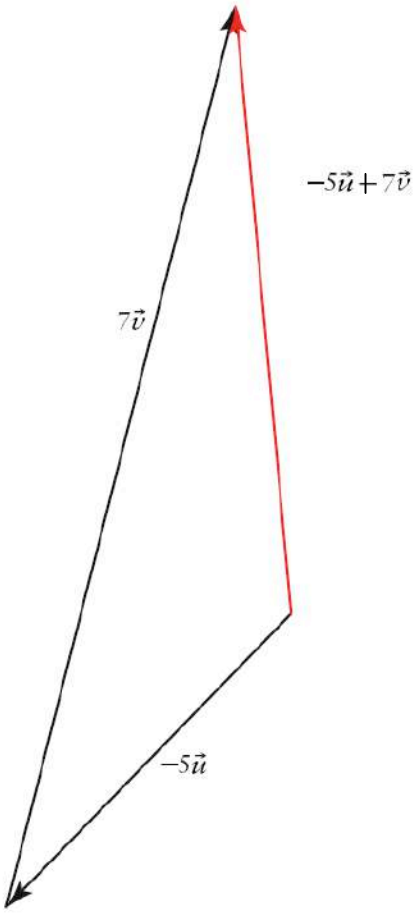
c)


$\vec{v}$
$\vec{v}$
$\vec{v}$
$6\vec{v}$
$\vec{v}$
$\vec{v}$
$\vec{v}$

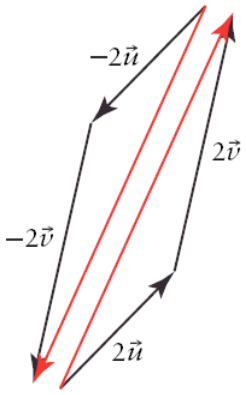
d)


$5\vec{u}$
$-3\vec{v}$
$5\vec{u}-3\vec{v}$

e)



f)



Chapter 6 Section 3 **Question 4 Page 334**

a) $\overrightarrow{CF} = -2\vec{u}$

b) $\overrightarrow{FB} = \vec{v} + \vec{u}$

c) $\overrightarrow{FD} = \overrightarrow{FC} + \overrightarrow{CD}$
$= 2\vec{u} - \vec{v}$

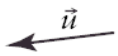
d) $\overrightarrow{CA} = \overrightarrow{CF} + \overrightarrow{FA}$
$= -2\vec{u} + \vec{v}$

e) $\overrightarrow{EB} = 2\vec{v}$

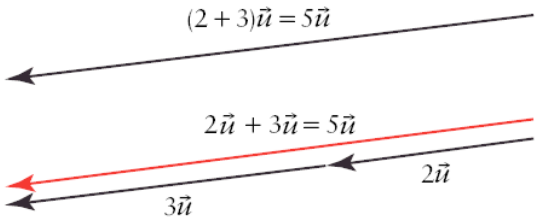
f) $\overrightarrow{BE} = -\overrightarrow{EB}$
$= -2\vec{v}$

Chapter 6 Section 3 **Question 5 Page 334**

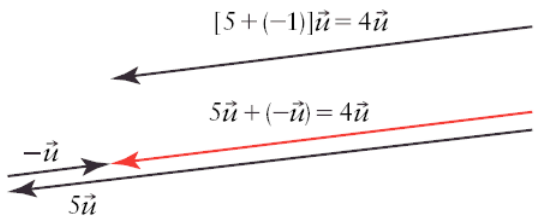
Answers may vary. For example, if this is $\vec{u}$,



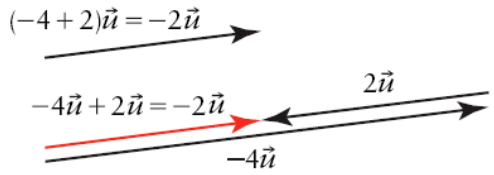
a)



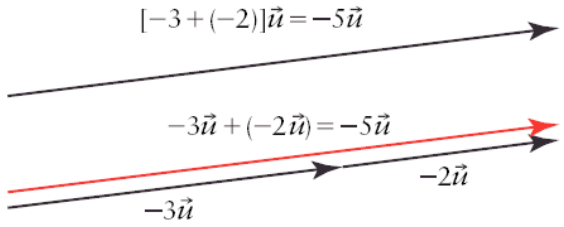
b)



c)



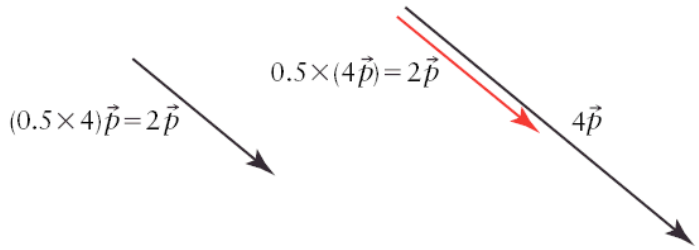
d)



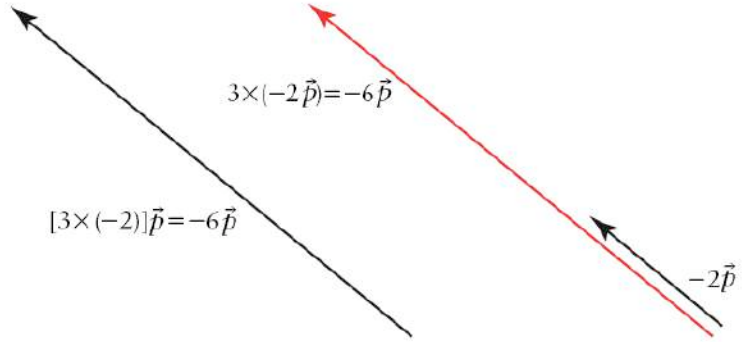
Chapter 6 Section 3 **Question 6 Page 334**

Answers may vary. For example, if this is $\vec{p}$,

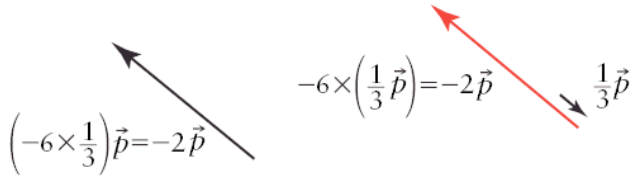
a)



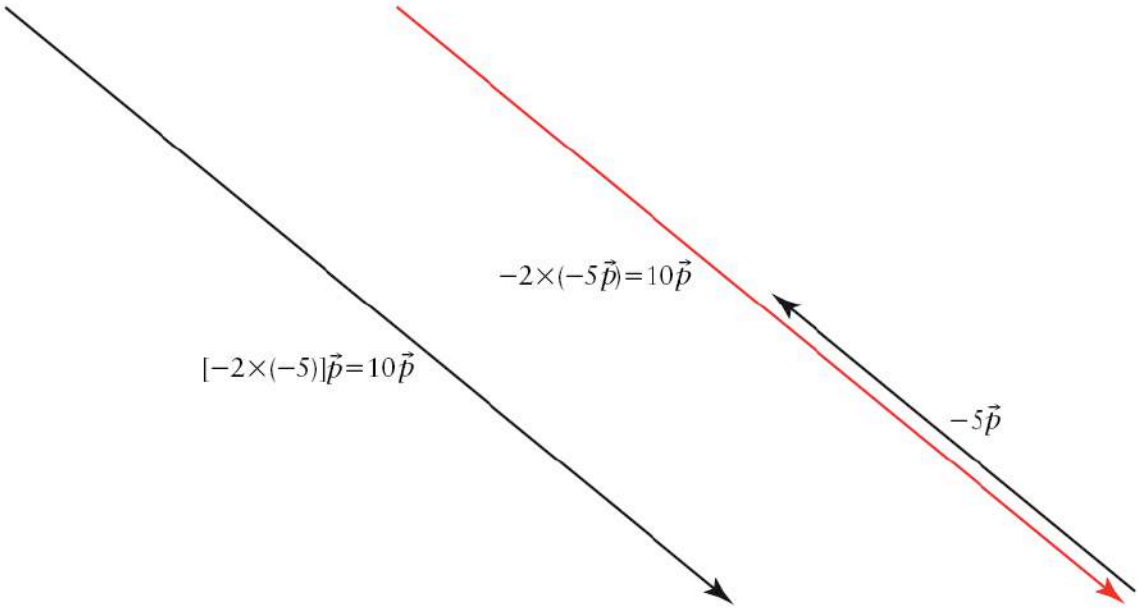
b)



c)

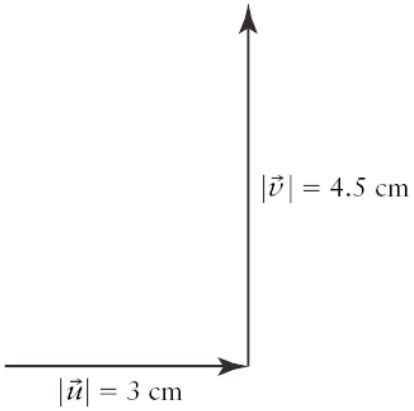


d)

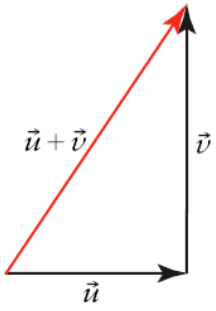


Chapter 6 Section 3 **Question 7 Page 334**

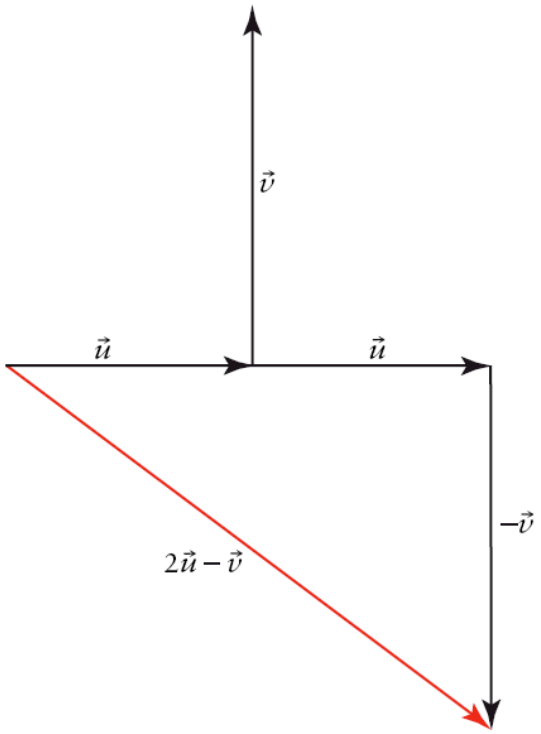
Answers may vary. For example, if this is $\vec{u}$ and $\vec{v}$,
(Note: diagrams are not exactly to scale.)



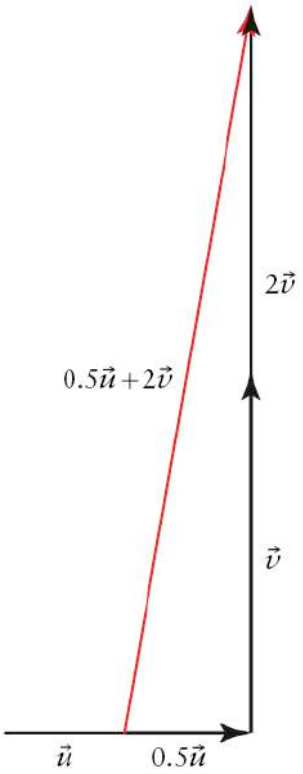
a)



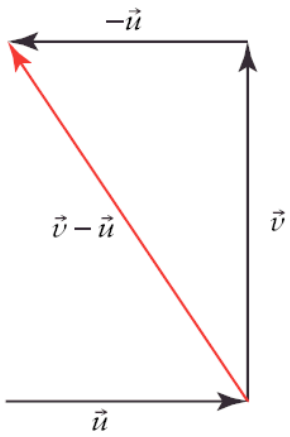
b)



c)


$2\vec{v}$
$0.5\vec{u}+2\vec{v}$
$\vec{v}$
$\vec{u}$
$0.5\vec{u}$

d)


$-\vec{u}$
$\vec{v}-\vec{u}$
$\vec{v}$
$\vec{u}$

Chapter 6 Section 3 **Question 8 Page 334**

The magnitude is 1.

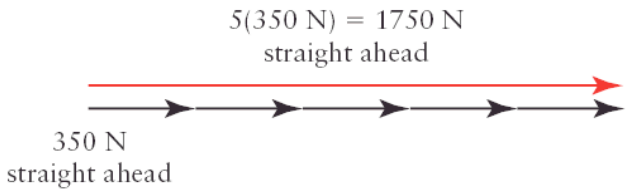
The magnitude of $\frac{1}{|\vec{v}|}\vec{v}$ is $\left|\frac{1}{|\vec{v}|}\vec{v}\right| = \left|\frac{1}{|\vec{v}|}\right| \times |\vec{v}|$

$= \frac{1}{|\vec{v}|} \times |\vec{v}|$

$= 1$

Chapter 6 Section 3 **Question 9 Page 335**

Answers may vary. For example: The five people are each pushing the disabled car along the road in the same direction with a force of 350 N straight ahead. The resulting vector could be found by adding the five forces of 350 N together, or by multiplying the force of 350 N by 5. The resulting force could be calculated by multiplying the scalar 5 by the force of 350 N as follows: 5(350) = 1750 N as shown in the diagram.



Chapter 6 Section 3 **Question 10 Page 335**

a) $\overrightarrow{F_E} = m \times \overrightarrow{g_E}$ or $\overrightarrow{F_E} = 9.8$ m (down)

b) $|\overrightarrow{W}| = 9.8(60)$

$= 588$

The force of gravity on a 60-kg person is 588 N downward.

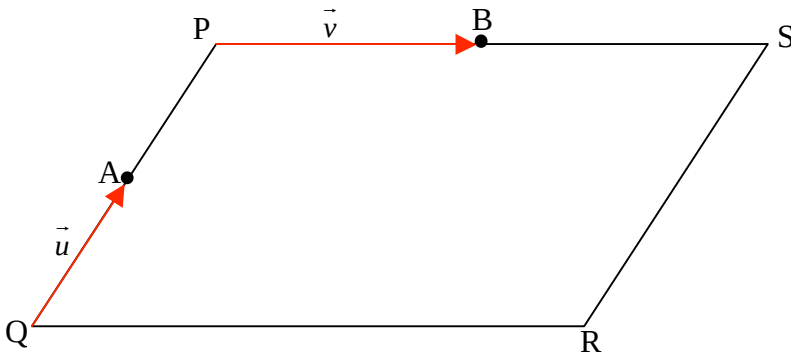
c) $\overrightarrow{F_M} = m \times \overrightarrow{g_M}$ or $\overrightarrow{F_M} = 1.63$ m (down)

d) $|\overrightarrow{W}| = 1.63(60)$

$= 97.8$

The force of gravity on a 60-kg person is 97.8 N downward.

Chapter 6 Section 3 **Question 11 Page 335**



a) $\overrightarrow{PS} = 2\vec{v}$

b) $\overrightarrow{AP} = \vec{u}$

c) $\overrightarrow{RS} = \overrightarrow{QP} = 2\vec{u}$

d) $\overrightarrow{AB} = \overrightarrow{AP} + \overrightarrow{PB}$
$= \vec{u} + \vec{v}$

e) $\overrightarrow{QS} = \overrightarrow{QP} + \overrightarrow{PS}$
$= 2\vec{u} + 2\vec{v}$

f) $\overrightarrow{AS} = \overrightarrow{AP} + \overrightarrow{PS}$
$= \vec{u} + 2\vec{v}$

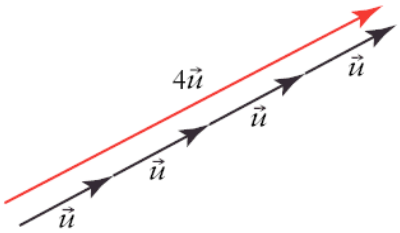
g) $\overrightarrow{BR} = \overrightarrow{BS} + \overrightarrow{SR}$
$= \vec{v} - 2\vec{u}$

h) $\overrightarrow{PR} = \overrightarrow{PS} + \overrightarrow{SR}$
$= 2\vec{v} - 2\vec{u}$

i) $\overrightarrow{RP} = -\overrightarrow{PR}$
$= -\left(2\vec{v} - 2\vec{u}\right)$
$= -2\vec{v} + 2\vec{u}$

Chapter 6 Section 3 **Question 12 Page 335**

Answers may vary. For example:



Chapter 6 Section 3 **Question 13 Page 335**

This can only be true if $k = 1$ when $\vec{u}$ is a non-zero vector.
If $\vec{u}$ is the zero vector, then any value of k will make the statement true; i.e., $k \in \mathbb{R}$.

Chapter 6 Section 3 **Question 14 Page 335**

Answers may vary. For example: The velocity of an airplane is 390 km/h at 15° above the horizontal indicates scalar multiplication by 3. Note that the direction does not change.

Chapter 6 Section 3 **Question 15 Page 335**

Answers may vary. For example:

a) Three people are pushing a box along the ground with a force of 10 N each, in the same direction. The total force is 30 N.

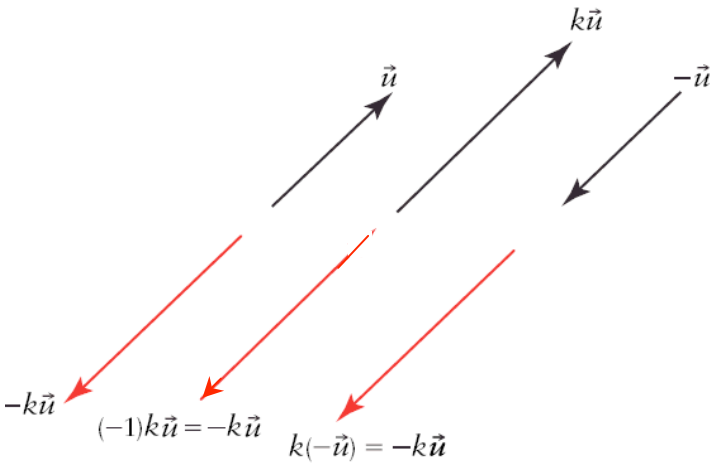
b) Mary's velocity on Highway 21 East was 80 km/h which was twice Doug's velocity on the same highway.

c) Acceleration due to gravity on Earth's surface is 9.8 m/s^2. Acceleration due to gravity on Planet X's surface, whose mass and radius is twice that of earth, is $\frac{2\times 9.8}{4}$ or 4.9 m/s^2.

d) Alana travelled 10 times farther, in the same direction, than Claire travelled.

Chapter 6 Section 3 **Question 16 Page 335**

Answers may vary. For example: $(-1)k\vec{u} = -k\vec{u}$ and $k\left(-\vec{u}\right) = -k\vec{u}$. Therefore, $(-1)k\vec{u} = k\left(-\vec{u}\right)$.



Chapter 6 Section 3 **Question 17 Page 335**

Since A, B, and C are collinear and B is the midpoint of AC, $\overrightarrow{BA} = \overrightarrow{BC}$

$$\begin{aligned}
\text{L.S.} &= \overrightarrow{OA} + \overrightarrow{OC} \\
&= \overrightarrow{OB} + \overrightarrow{BA} + \overrightarrow{OB} + \overrightarrow{BC} \\
&= \overrightarrow{OB} + \overrightarrow{OB} + \left(\overrightarrow{BA} + \overrightarrow{BC}\right) \\
&= 2\overrightarrow{OB} + \vec{0} \\
&= 2\overrightarrow{OB} \\
&= \text{R.S.}
\end{aligned}$$

Chapter 6 Section 3 **Question 18 Page 335**

a) Vectors $\vec{u}$ and $\vec{v}$ are in the same direction. Also $\left|2\vec{u}\right| = \left|3\vec{v}\right|$, so $\left|\vec{u}\right| = \frac{3}{2}\left|\vec{v}\right|$.

b) $\vec{u}$ and $\vec{v}$ are equal vectors since their difference is $\vec{0}$.

c) The vector equation simplifies to $\vec{u} + 5\vec{v} = \vec{u} + 5\vec{v}$. This equation is true for any vectors $\vec{u}$ and $\vec{v}$.

Chapter 6 Section 3 **Question 19 Page 335**

a) True. The vector equation simplifies to $7\overrightarrow{AA} = \vec{0}$ or $\overrightarrow{AA} = \vec{0}$. This is true statement for any point A.

b) Usually false. The vector equation simplifies to $-3\overrightarrow{AB} = -3\overrightarrow{BA}$ or $\overrightarrow{AB} = \overrightarrow{BA}$. This statement is usually false unless the points A and B are the same. In that case, both sides of the equation become $\vec{0}$ and the statement is true.

c) True. The vector equation simplifies to $-6\overrightarrow{BA} = 6\overrightarrow{AB}$ or $6\overrightarrow{AB} = 6\overrightarrow{AB}$. This is true statement for any points A and B.

d) True. The two vectors simplify to $2\overrightarrow{AB}$ and $3\overrightarrow{AB}$. These vectors are both positive scalar multiples of $\overrightarrow{AB}$ and so the original vectors are two collinear vectors pointing in the same direction.

e) True. Note that $-3\overrightarrow{BA} = 3\overrightarrow{AB}$. These vectors, $2\overrightarrow{AB}$ and $3\overrightarrow{AB}$, are both positive scalar multiples of $\overrightarrow{AB}$ and so they both point in the same direction.

Chapter 6 Section 3 **Question 20 Page 336**

Let O, A, B, and C be any four points such that $\overrightarrow{OA} + \overrightarrow{OC} = 2\overrightarrow{OB}$.

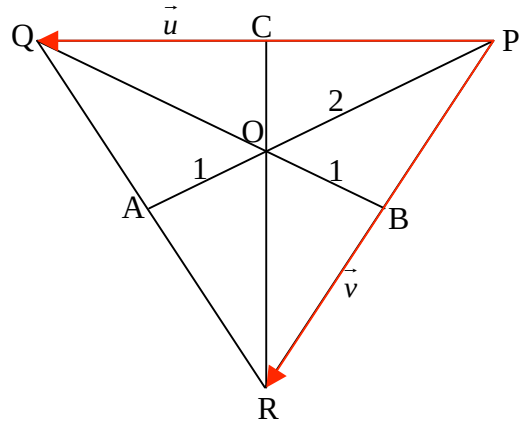
Recall that $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$ for any points O, P, and Q.

$$\overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{OB}$$
$$\overrightarrow{OA} - \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OC}$$
$$\overrightarrow{BA} = \overrightarrow{CB}$$

Since the vectors $\overrightarrow{BA}$ and $\overrightarrow{CB}$ are equal (in direction) and share a common point B, the points A, B, and C are collinear. Further, since the vectors are equal in length, B is the midpoint of AC.

Chapter 6 Section 3 **Question 21 Page 336**

The centroid, O, is the intersection point of the 3 medians of the triangle. Since this triangle is equilateral, the triangle is divided into 6 congruent smaller triangles that are the special 30°–60°–90° triangles. These triangles have sides in the ratio 1:2: $\sqrt{3}$. Therefore,



$$\begin{aligned}\overrightarrow{PO} &= \frac{2}{3}\overrightarrow{PA} \\ &= \frac{2}{3}\left(\overrightarrow{PQ} + \frac{1}{2}\overrightarrow{QR}\right) \\ &= \frac{2}{3}\left(\vec{u} + \frac{1}{2}\left(-\vec{u} + \vec{v}\right)\right) \\ &= \frac{1}{3}\vec{u} + \frac{1}{3}\vec{v}\end{aligned}$$

$$\begin{aligned}\overrightarrow{QO} &= \frac{2}{3}\overrightarrow{QB} \\ &= \frac{2}{3}\left(-\vec{u} + \frac{1}{2}\vec{v}\right) \\ &= -\frac{2}{3}\vec{u} + \frac{1}{3}\vec{v}\end{aligned}$$

$$\begin{aligned}\overrightarrow{RO} &= \frac{2}{3}\overrightarrow{RC} \\ &= \frac{2}{3}\left(-\vec{v} + \frac{1}{2}\vec{u}\right) \\ &= \frac{1}{3}\vec{u} - \frac{2}{3}\vec{v}\end{aligned}$$

Chapter 6 Section 3 **Question 22 Page 336**

a) The required point is C(−41, −7).

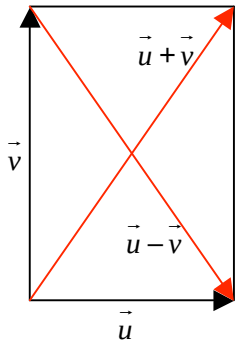
To get from point A to point B, go left 9 units and down 2 units. If $\overrightarrow{AC} = 5\overrightarrow{AB}$, multiply both the horizontal distance and the vertical distance by 5 units. Therefore, to get from point A to point C, go left 45 units and down 10 units. The coordinates of point C are C(4 − 45, 3 − 10) = C(−41, −7).

b) The required point is C(22, 7).

To get from point A to point B, go left 9 units and down 2 units. If $\overrightarrow{AC} = -2\overrightarrow{AB}$, multiply both the horizontal distance and the vertical distance by –2 units. Therefore, to get from point A to point C, go right 18 units and up 4 units. The coordinates of point C are C(4 + 18, 3 + 4) = C(22, 7).

Chapter 6 Section 3 **Question 23 Page 336**

Since the vectors are perpendicular to each other, use the Pythagorean theorem,



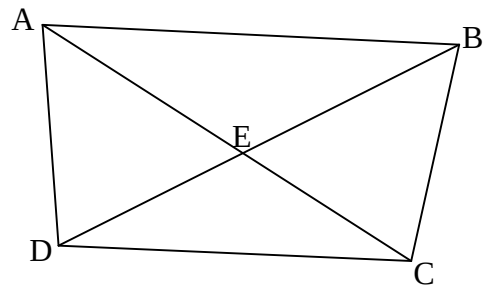
$$\left|\vec{u} + \vec{v}\right|^2 = \left|\vec{u}\right|^2 + \left|\vec{v}\right|^2$$

$$\left|\vec{u} - \vec{v}\right|^2 = \left|\vec{u}\right|^2 + \left|-\vec{v}\right|^2$$

Therefore,

$$\begin{aligned}\left|\vec{u} + \vec{v}\right|^2 + \left|\vec{u} - \vec{v}\right|^2 &= \left|\vec{u}\right|^2 + \left|\vec{v}\right|^2 + \left|\vec{u}\right|^2 + \left|\vec{v}\right|^2 \\ &= 2\left|\vec{u}\right|^2 + 2\left|\vec{v}\right|^2 \\ &= 2\left(\left|\vec{u}\right|^2 + \left|\vec{v}\right|^2\right)\end{aligned}$$

Chapter 6 Section 3 **Question 24 Page 336**

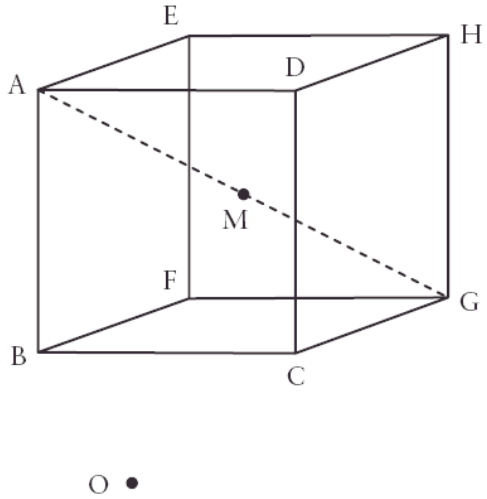


Let E be the point where the diagonals AC and BD of quadrilateral ABCD bisect each other.
Note that the diagram has been drawn to *not* look like a parallelogram. This helps to avoid making assumptions that are not given in the question.

From the given information, $\overrightarrow{AE} = \overrightarrow{EC}$, $\overrightarrow{EA} = \overrightarrow{CE}$, $\overrightarrow{BE} = \overrightarrow{ED}$, and $\overrightarrow{EB} = \overrightarrow{DE}$.
Therefore, $\overrightarrow{CD} = \overrightarrow{CE} + \overrightarrow{ED}$
$= \overrightarrow{EA} + \overrightarrow{BE}$
$= \overrightarrow{BA}$
Since one pair of opposite sides is equal and parallel, the quadrilateral ABCD is a parallelogram.
(Note that it is an easy vector proof to show that in a quadrilateral, if one pair of sides is equal and parallel, then the other pair of sides is equal and parallel.)

Chapter 6 Section 3 **Question 25 Page 336**

a) Let ABCDEFGH be a cube and let O be any point.



There are four diagonals to the cube: AG, BH, CE, and DF.
It appears as though they all intersect at their midpoints.
Let M be the midpoint of one diagonal, say AG.

$$\overrightarrow{OM} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AG}$$
$$= \overrightarrow{OA} + \frac{1}{2}\left(\overrightarrow{OG} - \overrightarrow{OA}\right)$$
$$= \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OG}$$

But, $\overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{BA}$ and $\overrightarrow{OG} = \overrightarrow{OH} + \overrightarrow{HG}$.

Re-write the expression for $\overrightarrow{OM}$ as follows:

$$\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OG}$$
$$= \frac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{BA} + \overrightarrow{OH} + \overrightarrow{HG}\right)$$
$$= \frac{1}{2}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{OH} \qquad \left(\overrightarrow{BA} + \overrightarrow{HG} = \vec{0}\right)$$

Examine the last expression carefully. It is the vector from O to the midpoint of BH.

Therefore $\overrightarrow{OM}$ goes to the midpoint of both AG and BH.

Similarly, $\overrightarrow{OM}$ goes to the midpoint of all diagonals.

b) Choose two opposite edges, say AB and HG. Let X and Y be the midpoints of AB and HG respectively. Let Z be the midpoint of XY.

$$\overrightarrow{OZ} = \overrightarrow{OX} + \frac{1}{2}\overrightarrow{XY}$$
$$= \overrightarrow{OX} + \frac{1}{2}\left(\overrightarrow{OY} - \overrightarrow{OX}\right)$$
$$= \frac{1}{2}\overrightarrow{OX} + \frac{1}{2}\overrightarrow{OY}$$

In a similar way, it can be shown that $\overrightarrow{OX} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB}$ and $\overrightarrow{OY} = \frac{1}{2}\overrightarrow{OH} + \frac{1}{2}\overrightarrow{OG}$.

Therefore, $\overrightarrow{OZ} = \frac{1}{2}\left(\frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB}\right) + \frac{1}{2}\left(\frac{1}{2}\overrightarrow{OH} + \frac{1}{2}\overrightarrow{OG}\right)$

$$= \frac{1}{2}\left(\frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OG} + \frac{1}{2}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{OH}\right)$$
$$= \frac{1}{2}\left(\overrightarrow{OM} + \overrightarrow{OM}\right) \qquad \text{(Proved in part a))}$$
$$= \frac{1}{2}\left(2\overrightarrow{OM}\right)$$
$$= \overrightarrow{OM}$$

Therefore Z and M are the same point. The line joining the midpoints of any two opposite edges of the cube has the same midpoint as the midpoint of any (space) diagonal of the cube.

Chapter 6 Section 3 **Question 26 Page 336**

Method 1:

Start with the given information.

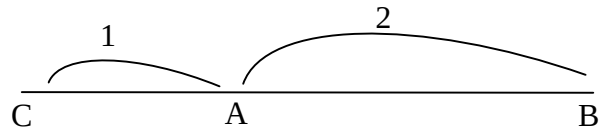
$$\begin{aligned}\overrightarrow{OA} &= \frac{1}{3}\overrightarrow{OB} + \frac{2}{3}\overrightarrow{OC} \\ &= \frac{1}{3}\overrightarrow{OB} + \overrightarrow{OC} - \frac{1}{3}\overrightarrow{OC} \\ \overrightarrow{OA} - \overrightarrow{OC} &= \frac{1}{3}\overrightarrow{OB} - \frac{1}{3}\overrightarrow{OC} \\ \overrightarrow{CA} &= \frac{1}{3}\overrightarrow{CB}\end{aligned}$$

The vectors $\overrightarrow{CA}$ and $\frac{1}{3}\overrightarrow{CB}$ are parallel vectors, both passing through point C.

Therefore points C, A, and B are collinear.

Also $\left|\overrightarrow{CA}\right| = \frac{1}{3}\left|\overrightarrow{CB}\right|$. This can be written as $\frac{\left|\overrightarrow{CA}\right|}{\left|\overrightarrow{CB}\right|} = \frac{1}{3}$.

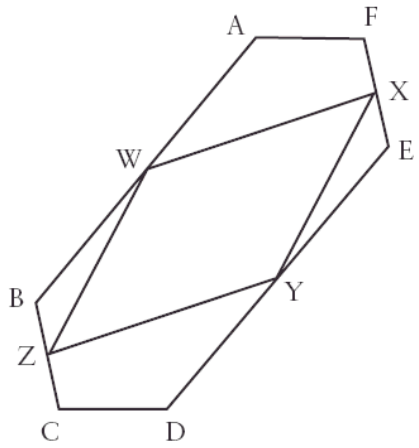
A is $\frac{1}{3}$ of the way along CB; or in other words, A divides CB in the ratio 1:2.



Method 2:

$$\begin{aligned}\left|\overrightarrow{CA}\right| &= \frac{1}{3}\left|\overrightarrow{CB}\right| \\ \left|\overrightarrow{CA}\right| &= \frac{1}{3}\left|\overrightarrow{CA} + \overrightarrow{CB}\right| \\ \left|\overrightarrow{CA}\right| &= \frac{1}{3}\left|\overrightarrow{CA}\right| + \frac{1}{3}\left|\overrightarrow{CB}\right| \\ \frac{2}{3}\left|\overrightarrow{CA}\right| &= \frac{1}{3}\left|\overrightarrow{AB}\right| \\ 2\left|\overrightarrow{CA}\right| &= \left|\overrightarrow{AB}\right| \\ \frac{\left|\overrightarrow{AB}\right|}{\left|\overrightarrow{CA}\right|} &= \frac{2}{1}\end{aligned}$$

Chapter 6 Section 3 **Question 27 Page 336**



Method 1:
Let W, X, Y and Z be midpoints of sides AB, BC, CD, DE, EF and FA in hexagon ABCDEF as shown.

Since opposite sides in ABCDEF are equal and parallel,
$\overrightarrow{AW} = \overrightarrow{WB} = \overrightarrow{EY} = \overrightarrow{YD}$

Consider one side of quadrilateral WXYZ.

$$\begin{aligned}\overrightarrow{WX} &= \overrightarrow{WB} + \overrightarrow{BE} + \overrightarrow{EX} \\ &= \frac{1}{2}\overrightarrow{AB} + \overrightarrow{BZ} + \overrightarrow{ZY} + \overrightarrow{YE} + \overrightarrow{EX} \\ &= \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} + \overrightarrow{ZY} + \frac{1}{2}\overrightarrow{DE} + \frac{1}{2}\overrightarrow{EF} \\ &= \left(\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{DE}\right) + \left(\frac{1}{2}\overrightarrow{BC} + \frac{1}{2}\overrightarrow{EF}\right) + \overrightarrow{ZY} \\ &= \vec{0} + \vec{0} + \overrightarrow{ZY} \\ &= \overrightarrow{ZY}\end{aligned}$$

Two opposite sides of the quadrilateral are equal and parallel.
Therefore, WXYZ is a parallelogram.

Method 2:
To prove that the quadrilateral formed by these four midpoints is a parallelogram, we need to show that one pair of opposite sides is equal and parallel.

Since opposite sides of the hexagon are equal and parallel and W, X, Y, and Z are midpoints of $\overrightarrow{BA}, \overrightarrow{FE}, \overrightarrow{ED}$, and $\overrightarrow{CB}$, then $\overrightarrow{BW} = \overrightarrow{WA} = \overrightarrow{DY} = \overrightarrow{YE}$ and $\overrightarrow{CZ} = \overrightarrow{ZB} = \overrightarrow{EX} = \overrightarrow{XF}$.

Now consider, $\overrightarrow{ZW}$.

$$\begin{aligned}\overrightarrow{ZW} &= \overrightarrow{ZB} + \overrightarrow{BW} \\ &= \overrightarrow{EX} + \overrightarrow{YE} \\ &= \overrightarrow{YX}\end{aligned}$$

Two opposite sides of the quadrilateral WXYZ are equal and parallel. Therefore, WXYZ is a parallelogram.

Chapter 6 Section 3 **Question 28 Page 336**

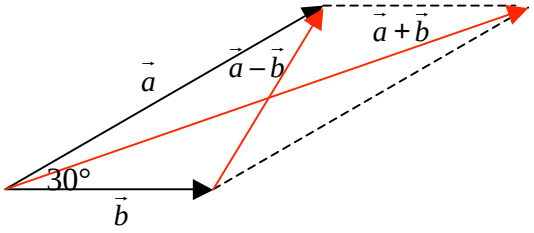
The correct answer is B.

a) $\vec{a}, \vec{b},$ and $\vec{a}+\vec{b}$ form a triangle for vector addition. The length of one side of a triangle is less than the sum of the lengths of the other two sides.

b) $\vec{a}-\vec{b}$ and $\vec{a}+\vec{b}$ are the diagonals of the parallelogram formed by $\vec{a}$ and $\vec{b}$.
Since the angle between the vectors is less than 90°, $\vec{a}+\vec{b}$ is the longer diagonal.

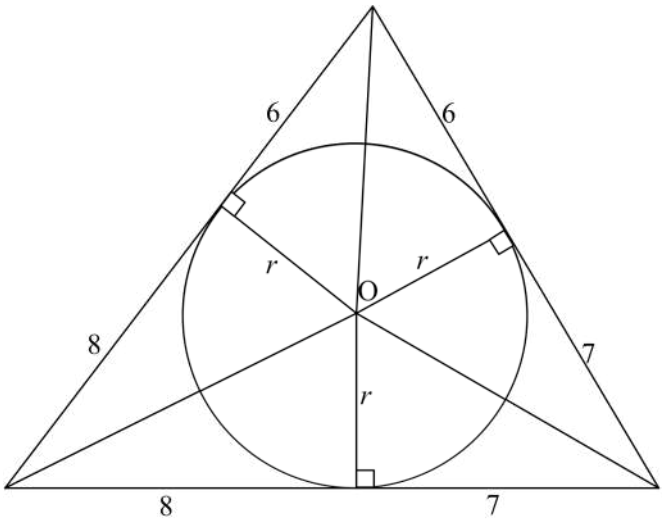
c) See part b).

d) See part b).



Chapter 6 Section 3 **Question 29 Page 336**

Let the radius be r.
Join the centre of the circle to the points where the circle touches the triangle.
The radii will meet the sides of the triangle at right angles.



You can calculate the area of the triangle in two different ways.

Method 1:
Consider the three triangles formed by joining the centre, O, to the three vertices. Each of these triangles has height r and base equal to one side of the triangle.

$$A = \frac{1}{2}15r + \frac{1}{2}13r + \frac{1}{2}14r$$
$$= 21r$$

Method 2:
Use Heron's formula.

If a triangle has sides of length a, b, and c and $s = \frac{a+b+c}{2}$ (the semi-perimeter), then

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

In this triangle, $s = \frac{13+14+15}{2}$

$$= 21$$

Thus, $A = \sqrt{21(21-13)(21-14)(21-15)}$
$$= \sqrt{21(8)(7)(6)}$$
$$= 84$$

Equating the two results, $21r = 84$, gives $r = 4$.

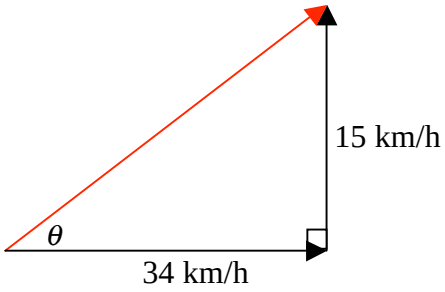
The radius of the circle is 4 cm.

Note: Method 2 could be replaced by using the cosine law of trigonometry to find one of the angles in the triangle. Then calculate the area by $A = ab\sin\theta$ where θ is the angle found and a and b are the two sides formed by the angle.

Chapter 6 Section 4 **Applications of Vector Addition**

Chapter 6 Section 4 **Question 1 Page 343**

a) Draw a diagram of the situation.



Since the velocities are perpendicular, use the Pythagorean theorem to find the magnitude of the resultant.

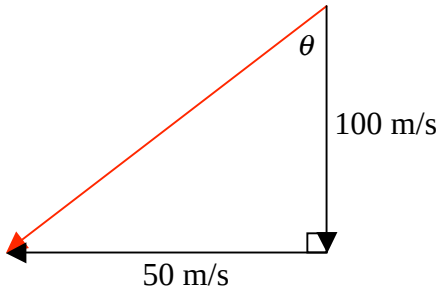
$$\left|\vec{R}\right|^2 = 34^2 + 15^2$$
$$= 1381$$
$$\left|\vec{R}\right| \doteq 37.2$$

To find the resultant velocity, use trigonometry. Let θ represent the angle of $\vec{R}$ to the east direction.

$$\tan\theta = \frac{15}{34}$$
$$\theta = \tan^{-1}\left(\frac{15}{34}\right)$$
$$\doteq 23.8^\circ$$

The resultant has a magnitude of about 37 km/h and a direction about 24° north of east (N66°E)

b) Draw a diagram of the situation.



Since the velocities are perpendicular, use the Pythagorean theorem to find the magnitude of the resultant.

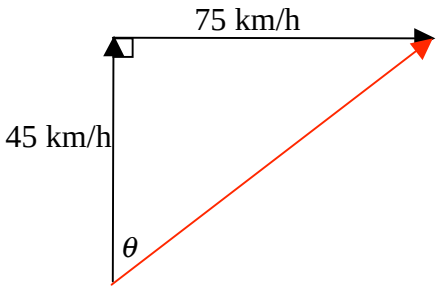
$$\left|\vec{R}\right|^2 = 100^2 + 50^2$$
$$= 12\,500$$
$$\left|\vec{R}\right| \doteq 111.8$$

To find the resultant velocity, use trigonometry. Let θ represent the angle of $\vec{R}$ to the south direction.

$$\tan\theta = \frac{50}{100}$$
$$\theta = \tan^{-1}\left(\frac{50}{100}\right)$$
$$\doteq 26.6^\circ$$

The resultant has a magnitude of about 112 km/h and a direction about 27° west of south (S27°W).

c) Draw a diagram of the situation.



Since the velocities are perpendicular, use the Pythagorean theorem to find the magnitude of the resultant.

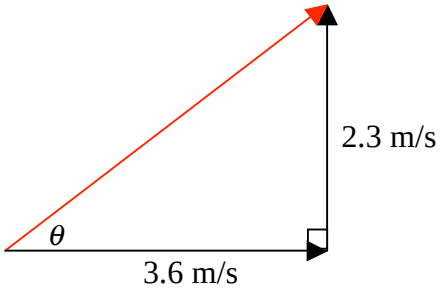
$$\left|\vec{R}\right|^2 = 45^2 + 75^2$$
$$= 7650$$
$$\left|\vec{R}\right| \doteq 87.5$$

To find the resultant velocity, use trigonometry. Let θ represent the angle of $\vec{R}$ to the vertical direction.

$$\tan\theta = \frac{75}{45}$$
$$\theta = \tan^{-1}\left(\frac{75}{45}\right)$$
$$\doteq 59.0^\circ$$

The resultant has a magnitude of about 88 km/h and a direction about 59° to the vertical direction.

d) Draw a diagram of the situation.



Since the velocities are perpendicular, use the Pythagorean theorem to find the magnitude of the resultant.

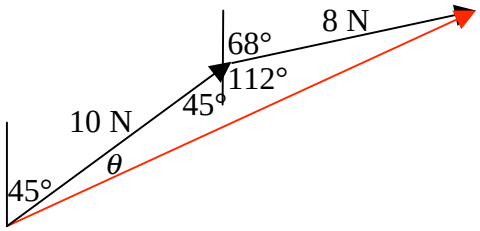
$$\left|\vec{R}\right|^2 = 3.6^2 + 2.3^2$$
$$= 18.25$$
$$\left|\vec{R}\right| \doteq 4.3$$

To find the resultant velocity, use trigonometry. Let θ represent the angle of $\vec{R}$ to the horizontal direction.

$$\tan\theta = \frac{2.3}{3.6}$$
$$\theta = \tan^{-1}\left(\frac{2.3}{3.6}\right)$$
$$\doteq 32.6^\circ$$

The resultant has a magnitude of about 4.3 km/h and a direction about 32.6° to the horizontal direction.

e) Draw a diagram of the situation.



Since the forces are not perpendicular, use the cosine law to find the magnitude of the resultant. The angle between the forces in the diagram is 157°.

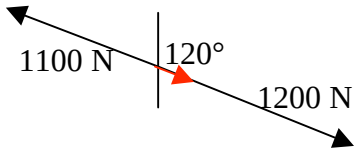
$$\left|\vec{R}\right|^2 = 10^2 + 8^2 - 2(10)(8)\cos 157^0$$
$$\doteq 311.28$$
$$\left|\vec{R}\right| \doteq 17.6$$

To find the resultant force, use the sine law. Let θ represent the angle of $\vec{R}$ to the 10 N force.

$$\frac{\sin\theta}{8} = \frac{\sin 157^o}{17.6}$$
$$\sin\theta = \frac{8\sin 157^o}{17.6}$$
$$\theta = \sin^{-1}\left(\frac{8\sin 157^o}{17.6}\right)$$
$$\doteq 10.2^o$$

The resultant force has a magnitude of about 18 N and a direction of about 055°.

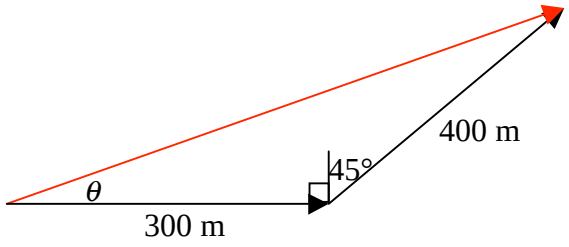
f) Draw a diagram of the situation.



The directions of the forces are opposite to each other.
To find the resultant force, simply subtract. The magnitude of the resultant is 100 N.
The direction of the resultant is the direction of the larger force.

The resultant force has a magnitude of 100 N and a direction of 120°.

g) Draw a diagram of the situation.



Use the cosine law to find the magnitude of the resultant displacement.
The angle between the displacements in the diagram is 135°.

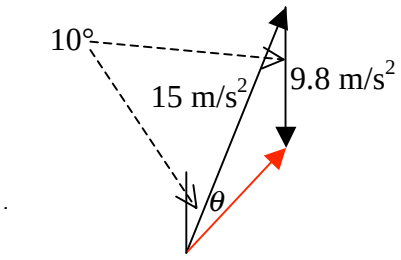
$$\left|\vec{R}\right|^2 = 300^2 + 400^2 - 2(300)(400)\cos 135^0$$
$$\doteq 419\,705$$
$$\left|\vec{R}\right| \doteq 647.8$$

To find the direction of the resultant, use the sine law. Let θ represent the angle of $\vec{R}$ to the east direction.

$$\frac{\sin\theta}{400} = \frac{\sin 135^o}{647.8}$$
$$\sin\theta = \frac{400\sin 135^o}{647.8}$$
$$\theta = \sin^{-1}\left(\frac{400\sin 135^o}{647.8}\right)$$
$$\doteq 25.9^o$$

The resultant displacement has a magnitude of about 648 m and a direction of about 25.9° north of east (N64.1°E).

h) Draw a diagram of the situation.



Use the cosine law to find the magnitude of the resultant acceleration.
The angle between the accelerations in the diagram is 10°.

$$\left|\vec{R}\right|^2 = 15^2 + 9.8^2 - 2(15)(9.8)\cos 10^0$$
$$\doteq 31.51$$
$$\left|\vec{R}\right| \doteq 5.6$$

To find the direction of the resultant, use the sine law. Let θ represent the angle of $\vec{R}$ to 15 m/s^2 acceleration.

$$\frac{\sin\theta}{9.8} = \frac{\sin 10^\circ}{5.6}$$

$$\sin\theta = \frac{9.8\sin 10^\circ}{5.6}$$

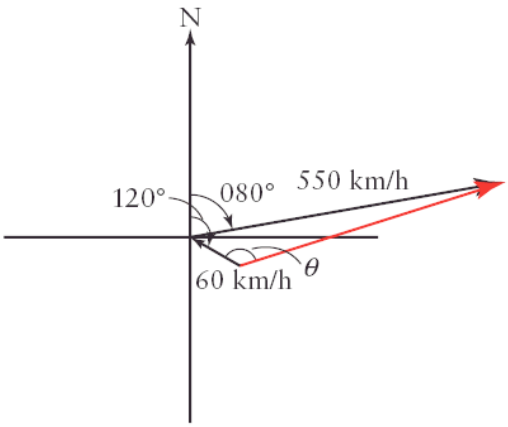
$$\theta = \sin^{-1}\left(\frac{9.8\sin 10^\circ}{5.6}\right)$$

$$\doteq 17.7^\circ$$

The resultant acceleration has a magnitude of about 5.6 m/s^2 and a direction of about 27.7° from vertical or 62.3° above the horizontal.

Chapter 6 Section 4 **Question 2 Page 343**

a) Draw a diagram of the situation.



b) Use the cosine law to find the magnitude of the ground velocity.
The angle between the velocities in the diagram is 40°.

$$\left|\vec{R}\right|^2 = 550^2 + 60^2 - 2(550)(60)\cos 40^0$$

$$\doteq 255\,541$$

$$\left|\vec{R}\right| \doteq 505.5$$

To find the direction of the ground velocity, use the sine law.
Let θ represent the angle between the wind vector and the ground velocity.

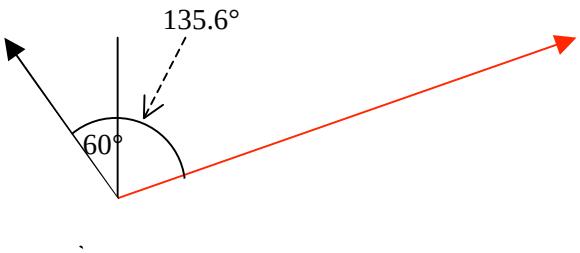
$$\frac{\sin\theta}{550} = \frac{\sin 40^\circ}{505.5}$$

$$\sin\theta = \frac{550\sin 40^\circ}{505.5}$$

$$\theta = \sin^{-1}\left(\frac{550\sin 40^\circ}{505.5}\right)$$

$$\doteq 135.6^\circ$$

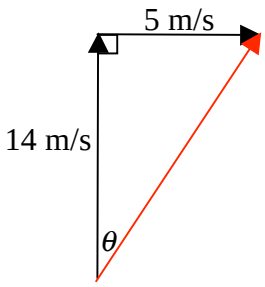
Note that the angle is obtuse, which is obvious from the diagram.
The angle between the wind vector and the north direction is 60° (180° – 120°).
Determine the angle between the ground velocity and the north direction.
To get this angle, calculate 135.6° – 60°.



The ground velocity has a magnitude of about 506 km/h and a direction of about 076°.

Chapter 6 Section 4 **Question 3 Page 343**

Draw a diagram of the situation.



a) Since the velocities are perpendicular, use the Pythagorean theorem to find the magnitude of the resultant velocity.

$$\begin{aligned} \left|\vec{R}\right|^2 &= 14^2 + 5^2 \\ &= 221 \\ \left|\vec{R}\right| &\doteq 14.9 \end{aligned}$$

The magnitude of the velocity of the boat relative to the shore is about 14.9 m/s.

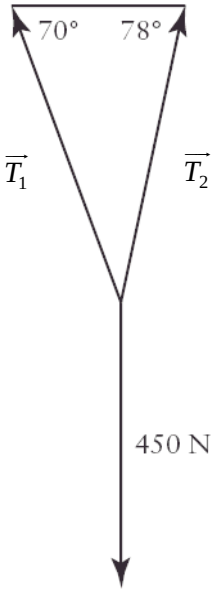
b) To find the direction of the resultant velocity, use trigonometry.
Let θ represent the angle of $\vec{R}$ to the boat's direction.

$$\begin{aligned} \tan\theta &= \frac{5}{14} \\ \theta &= \tan^{-1}\left(\frac{5}{14}\right) \\ &\doteq 19.7^\circ \end{aligned}$$

To find the angle relative to the shore, calculate 90° – 19.7° = 70.3°.
The direction of the boat's motion is about 70.3° relative to the shore.

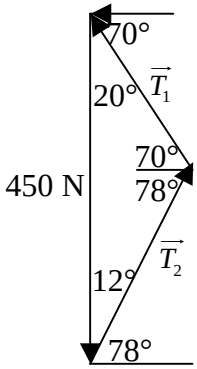
Chapter 6 Section 4 **Question 4 Page 343**

a) Draw the vector diagram.



b) It may be more useful to show the vectors forming an addition triangle since their sum must be the zero vector.

Label the tension vectors as $\vec{T}_1$ and $\vec{T}_2$ as shown.



To find $\vec{T}_1$, use the sine law.

$$\frac{\left|\vec{T}_1\right|}{\sin 12^\circ} = \frac{450}{\sin 148^\circ}$$

$$\left|\vec{T}_1\right| = \frac{450 \sin 12^\circ}{\sin 148^\circ}$$

$$\doteq 176.6$$

To find $\vec{T_2}$, use the sine law a second time.

$$\frac{|\vec{T_2}|}{\sin 20^\circ} = \frac{450}{\sin 148^\circ}$$

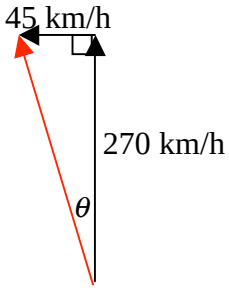
$$|\vec{T_2}| = \frac{450 \sin 20^\circ}{\sin 148^\circ}$$

$$\doteq 290.4$$

The tensions are about 176.6 N at 110° to the horizontal and about 290.4 N at 78° to the horizontal.

Chapter 6 Section 4 **Question 5 Page 343**

Draw a diagram of the situation.



a) Since the velocities are perpendicular, use the Pythagorean theorem to find the magnitude of the resultant velocity.

$$|\vec{R}|^2 = 270^2 + 45^2$$

$$= 74\,925$$

$$|\vec{R}| \doteq 273.72$$

To find the direction of the resultant velocity, use trigonometry.

Let θ represent the angle of $\vec{R}$ to the plane's heading.

$$\tan\theta = \frac{45}{270}$$

$$\theta = \tan^{-1}\left(\frac{45}{270}\right)$$

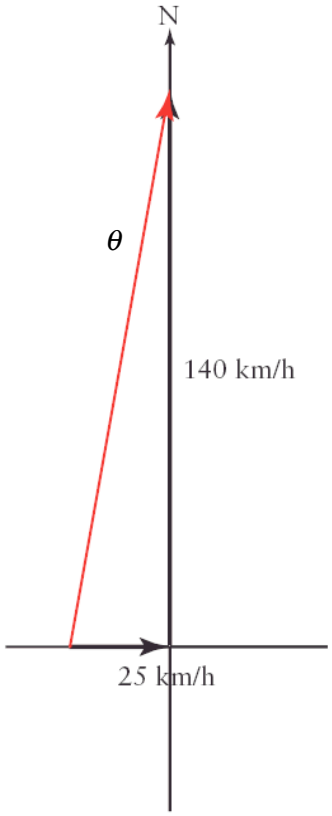
$$\doteq 9.46^\circ$$

The resultant ground velocity of the plane is about 273.7 km/h on a bearing of 350.5°.

b) Nancy is travelling at a speed of 273.7 km/h. In 2 h, she will travel 547.4 km, which is more than enough range to make her delivery on time.

Chapter 6 Section 4 **Question 6 Page 343**

a) Draw a diagram of the situation.



b) Since the velocities are perpendicular, use the Pythagorean theorem to find the magnitude of the resultant velocity.

$$\left|\vec{R}\right|^2 = 140^2 + 25^2$$
$$= 20\ 225$$
$$\left|\vec{R}\right| \doteq 142.21$$

To find the direction of the resultant velocity, use trigonometry.

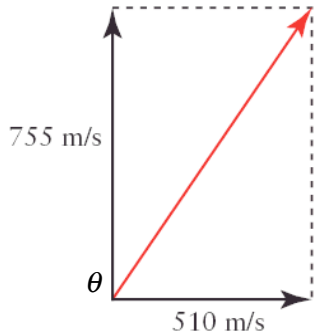
Let θ represent the angle of $\vec{R}$ to north as shown.

$$\tan\theta = \frac{25}{140}$$
$$\theta = \tan^{-1}\left(\frac{25}{140}\right)$$
$$\doteq 10.12^\circ$$

The resultant velocity of the ball is about 142.2 km/h on a bearing of 010.1° (N10.1°E).

Chapter 6 Section 4 **Question 7 Page 344**

a) Draw a diagram of the situation.



b) Since the velocities are perpendicular, use the Pythagorean theorem to find the magnitude of the resultant velocity.

$$\begin{aligned}\left|\vec{R}\right|^2 &= 755^2 + 510^2 \\ &= 830\,125 \\ \left|\vec{R}\right| &\doteq 911.11\end{aligned}$$

To find the direction of the resultant velocity, use trigonometry.

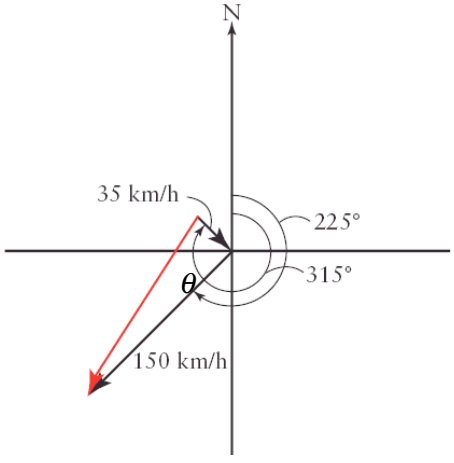
Let θ represent the angle of $\vec{R}$ to the horizontal as shown.

$$\begin{aligned}\tan\theta &= \frac{755}{510} \\ \theta &= \tan^{-1}\left(\frac{755}{510}\right) \\ &\doteq 55.96^\circ\end{aligned}$$

The resultant velocity of the rocket is about 911.1 m/s at angle of about 56° to the horizontal.

Chapter 6 Section 4 **Question 8 Page 344**

a) Draw a diagram of the situation.



Since 315° – 225° = 90°, the vector triangle is a right-angled triangle.
Use the Pythagorean theorem to find the magnitude of the resultant velocity.

$$\left|\vec{R}\right|^2 = 150^2 + 35^2$$
$$= 23\,725$$
$$\left|\vec{R}\right| \doteq 154.03$$

To find the direction of the resultant velocity, use trigonometry.
Let θ represent the angle of $\vec{R}$ to the wind vector, as shown.

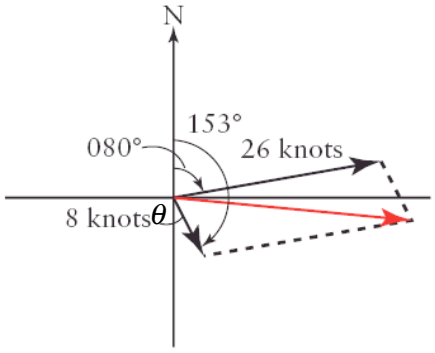
$$\tan\theta = \frac{150}{35}$$
$$\theta = \tan^{-1}\left(\frac{150}{35}\right)$$
$$\doteq 76.87^\circ$$

The direction of the wind is actually 135°.
The bearing of $\vec{R}$ then is 135° + 76.9° = 211.9°

The resultant velocity of the small aircraft is about 154.0 km/h on a bearing of 211.9°.

Chapter 6 Section 4 **Question 9 Page 344**

a)



b) The angle between the vectors when they are tail to tail is 153° – 80° = 73°.
Therefore the angle between them when they are head to tail is 180° – 73° = 107°.

Use the cosine law to find the magnitude of the ground velocity.

$$\left|\vec{R}\right|^2 = 26^2 + 8^2 - 2(26)(8)\cos 107^0$$
$$\doteq 861.63$$
$$\left|\vec{R}\right| \doteq 29.35$$

To find the direction of the ground velocity, use the sine law.
Let θ represent the angle between the current vector and the ground velocity, as shown.

$$\frac{\sin\theta}{26} = \frac{\sin 107^\circ}{29.35}$$

$$\sin\theta = \frac{26\sin 107^\circ}{29.35}$$

$$\theta = \sin^{-1}\left(\frac{26\sin 107^\circ}{29.35}\right)$$

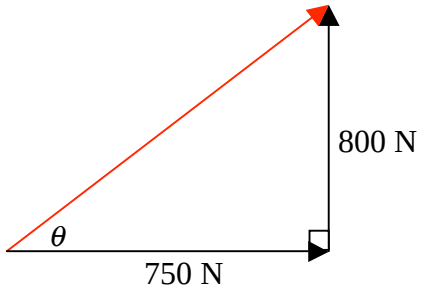
$$\doteq 57.90^\circ$$

To find the resultant bearing, start with the current bearing.
153° – 57.9° = 95.1°

The ground velocity of the cruise ship is 29.4 knots at a bearing of 095.1°.

Chapter 6 Section 4 **Question 10 Page 344**

a) Draw a diagram of the situation.



Since the forces are perpendicular, use the Pythagorean theorem to find the magnitude of the resultant force.

$$\left|\vec{R}\right|^2 = 750^2 + 800^2$$

$$= 1\ 202\ 500$$

$$\left|\vec{R}\right| \doteq 1096.59$$

To find the direction of the resultant force, use trigonometry.
Let θ represent the angle of $\vec{R}$ to the horizontal as shown.

$$\tan\theta = \frac{800}{750}$$

$$\theta = \tan^{-1}\left(\frac{800}{750}\right)$$

$$\doteq 46.847^\circ$$

The resultant force is about 1096.6 N at angle of 46.8° to the horizontal.

b) If the vertical force is doubled, the resultant will have a greater magnitude and will act at a steeper angle to the horizontal.
Specifically,

$$\left|\vec{R}\right|^2 = 750^2 + 1600^2$$
$$= 3\,122\,500$$
$$\left|\vec{R}\right| \doteq 1767.06$$

$$\tan\theta = \frac{1600}{750}$$
$$\theta = \tan^{-1}\left(\frac{1600}{750}\right)$$
$$\doteq 64.89^\circ$$

The resultant force will be about 1767.1 N at angle of 64.9° to the horizontal.

c) In this case, the resultant would be doubled in magnitude, but the angle would be unchanged.
Verifying,

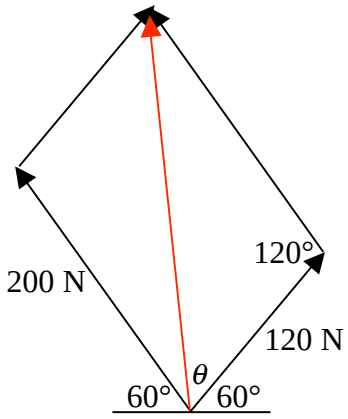
$$\left|\vec{R}\right|^2 = 150^2 + 1600^2$$
$$= 4\,810\,000$$
$$\left|\vec{R}\right| \doteq 2193.17$$

$$\tan\theta = \frac{1600}{750}$$
$$\theta = \tan^{-1}\left(\frac{1600}{1500}\right)$$
$$\doteq 46.847^\circ$$

The resultant force will be about 2193.2 N at angle of 46.7° to the horizontal.

Chapter 6 Section 4 **Question 11 Page 344**

a) Draw a diagram of the situation. Note that both forces make 60° angles with the line between the centres of the two goals but in different directions.



Since the bottom interior angle of the parallelogram in the diagram measures 60°, the angle between the two forces when they are head to tail is 120°.

Use the cosine law to find the magnitude of the resultant force.

$$\left|\vec{R}\right|^2 = 120^2 + 200^2 - 2(120)(200)\cos 120^0$$
$$\doteq 78\ 400$$
$$\left|\vec{R}\right| \doteq 280$$

To find the direction of the resultant force, use the sine law.
Let θ represent the angle between the resultant and the force of Emily's kick, as shown.

$$\frac{\sin\theta}{200} = \frac{\sin 120^o}{280}$$
$$\sin\theta = \frac{200\sin 120^o}{280}$$
$$\theta = \sin^{-1}\left(\frac{200\sin 120^o}{280}\right)$$
$$\doteq 38.21^o$$

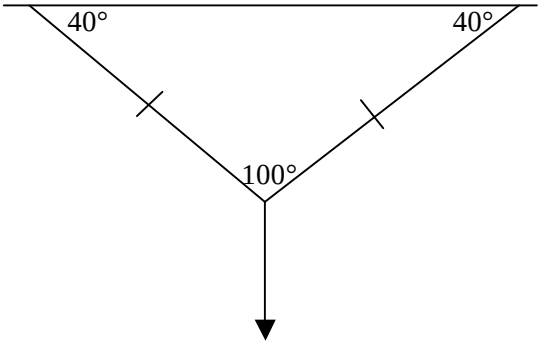
The magnitude and direction of the resultant force is 280 N in a direction making an angle of 98.2° with the line joining the centres of the goals.

Chapter 6 Section 4 **Question 12 Page 344**

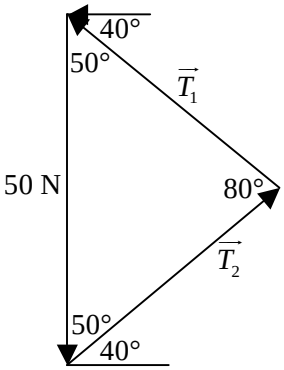
First draw a diagram of the physical situation. Note that the wires are of equal length and so the triangle is isosceles, and in particular the angles at the top are each 40°.

The tensions are in the direction of the wires, pointing upward. Label these tensions as $\vec{T_1}$ and $\vec{T_2}$.

Now draw a vector diagram, which is not the same as the physical diagram.



Physical diagram



Vector diagram

To find $\vec{T_1}$, use the sine law.

$$\frac{\left|\vec{T_1}\right|}{\sin 50^\circ} = \frac{50}{\sin 80^\circ}$$

$$\left|\vec{T_1}\right| = \frac{50 \sin 50^\circ}{\sin 80^\circ}$$

$$\doteq 38.89$$

To find $\vec{T_2}$, use the sine law a second time.

$$\frac{\left|\vec{T_2}\right|}{\sin 50^\circ} = \frac{50}{\sin 80^\circ}$$

$$\left|\vec{T_2}\right| = \frac{50 \sin 50^\circ}{\sin 80^\circ}$$

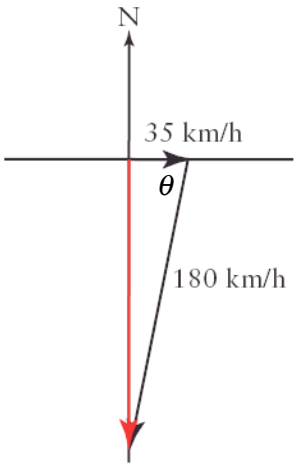
$$\doteq 38.89$$

Note that calculations are identical, this time.

The tensions are about 38.9 N at 40° above the horizontal.

Chapter 6 Section 4 **Question 13 Page 344**

a) First draw a diagram of the situation.



b) To find the heading that Tanner must adopt, use trigonometry.
Let θ represent the angle of the heading direction to the horizontal, as shown.

$$\cos\theta = \frac{35}{180}$$

$$\theta = \tan^{-1}\left(\frac{35}{180}\right)$$

$$\doteq 78.79^{\circ}$$

To translate this to a bearing, calculate 270° – 78.8° = 191.2°
Tanner needs a bearing of 191.2° to reach his destination.

c) To calculate the magnitude of Tanner's resultant velocity, use the Pythagorean theorem.

$$\left|\vec{R}\right|^2 = 180^2 - 35^2$$

$$= 31\,175$$

$$\left|\vec{R}\right| \doteq 176.56$$

Tanner will travel at a speed of 176.6 km/h.

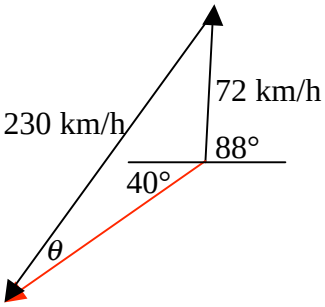
To travel 200 km, he needs $\frac{200}{176.6}$ or 1.33 h which is approximately 1 h 8 min.

He will arrive in sufficient time.

Because two of the vectors in this problem were acting at right angles to each other, you could use the Pythagorean theorem and simple trigonometric ratios to determine the necessary information.

Chapter 6 Section 4 **Question 14 Page 344**

a) First draw a diagram of the situation. The red vector is the resultant of her heading vector added to the wind vector. The largest angle in the vector triangle can be calculated to be 92° + 40° = 132°.



To find the heading when the airplane will be flying at 230 km/h, determine the size of angle θ using the sine law.

$$\frac{\sin\theta}{72} = \frac{\sin 132^\circ}{230}$$

$$\sin\theta = \frac{72\sin 132^\circ}{230}$$

$$\theta = \sin^{-1}\left(\frac{72\sin 132^\circ}{230}\right)$$

$$\doteq 13.45^\circ$$

The heading is 13.5° less than the resultant which has a bearing of 230°. Thus the heading in this case is 216.5°.

To find the heading when the airplane will be flying at 300 km/h, again determine the size of angle θ using the sine law.

$$\frac{\sin\theta}{72} = \frac{\sin 132^\circ}{300}$$

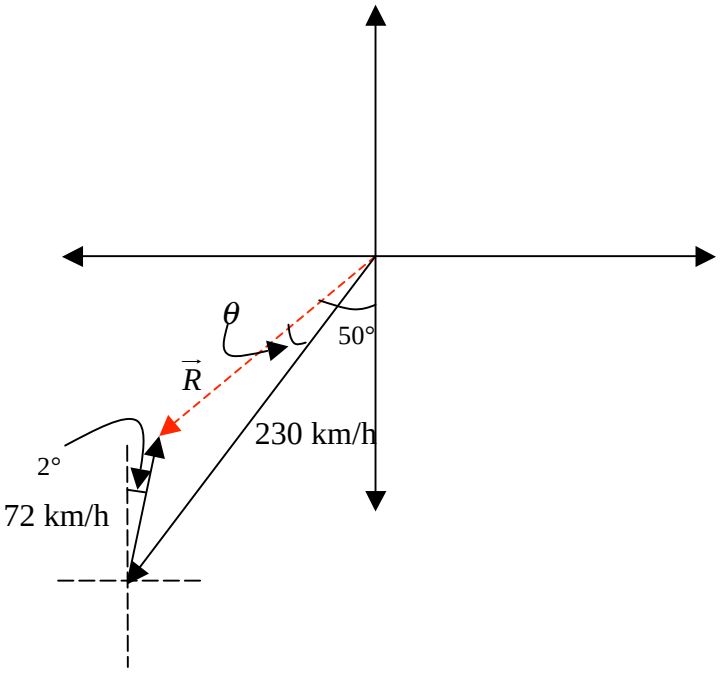
$$\sin\theta = \frac{72\sin 132^\circ}{300}$$

$$\theta = \sin^{-1}\left(\frac{72\sin 132^\circ}{300}\right)$$

$$\doteq 10.27^\circ$$

The heading is 10.3° less than the resultant which has a bearing of 230°. Thus the heading in this case is 219.7°.

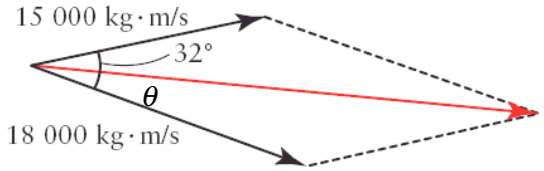
An alternative diagram for the problem is shown.



b) When the airplane flies faster, the effect of the wind on the heading is less and the heading is closer to the direction of the resultant vector. If the airplane could fly very fast, the heading would approach closer
to 230°.

Chapter 6 Section 4 **Question 15 Page 344**

a) First draw a diagram of the situation.



b) To find the magnitude of the resultant, use the cosine law in the lower triangle in the diagram. Note that the angle between the momentum vectors when they are laced head to tail is 180° – 32° = 148°.

$$\left|\vec{R}\right|^2 = 18\,000^2 + 15\,000^2 - 2(18\,000)(15\,000)\cos 148^0$$
$$\doteq 1\,006\,945\,972$$
$$\left|\vec{R}\right| \doteq 31\,732.4$$

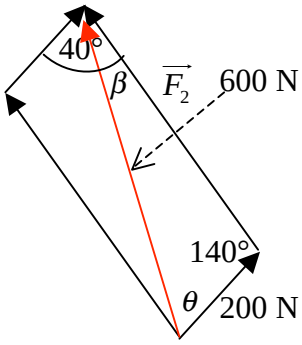
To find the direction of the resultant, use the sine law to determine angle θ.

$$\frac{\sin\theta}{15\,000} = \frac{\sin 148^o}{31\,732}$$
$$\sin\theta = \frac{15\,000\sin 148^o}{31\,732}$$
$$\theta = \sin^{-1}\left(\frac{15\,000\sin 148^o}{31\,732}\right)$$
$$\doteq 14.51^o$$

The resultant momentum is about 32 000 kg·m/s at angle of 14.5° to the 18 000 kg·m/s vector.

Chapter 6 Section 4 **Question 16 Page 345**

First draw a diagram of the situation. Note the larger angle in force parallelogram is 140°. Label interior angles as θ and β, as shown.



a) Use the sine law to find β.

$$\frac{\sin\beta}{200} = \frac{\sin 140^o}{600}$$
$$\sin\beta = \frac{200\sin 140^o}{600}$$
$$\beta = \sin^{-1}\left(\frac{200\sin 140^o}{600}\right)$$
$$\doteq 12.37^o$$

Since $\theta = 180^o - 12.37^o - 140^o = 27.63^o$, the resultant makes an angle of approximately 27.6° with the 200 N force.

b) Use the sine law a second time to determine the magnitude of the second force, $\overrightarrow{F_2}$.

$$\frac{\left|\overrightarrow{F_2}\right|}{\sin 27.63^\circ} = \frac{600}{\sin 140^\circ}$$

$$\left|\overrightarrow{F_2}\right| = \frac{600 \sin 27.63^\circ}{\sin 140^\circ}$$

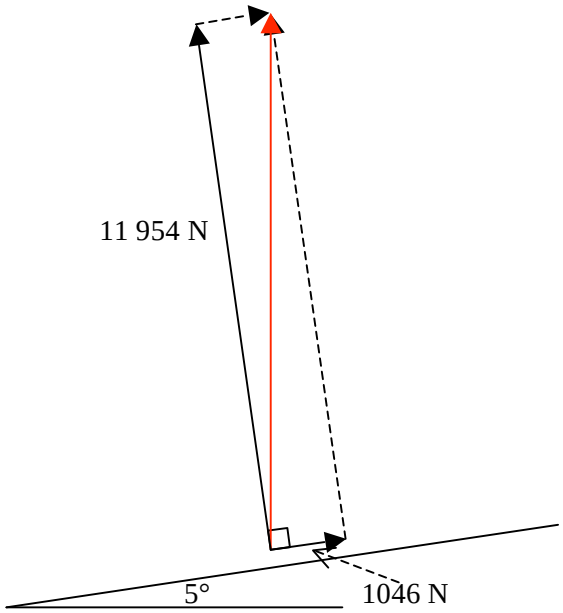
$$\left|\overrightarrow{F_2}\right| \doteq 432.89$$

The magnitude of the second force is about 433 N.

Chapter 6 Section 4 **Question 17 Page 345**

a) Answers may vary. For Example: In any situation, choose the most convenient directions for rectangular components. In this situation, the rectangular components are not vertical and horizontal since the horizontal brake force is parallel to the hill and the vertical force is perpendicular to the hill.

b) Draw a diagram of the situation.



Since the forces are perpendicular, use the Pythagorean theorem to find the magnitude of the resultant force.

$$\left|\overrightarrow{R}\right|^2 = 11\,954^2 + 1046^2$$

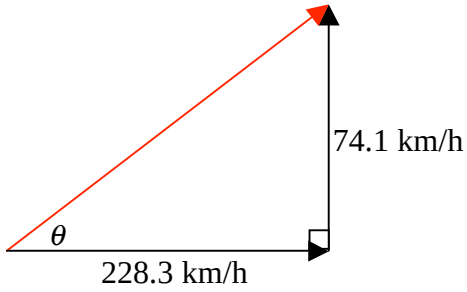
$$= 143\,992\,232$$

$$\left|\overrightarrow{R}\right| \doteq 11\,999.7$$

The weight of the car is found by taking the opposite of this resultant vector. Both vectors have the same magnitude. Thus, the weight of the car is about 12 000 N.

Chapter 6 Section 4 **Question 18 Page 345**

a) Draw a diagram of the situation.



Use the Pythagorean theorem to find the magnitude of the take-off velocity.

$$\left|\vec{R}\right|^2 = 228.3^2 + 74.1^2$$
$$= 57\ 611.7$$
$$\left|\vec{R}\right| \doteq 240.0$$

The take-off velocity is about 240 km/h in a direction θ above the runway. In 3 min, the jet will travel $\frac{3 \times 240}{60}$ or 12 km.

Use trigonometric ratios to find θ.

$$\tan\theta = \frac{74.1}{228.3}$$
$$= 2.4$$
$$\theta \doteq \tan^{-1}(0.3246)$$
$$\theta \doteq 17.98^\circ$$

After 3 min, the jet's displacement will be about 12 km at 18° to the horizontal.

Chapter 6 Section 4 **Question 19 Page 345**

This question involves finding the resultant of the sum of four vectors. This resultant can be found using trigonometry and the sine and cosine laws as in the other questions in this exercise set, but our previous solutions have only involved adding two vectors at a time which would lead to many calculations in this case.

Use a more efficient approach, introduced in section 6.1. Use *The Geometer's Sketchpad*®.

Choose an appropriate scale of 1 cm = 10 km.
Let A be the starting helicopter pad.

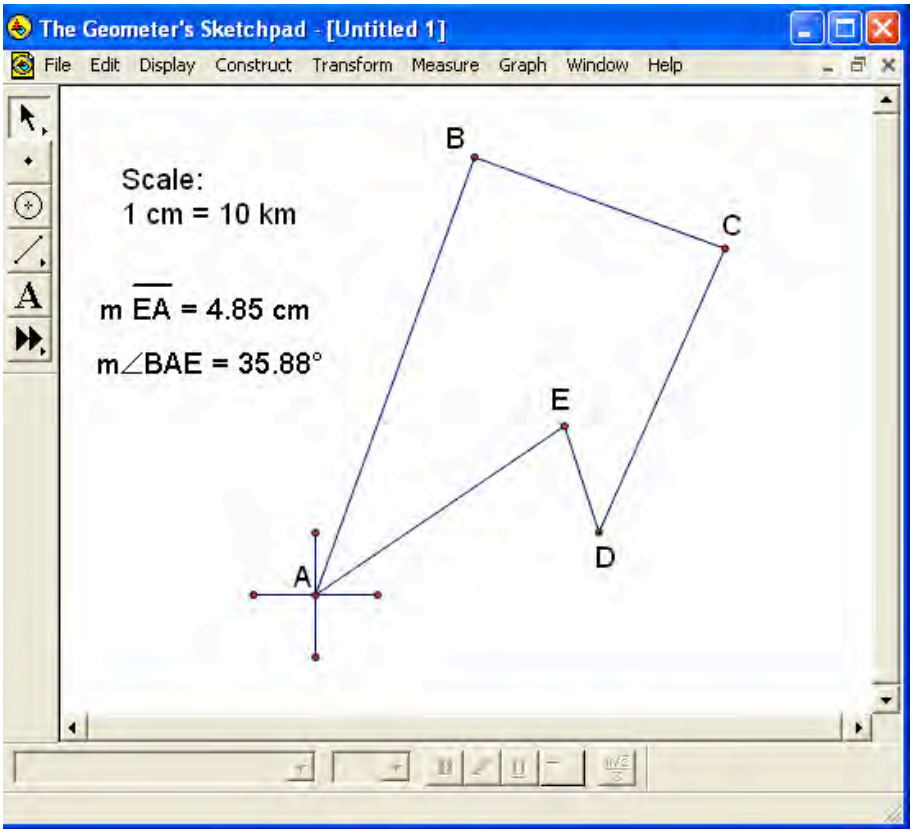
Translate the point A as follows:
7.5 cm at 70°; then 4.3 cm at –20°; then 50 cm at –114°; and finally 1.8 cm at 108°

Note that all rotation angles are measured from the horizontal (right) in a counterclockwise positive direction.
This creates the intermediate travel points B, C, D, and E; E is the rescue point.
Using the measuring tools of *The Geometer's Sketchpad*®, EA = 4.85 cm and $\angle BAE \doteq 35.9^{\circ}$.

$4.85 \times 10 = 48.5$

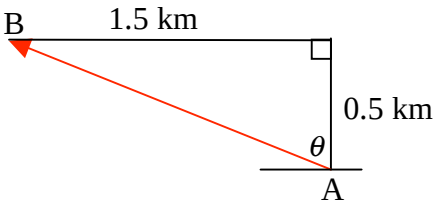
$35.9° + 20° = 55.9°$

Thus, the displacement from the helicopter pad is about 48.5 km at a bearing of N55.9°E.



Chapter 6 Section 4 **Question 20 Page 345**

a) Draw a diagram of the physical situation.

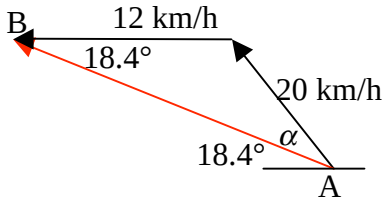


Use trigonometry to determine the direction from A to B, first determine the direction of the resultant velocity.

$$\tan\theta = \frac{1.5}{0.5}$$
$$\theta = \tan^{-1}(3)$$
$$\doteq 71.57$$

But, 90° – 71.57° = 18.43°.
The captain needs to head at an angle of about 18.4° relative to the shore of the river.

Redraw the diagram to involve velocities.



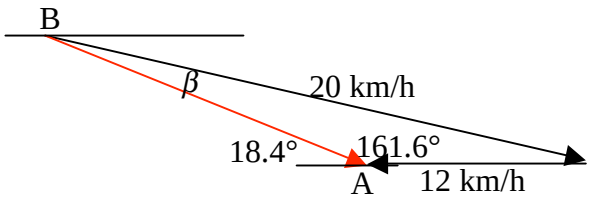
To find α, use the sine law

$$\frac{\sin\alpha}{12} = \frac{\sin 18.4^\circ}{20}$$
$$\sin\alpha = \frac{12\sin 18.4^\circ}{20}$$
$$\alpha = \sin^{-1}\left(\frac{12\sin 18.4^\circ}{20}\right)$$
$$\doteq 10.92^\circ$$

But, 18.4° + 10.9° = 29.3°.

The required heading for the trip from A to B is 29.3° to the shore.

b) Let β be the angle between the heading and the resultant as shown.



Use the sine law to solve for β.

$$\frac{\sin\beta}{12} = \frac{\sin 161.6^\circ}{20}$$

$$\sin\beta = \frac{12\sin 161.6^\circ}{20}$$

$$\beta = \sin^{-1}\left(\frac{12\sin 161.6^\circ}{20}\right)$$

$$\doteq 10.92^\circ$$

Then, 18.4° – 10.9° = 7.5°.

The required heading for the trip from B to A is 7.5° to the shore.

Chapter 6 Section 4 **Question 21 Page 345**

Solutions for Achievement Checks are shown in the Teacher Resource.

Chapter 6 Section 4 **Question 22 Page 345**

Since the forces are mutually perpendicular, use the Pythagorean theorem to find the magnitude of the resultant of the first two forces.

$$\left|\overrightarrow{R_{1,2}}\right|^2 = 35^2 + 45^2$$

$$= 3250$$

$$\left|\overrightarrow{R_{1,2}}\right| = \sqrt{3250}$$

This resultant force is in the plane of $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$. Therefore it is perpendicular to $\overrightarrow{F_3}$.

Use the Pythagorean theorem to find the magnitude of the resultant of $\overrightarrow{R_{1,2}}$ and $\overrightarrow{F_3}$.

$$\left|\overrightarrow{R_{1,2,3}}\right|^2 = \sqrt{3250}^2 + 25^2$$

$$\left|\overrightarrow{R_{1,2,3}}\right| = 62.249$$

$$\left|\vec{R}\right| = 62.249$$

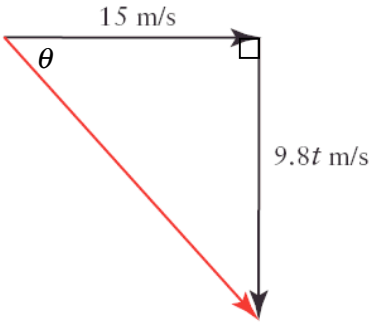
Chapter 6 Section 4 **Question 23 Page 345**

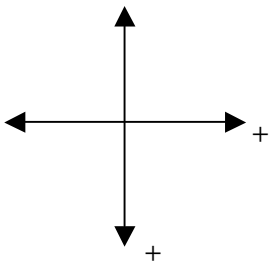
a) It is useful to consider the motion of the ball as a resultant of two vectors (forces), one acting in the horizontal direction and the other acting in vertical direction.

In the horizontal direction, air resistance is zero and there are no forces opposing the motion. Hence, the 15 m/s vector remains constant.

Since the vertical acceleration is 9.8 m/s^2, it is reasonable that the vertical velocity function is of the form $v = v_0 + 9.8t$ where all vectors are pointing down. (This is reasonable since the derivative of this function must be the constant vector with magnitude 9.8 down.) Since the ball is thrown starting at the top of the cliff, it is also reasonable to assume that the initial vertical velocity is zero. Hence $v_0 = 0$ and $v = 9.8t$.

Since the velocities are perpendicular, use the Pythagorean theorem to find the magnitude of the resultant velocity.





$$\left|\vec{R}\right|^2 = 15^2 + \left(9.8t\right)^2$$
$$= 225 + 96.04t^2$$
$$\left|\vec{R}\right| = \sqrt{225 + 96.04t^2}$$

Use trigonometry to find the direction of this velocity.

$$\tan\theta = \frac{9.8t}{15}$$
$$\theta = \tan^{-1}\left(\frac{9.8t}{15}\right)$$

The vector (model) representing the resultant velocity has magnitude $\sqrt{225 + 96.04t^2}$ and direction that makes an angle θ downward to the horizontal where $\theta = \tan^{-1}\left(\frac{9.8t}{15}\right)$.

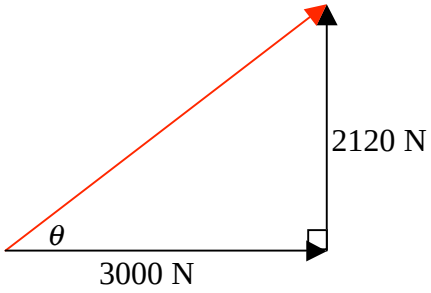
b) After 2 s, the magnitude of the velocity is $\sqrt{225+96.04(2)^2}$ or approximately 24.7 m/s and the angle the velocity is making (downward) with the horizontal direction is $\theta = \tan^{-1}\left(\frac{9.8(2)}{15}\right)$

$\doteq 52.6^{\circ}$

Chapter 6 Section 4 **Question 24 Page 345**

Since the thrust and drag forces are both horizontal, add their magnitudes. The result is 3000 N forward.

The mass of the airplane converts to the force of gravity acting downward. $\vec{F} = 600\vec{g}$ where $\vec{g}$ is 9.8 m/s^2 downward. Thus the force of gravity is 5880 N downward.
The lift is directly upward and so the total vertical force is 8000 N – 5880 N = 2120 N upward.



Since the forces are perpendicular, use the Pythagorean theorem to find the magnitude of the resultant force.

$|\vec{R}|^2 = 2120^2 + 3000^2$

$= 13\,494\,400$

$|\vec{R}| \doteq 3673.47$

To find the direction of the resultant force, use trigonometry.
Let θ represent the angle of $\vec{R}$ to the horizontal as shown.

$\tan\theta = \frac{2120}{3000}$

$\theta = \tan^{-1}\left(\frac{2120}{3000}\right)$

$\doteq 35.247^{\circ}$

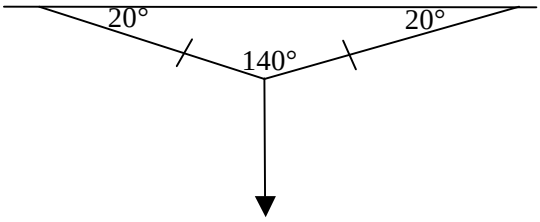
The resultant force has magnitude about 3673.5 N and acts at an angle of about 35.2° upward from the horizontal.

Chapter 6 Section 4 **Question 25 Page 346**

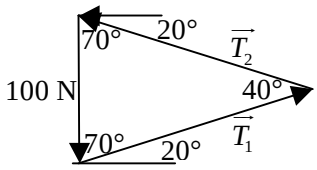
Consider a point on the outside of the wheel.
The angular acceleration is 200 revolutions per minute squared.
One revolution is 9π cm. ($C = \pi d$)
Thus, the given point is accelerating at $9\pi(240)$ or approximately 6786 cm/min^2.
The direction of the acceleration is constantly changing. Centripetal acceleration is usually considered to be outward from the centre of rotation.

Chapter 6 Section 4 **Question 26 Page 346**

a) First draw the physical diagram. Then draw a vector diagram, which is not the same diagram



Physical diagram



Vector diagram

To find $\vec{T}_1$, use the sine law.

$$\frac{|\vec{T}_1|}{\sin 70^\circ} = \frac{100}{\sin 40^\circ}$$

$$|\vec{T}_1| = \frac{100 \sin 70^\circ}{\sin 40^\circ}$$

$$\doteq 146.19$$

Since the triangle is isosceles, we know that the magnitudes of both tension vectors will be the same, about 146.2 N. the direction of the tension vectors is along each rope, at a 20° angle to the horizontal.

b) If one rope were 1.5 times as long as the other, then the tension in the shorter rope would be less than the tension in the longer rope.
First find the distance, d, between the rope anchor points. Use the cosine law in the physical diagram above. Assume the ropes have length 1 unit to start.

$$d^2 = 1^2 + 1^2 - 2(1)(1)\cos 140^\circ$$

$$\doteq 3.5321$$

$$d \doteq 1.8794$$

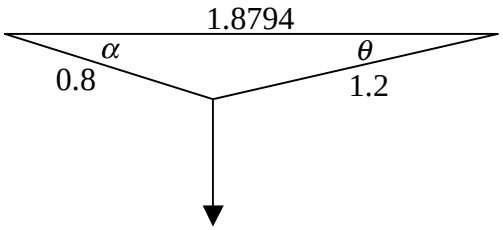
Now let one rope be 1.5 times as long as the other (x and $1.5x$).

$$x + 1.5x = 1 + 1$$

$$2.5x = 2$$

$$x = 0.8$$

The new ropes are 0.8 units and 1.2 units long.



Use the cosine law to find θ.

$$0.8^2 = 1.8794^2 + 1.2^2 - 2(1.8794)(1.2)\cos\theta$$

$$\cos\theta \doteq \frac{-4.3321}{-1.92}$$

$$\theta \doteq \cos^{-1}\left(\frac{-4.3321}{-4.5106}\right)$$

$$\theta \doteq 16.17^\circ$$

Then use the cosine law to find α.

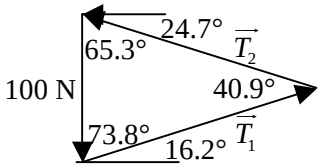
$$1.2^2 = 0.8^2 + 1.8794^2 - 2(0.8)(1.8794)\cos\alpha$$

$$\cos\alpha \doteq \frac{-2.732}{-3.007}$$

$$\alpha \doteq \cos^{-1}\left(\frac{-2.732}{-3.007}\right)$$

$$\alpha \doteq 24.69^\circ$$

Draw the new vector diagram.



To find $\vec{T}_1$, use the sine law.

$$\frac{\left|\vec{T}_1\right|}{\sin 65.3^\circ} = \frac{100}{\sin 40.9^\circ}$$

$$\left|\vec{T}_1\right| = \frac{100 \sin 65.3^\circ}{\sin 40.9^\circ}$$

$$\doteq 138.76$$

To find $\vec{T}_2$, use the sine law.

$$\frac{\left|\vec{T}_2\right|}{\sin 73.8^\circ} = \frac{100}{\sin 40.9^\circ}$$

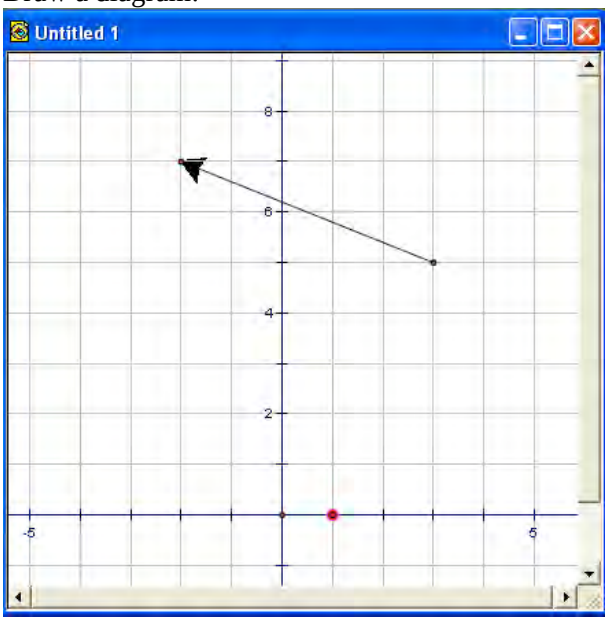
$$\left|\vec{T}_2\right| = \frac{100 \sin 73.8^\circ}{\sin 40.9^\circ}$$

$$\doteq 146.67$$

The new tensions are 138.8 N in the longer rope and 146.7 N in the shorter rope.

Chapter 6 Section 4 **Question 27 Page 346**

Draw a diagram.



For this line segment (vector) the rise and run are 2 and –5 respectively.
The magnitude of the vector is

$$\sqrt{2^2 + (-5)^2} = \sqrt{29}$$
$$\doteq 5.4$$

The direction is $\theta = \tan^{-1}\left(\frac{2}{-5}\right)$

$$\doteq -21.8^{\circ}$$

The vector has magnitude about 5.4 units and a direction of 158.2° to the horizontal (right). This direction can also be expressed as 21.8° above the horizontal (left).

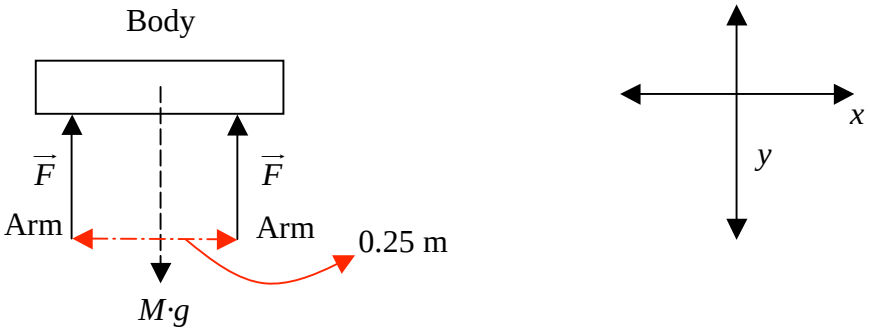
Chapter 6 Section 4 **Question 28 Page 346**

Answers may vary. For example:

When doing push-ups, a smaller muscular force is required in situation a) when your hands are 0.25 m apart. If your hands are straight down the angle between your arms is zero and the muscular force required would be equivalent to the weight of your body. As the distance between your hands is increased, the angle between your arms will also increase. $F\cos\theta$, where θ is half the angle in between your arms and F is the muscular force, will now equal the weight of your body. As θ increases, the muscular force required will increase and a greater muscular force will be required in situation b) when your hands are 0.5 m apart.

For example:

a) The following diagram represents the case when the hands are 0.25 m apart. The force of gravity acts downwards (mass = M kg, weight = $M \cdot g$ N where g = 9.8 m/s^2). Two arms are opposing the force of gravity.



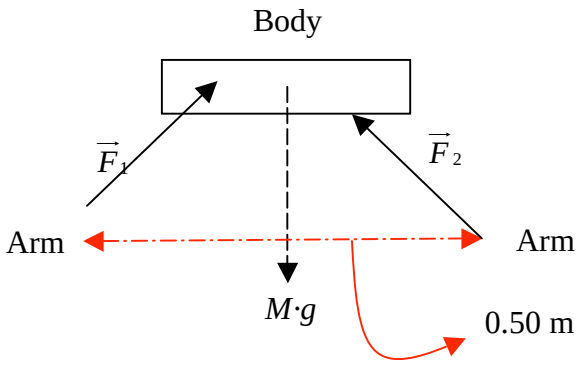
For sake of simplicity, assume that while moving the body up, there is no acceleration and both arms exert equal force (otherwise there while be rotational motion).

To understand how the forces interact, consider the x- and y-components of all the forces. Since the body is in equilibrium (you do not fall on the ground), the forces must balance each other.

$$M \cdot g = 2 \cdot \vec{F}$$

$$\vec{F} = \frac{M \cdot g}{2}$$

b) The following diagram represents the case when the hands are 0.50 m apart. The force of gravity acts downwards (mass = M kg, weight = $M \cdot g$ N where g = 9.8 m/s^2). Two arms are opposing the force of gravity and have a component each in the x-direction.



In this case, the left arm is exerting a force $\vec{F}_1$ and the right arm is exerting a force $\vec{F}_2$ with $\left|\vec{F}_1\right| = \left|\vec{F}_2\right|$. Both these forces have a component in the x-direction and a component in the y-direction. Since the body is in equilibrium, the forces must balance each other. To overcome the force of gravity, the sum of y-components of the two forces must equal the force of gravity.

$$\left|\vec{F}_1\right| = \left|\vec{F}_2\right|$$
$$\left|\vec{F}_1\right|\cos\theta + \left|\vec{F}_2\right|\cos\theta = M \cdot g$$
$$2\left|\vec{F}_1\right|\cos\theta = M \cdot g$$
$$\left|\vec{F}_1\right| = \frac{M \cdot g}{2\cos\theta}$$
$$\left|\vec{F}_1\right| = \left|\vec{F}_2\right|$$

From trigonometric relations, $-1 \le \cos\theta \le 1$.

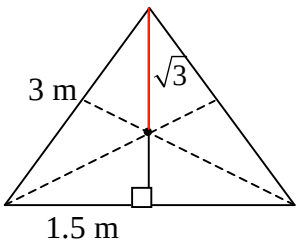
$$\vec{F} = \frac{M \cdot g}{2}$$
$$\vec{F}_1 = \frac{M \cdot g}{2\cos\theta}$$

Therefore, $\vec{F}_1 \ge \vec{F}$.

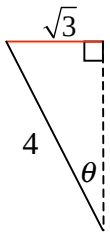
Chapter 6 Section 4 **Question 29 Page 346**

a) Since the chains each have the same length, you can assume that the tension in each one is equal in magnitude. For example, if the chains were all attached at the same point in the ceiling the tension in each would be 800 N. You would expect the tension in each chain to be greater than 800 N.

First, determine some lengths and angles.
Consider the equilateral triangle, viewed from above. The dolphin will be located exactly below the centroid.
The sides of the equilateral triangle are 3 m long. The perpendicular to the base shown intersects the base at right angles and is 1.5 m long (properties of an equilateral triangle).
Using the Pythagorean triangle, the length of the perpendicular is $\sqrt{3^2 - \left(\frac{3}{2}\right)^2} = \frac{3\sqrt{3}}{2}$.
The medians of a triangle intersect at a point $\frac{2}{3}$ of the way along each median.
Therefore, the centroid is located $\sqrt{3}$ units from each vertex.



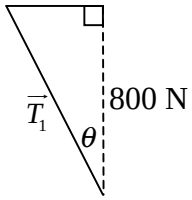
Now consider a side view of the situation, but only involving one chain.



Use trigonometry to calculate the angle in this triangle.

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{4}\right)$$
$$\doteq 23.41^{\circ}$$

There is a similar force triangle.



Use trigonometry to calculate the magnitude of the tension.

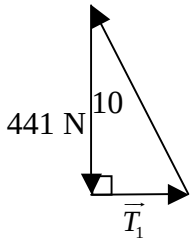
$$\frac{\left|\vec{T}_1\right|}{800} = \sec 23.41^{\circ}$$
$$\doteq 871.76$$

The tension in each chain is about 871.8 N.

b) Answers may vary. Using only one or two chains would allow the sculpture to swing with disturbances in the air. Using 3 anchor points creates a stable configuration that will not swing.

Chapter 6 Section 4 **Question 30 Page 346**

Since the ladder has four legs, we can assume that the weight held up by each one is equal in magnitude. Therefore, consider only one leg and one-half of a brace.
The heavy object has a (gravitational) force of 180×9.8 or 1746 N. The force applied to each leg is 441 N.



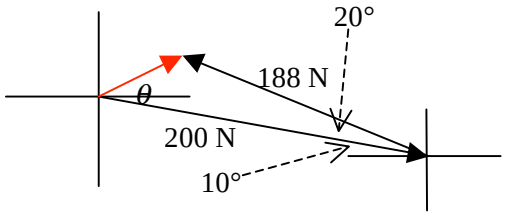
Use trigonometry to find the magnitude of the tension in one-half of the brace.

$$\tan 10^\circ = \frac{\left|\vec{T}_1\right|}{441}$$
$$\left|\vec{T}_1\right| = 441 \tan 10^\circ$$
$$\doteq 77.76$$

Therefore, the tension is each crosspiece is 77.76×2 or approximately 155.5 N.

Chapter 6 Section 4 **Question 31 Page 346**

Draw a diagram of the forces.



Use the cosine law to calculate the magnitude of the resultant.

$$\left|\vec{R}\right|^2 = 188^2 + 200^2 - 2(188)(200)\cos 20^0$$
$$\doteq 4679.1$$
$$\left|\vec{R}\right| \doteq 68.40$$

Use the sine law to find the direction of the resultant.

$$\frac{\sin\theta}{188} = \frac{\sin 20^\circ}{68.40}$$

$$\sin\theta = \frac{188\sin 20^\circ}{68.40}$$

$$\theta = \sin^{-1}\left(\frac{188\sin 20^\circ}{68.40}\right)$$

$$\doteq 70.06^\circ$$

To find the bearing, calculate 100° – 70.06° = 29.94°.

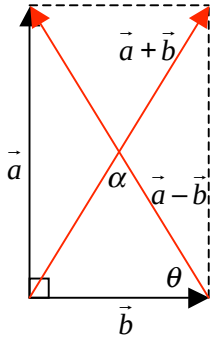
The sailboat will move upwind with a force of about 68.4 N at a bearing of about 029.9°.

Chapter 6 Section 4 Question 32 Page 346

Method 1:

Draw a diagram. Note the vectors form a rectangle.

Let α be the angle between $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.



$$\tan\theta = \frac{2}{1}$$

$$\theta \doteq 63.4^\circ$$

The triangle involving α is isosceles and has angles $\theta, \theta,$ and α.

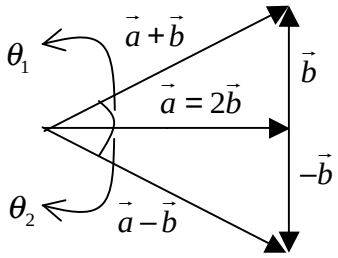
Therefore, $\alpha = 180^\circ - \theta - \theta$

$$= 180^\circ - 63.4^\circ - 63.4^\circ$$

$$\doteq 53.2^\circ$$

The best answer is D.

Method 2:



Since $\vec{a} \perp \vec{b}$, $\left|\vec{a}+\vec{b}\right| = \left|\vec{a}-\vec{b}\right|$ and $\theta_1 = \theta_2$.

$$\tan\theta_1 = \frac{\left|\vec{b}\right|}{\left|\vec{a}\right|}$$

$$= \left|\frac{\vec{b}}{2\vec{b}}\right|$$

$$= \frac{1}{2}$$

$$\theta_1 = \tan^{-1}\left(\frac{1}{2}\right)$$

$$\doteq 26.565^\circ$$

$$= \theta_2$$

Therefore, the angle between $\vec{a}$ and $\vec{b}$ is 53.1°. Note in this question the magnitude of $\left|\vec{a}-\vec{b}\right|$ is not required.

Chapter 6 Section 4 **Question 33 Page 346**

Find the intersection points.

$$x^2 + y^2 - 4x + 8y + 11 = x^2 - 4x - 4y - 24$$
$$y^2 + 12y + 35 = 0$$
$$(y + 5)(y + 7) = 0$$
$$y = -5, -7$$

If $y = -5$,

$$x^2 + 25 - 4x - 40 + 11 = 0$$
$$x^2 - 4x - 4 = 0$$
$$x = \frac{4 \pm \sqrt{16 + 16}}{2}$$
$$= \frac{4 \pm 4\sqrt{2}}{2}$$
$$= 2 \pm 2\sqrt{2}$$

If $y = -7$,

$$x^2 + 49 - 4x - 56 + 11 = 0$$
$$x^2 - 4x + 4 = 0$$
$$(x - 2)(x - 2) = 0$$
$$x = 2, 2$$

The three intersection points are $(2, -7)$, $\left(2 + 2\sqrt{2}, -5\right)$, and $\left(2 - 2\sqrt{2}, -5\right)$.

Note that the last two points form a horizontal segment that can be considered as the base for the triangle. The length of this segment is $4\sqrt{2}$ since

$$2 + 2\sqrt{2} - \left(2 - 2\sqrt{2}\right) = 2 + 2\sqrt{2} - 2 + 2\sqrt{2}$$
$$= 4\sqrt{2}$$

The height of the triangle (to this base) is 2 since $-5 - (-7) = 2$.

$$A = \frac{1}{2}bh$$
$$= \frac{1}{2}4\sqrt{2} \times 2$$
$$= 4\sqrt{2}$$

The area of the triangle is $4\sqrt{2}$ square units.

Chapter 6 Section 5: **Resolution of Vectors into Rectangular Components**

Chapter 6 Section 5 **Question 1 Page 349**

a) $\left|\overrightarrow{F_h}\right| = 560\cos 21^\circ$

$\doteq 522.8$

The horizontal component is approximately 522.8 N.

$\left|\overrightarrow{F_v}\right| = 560\sin 21^\circ$

$\doteq 200.7$

The vertical component is approximately 200.7 N.

b) $\left|\overrightarrow{F_h}\right| = 21\cos 56^\circ$

$\doteq 11.7$ N

The horizontal component is approximately 11.7 N.

$\left|\overrightarrow{F_v}\right| = 21\sin 56^\circ$

$\doteq 17.4$ N

The vertical component is approximately 17.4 N.

c) $\left|\overrightarrow{F_h}\right| = 1200\cos 43^\circ$

$\doteq 877.6$

The horizontal component is approximately 877.6 N.

$\left|\overrightarrow{F_v}\right| = 1200\sin 43^\circ$

$\doteq 818.4$

The vertical component is approximately 818.4 N.

d) $\left|\overrightarrow{F_h}\right| = 17\cos 75^\circ$

$\doteq 4.4$

The horizontal component is approximately 4.4 N.

$\left|\overrightarrow{F_v}\right| = 17\sin 75^\circ$

$\doteq 16.4$

The vertical component is approximately 16.4 N.

e) $\left|\overrightarrow{F_h}\right| = 400\cos 78^\circ$

$\doteq 83.2$

The horizontal component is approximately 83.2 N.

$\left|\overrightarrow{F_v}\right| = 400\sin 78^\circ$

$\doteq 391.3$

The vertical component is approximately 391.3 N.

Chapter 6 Section 5 **Question 2 Page 349**

a) $\left|\overrightarrow{F_h}\right| = 15\cos 56^\circ$

$\doteq 8.4$

The horizontal component is approximately 8.4 km/h.

$\left|\overrightarrow{F_v}\right| = 15\sin 56^\circ$

$\doteq 12.4$

The vertical component is approximately 12.4 km/h.

b) $\left|\overrightarrow{F_h}\right| = 120\cos(-35^\circ)$

$\doteq 98.3$

The horizontal component is approximately 98.3 m.

$\left|\overrightarrow{F_v}\right| = 120\sin(-35^\circ)$

$\doteq -68.8$

The vertical component is approximately –68.6 m.

c) $\left|\overrightarrow{F_h}\right| = 880\cos 70^\circ$

$\doteq 301.0$

The horizontal component is approximately 301.0 km/h.

$\left|\overrightarrow{F_v}\right| = 880\sin 70^\circ$

$\doteq 826.9$

The vertical component is approximately 826.0 km/h.

d) $\left|\overrightarrow{F_h}\right| = 135\cos 50^\circ$

$\doteq 86.8$

The horizontal component is approximately 86.8 m/s^2.

$\left|\overrightarrow{F_v}\right| = 135\sin 50^\circ$

$\doteq 103.4$

The vertical component is approximately 103.4 m/s^2.

Chapter 6 Section 5 **Question 3 Page 350**

a) For the components to be equal, we need the angle to be such that $\sin\theta = \cos\theta$.

One possible angle is $\theta = 45^\circ$.

$\left|\overrightarrow{F_h}\right| = 100\cos 45^\circ$

$\doteq 70.7$

The horizontal component is approximately 70.7 N.

$\left|\overrightarrow{F_v}\right| = 100\sin 45^\circ$

$\doteq 70.7$

The vertical component is approximately 70.7 N.

b) Yes. There is another possible angle, $\theta = 225^\circ$ since

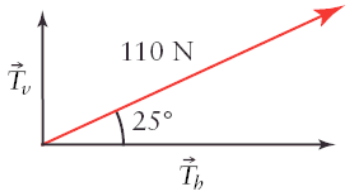
$\sin 225^\circ = \cos 225^\circ$

$\doteq -0.70711$

Note that each rectangular component would be negative in this case.

Chapter 6 Section 5 **Question 4 Page 350**

a) Draw a vector diagram. The components of the (red) tension vector are indicated.



b) $\left|\overrightarrow{F_h}\right| = 110\cos 25^\circ$

$\doteq 99.7$

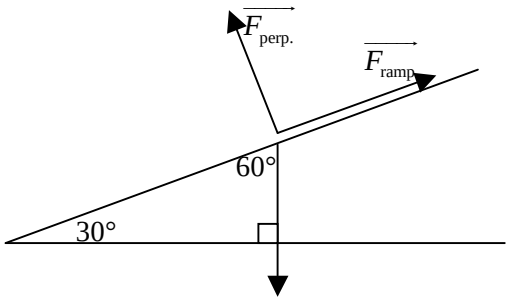
The horizontal component is approximately 99.7 N.

$\left|\overrightarrow{F_v}\right| = 110\sin 25^\circ$

$\doteq 46.5$

The vertical component is approximately 46.5 N.

Chapter 6 Section 5 **Question 5 Page 350**



$$\left|\overrightarrow{F_{ramp}}\right| = 35\cos 60^{o}$$
$$\doteq 17.5$$
The component parallel to the ramp is approximately 17.5 N.

$$\left|\overrightarrow{F_{perp.}}\right| = 35\sin 60^{o}$$
$$\doteq 30.3$$
The component perpendicular to the ramp is approximately 30.3 N.

Chapter 6 Section 5 **Question 6 Page 350**

$$\left|\overrightarrow{F_{h}}\right| = 120\cos 80^{o}$$
$$\doteq 20.8$$
The horizontal component is approximately 20.8 m/s.

$$\left|\overrightarrow{F_{v}}\right| = 120\sin 80^{o}$$
$$\doteq 118.2$$
The vertical component is approximately 118.2 m/s.

Chapter 6 Section 5 **Question 7 Page 350**

$$\left|\overrightarrow{F_{h}}\right| = 29\ 000\cos 2.7^{o}$$
$$\doteq 28\ 967.8$$
$$\doteq 29\ 000$$
The horizontal component is approximately 29 000 km/h.

$$\left|\overrightarrow{F_{v}}\right| = 29\ 000\sin 2.7^{o}$$
$$\doteq 1366.1$$
$$\doteq 1400$$
The vertical component is 1400 km/h.

Chapter 6 Section 5 **Question 8 Page 350**

The horizontal groundspeed is

$$\left|\overrightarrow{F_h}\right| = 600\cos 14^\circ$$
$$\doteq 582.2$$

The horizontal groundspeed is approximately 582.2 km/h.

$$\left|\overrightarrow{F_v}\right| = 600\sin 14^\circ$$
$$\doteq 145.2$$

The rate of climb is approximately 145.2 km/h.

Chapter 6 Section 5 **Question 9 Page 350**

For this question, neglect any altitude the plane may have.

$$\left|\overrightarrow{F_N}\right| = 125\cos 24.3^\circ$$
$$\doteq 113.9$$

The north component is approximately 113.9 km.

$$\left|\overrightarrow{F_E}\right| = 125\sin 24.3^\circ$$
$$\doteq 51.4$$

The east component is approximately 51.4 km.

Chapter 6 Section 5 **Question 10 Page 350**

Since the lengths of the wires are equal, they form an isosceles triangle. Hence the magnitude of both wires will be equal.

The two tension vectors are perpendicular to each other. The force due to gravity is 20×9.8 or 196 N.

$$\left|\overrightarrow{T_1}\right| = 196\cos 45^\circ$$
$$\doteq 138.6$$

The first tension vector will have a magnitude of approximately 138.6 N.

$$\left|\overrightarrow{T_2}\right| = 196\sin 45^\circ$$
$$\doteq 138.6$$

The second tension vector will have a magnitude of approximately 138.6 N.

The directions of these tension vectors will be upward making a 45° angle with the horizontal.

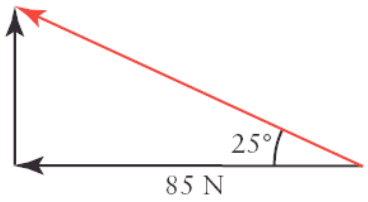
Chapter 6 Section 5 **Question 11 Page 350**

a) The horizontal component of the force is $\left|\vec{F}\right|\cos\theta$. You could reduce the 60° angle since this expression gets larger as θ gets smaller.

b) Answers may vary. With a smaller angle, you may have to bend over to control the mower which may be a dangerous operation.

Chapter 6 Section 5 **Question 12 Page 350**

a) Draw a diagram.



b) i) The magnitude of the total force on the handle can be found using trigonometry.

$$\frac{85}{\left|\vec{F}\right|} = \cos 25^{\circ}$$

$$\left|\vec{F}\right| = \frac{85}{\cos 25^{\circ}}$$

$$\doteq 93.79$$

The total force on the handle is about 93.8 N.

ii) Use trigonometry to find the vertical component of the force on the handle.

$$\left|\vec{F_v}\right| = 93.79 \sin 25^{\circ}$$

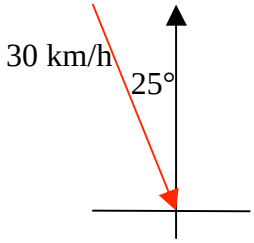
$$\doteq 39.64$$

The vertical component is about 39.6 N.

Chapter 6 Section 5 **Question 13 Page 350**

a) Draw a diagram for each situation.

Consider the due north runway.



$\left|\overrightarrow{F_N}\right| = 30\cos 25^\circ$

$\doteq 27.2$

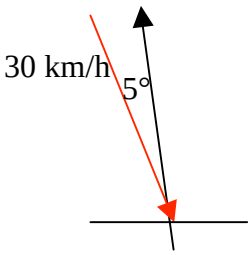
The north component of the wind is approximately 27.2 km/h.

$\left|\overrightarrow{F_E}\right| = 30\sin 25^\circ$

$\doteq 12.7$

The east component of the wind is approximately 12.7 km/h.

Consider the runway at 330°.



$\left|\overrightarrow{F_{\text{parallel}}}\right| = 30\cos 5^\circ$

$\doteq 29.9$

The component of the wind parallel to this runway is approximately 29.9 km/h.

$\left|\overrightarrow{F_{\text{perp.}}}\right| = 30\sin 5^\circ$

$\doteq 2.6$

The component of the wind perpendicular to this runway is approximately 2.6 km/h.

b) For the due north runway, the required groundspeed is 160 km/h – 27.2 km/h = 132.8 km/h.
For the runway at 330°, the required groundspeed is 160 km/h – 29.9 km/h = 130.1 km/h.

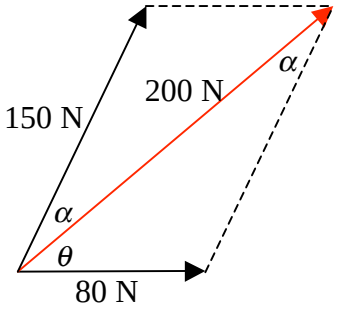
c) A pilot should use the 330° runway on this day as the wind is blowing almost directly at this runway. The wind only makes a 5° angle with the direction of takeoff. This runway will require the least groundspeed for takeoff. With the due north runway, the wind makes an angle of 25°.

d) Yes. Either runway could be used on this day as the crosswind is only 12.7 km/h for the due north runway and 2.6 km/h for the 330° runway.

Chapter 6 Section 5 **Question 14 Page 351**

If 150 N and 80 N are rectangular components for the 200 N force, then $\sqrt{150^2 + 80^2} = 200$.
However, L.S. = 170.

Draw a diagram.



Use the cosine law to find θ.

$$150^2 = 80^2 + 200^2 - 2(80)(200)\cos\theta$$

$$\cos\theta = \frac{-23\,900}{-32\,000}$$

$$\theta = \cos^{-1}\left(\frac{-23\,900}{-32\,000}\right)$$

$$\doteq 41.68^\circ$$

Use the cosine law to find α.

$$80^2 = 150^2 + 200^2 - 2(150)(200)\cos\alpha$$

$$\cos\alpha = \frac{-56\,100}{-60\,000}$$

$$\alpha = \cos^{-1}\left(\frac{-56\,100}{-60\,000}\right)$$

$$\doteq 20.77^\circ$$

$$\theta + \alpha = 41.68 + 20.77$$

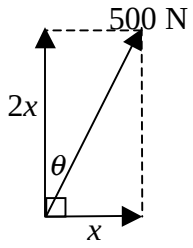
$$= 62.45$$

$$\doteq 62.5^\circ$$

The directions of the two components of the force, when placed tail to tail, are at an angle of 62.5° to each other.

Chapter 6 Section 5 **Question 15 Page 351**

Method 1:
Draw a diagram.



Use the Pythagorean theorem to solve for x.

$$(2x)^2 + x^2 = 500^2$$
$$5x^2 = 250\ 00$$
$$x^2 = 50\ 000$$
$$x \doteq 223.61$$

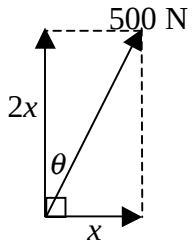
The two components have magnitudes of 223.6 N and 447.2 N.

Use trigonometry to find the required angle.

$$\tan\theta = \frac{1}{2}$$
$$\theta = \tan^{-1}(0.5)$$
$$\theta \doteq 26.57^{\circ}$$

The greater component makes an angle of about 26.6° with the 500-N force.

Method 2:
Draw a diagram.



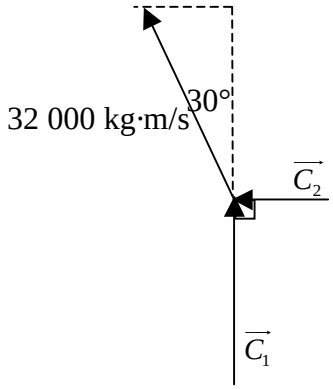
Note: To solve this question, you do not need to solve for x. From the diagram above, you know that

$$\tan\theta = \frac{x}{2x}$$
$$= \frac{1}{2}$$
$$\theta = \tan^{-1}\left(\frac{1}{2}\right)$$
$$\doteq 26.57°$$

Thus, the greater component makes an angle of 26.6° with the 500 N force.

Chapter 6 Section 5 **Question 16 Page 351**

Draw a diagram.



The two momentum vectors are components of the resultant.

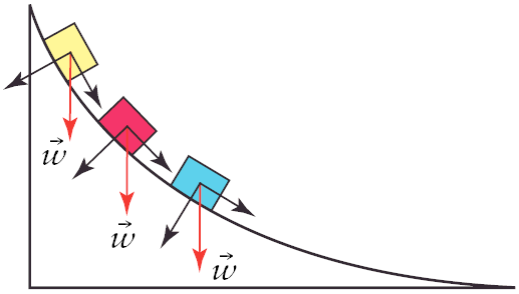
$|\vec{C_1}| = 32\,000\cos 30^\circ$ $|\vec{C_2}| = 32\,000\sin 30^\circ$

$\doteq 27\,712.8$ $\doteq 16\,000$

The momentums of the cars before the collision were 16 000 kg·m/s west and 27 712.8 kg·m/s north.

Chapter 6 Section 5 **Question 17 Page 351**

a) Draw a diagram. The weight vector at each position is indicated by $\vec{w}$.
(Note: this diagram is not to scale as the red vector should be the sum of the two component black vectors at each stop.)



b) The box will have the greatest acceleration in the first position (yellow box). This is because the component of the weight parallel to the ramp will be greatest at this time. As the box moves down the ramp, the angle between the weight vector and the parallel component vector increases. Since the magnitude varies with $\cos\theta$, the parallel component decreases as the box travels down the ramp. A greater force vector results in a greater acceleration.

Chapter 6 Section 5 **Question 18 Page 351**

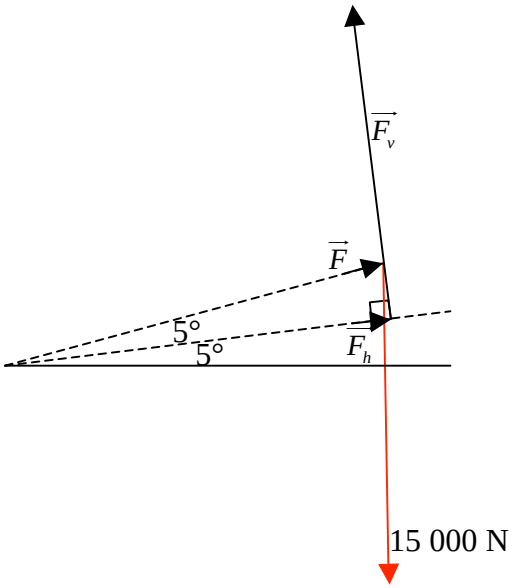
Answers may vary.

For example:
One of Newton's laws of motion states: "A physical body will remain at rest, or continue to move at a constant velocity unless an outside net force acts upon it." Assume this projectile is not under the influence of any external forces at the beginning of it flight except gravity. Air resistance is a force that acts on all projectiles but we are neglecting it in this situation. Suppose the projectile begins with some horizontal velocity. The projectile will then travel in a parabolic trajectory. Since there are no additional horizontal forces, there will be no horizontal acceleration and therefore the horizontal velocity will be constant. The only vertical force on the projectile is the force of gravity. The vertical velocity will not remain constant. The acceleration due to gravity will be 9.8 m/s^2. Therefore the vertical velocity will change by 9.8 m/s each second.

For example:
Newton's First Law of Motion states that "A body at rest will remain at rest and a body in motion will remain in motion unless a non-zero net external force acts on the body". This is also known as the Law of Inertia. The body is moving in two directions, horizontal and vertical. The type of motion body undergoes will depend on the forces acting in these directions.
In the initial state, assume that there are no forces acting on the body except gravity and the body is projected at an angle θ with an initial velocity $\left|\vec{u}\right|$. Since air resistance is neglected there is no force acting on the body in the horizontal direction. Hence the body will undergo uniform motion in the horizontal direction, i.e. the speed of the object will remain constant. In the vertical direction, the force of gravity constantly acts on the body. The acceleration due to gravity near the earth's surface is 9.8 m/s^2. Hence the speed of the object will change continuously in the vertical direction at 9.8 m/s each second.

Chapter 6 Section 5 **Question 19 Page 351**

Draw a diagram of the situation.



Resolve the equilibrant of the 15 000-N force into two forces one parallel to the slope of the road and one perpendicular to the slope.

$$\left|\overrightarrow{F_h}\right| = 15\,000 \sin 5^\circ$$
$$\doteq 1307.3$$

The parallel component is approximately 1307.3 N.

$$\left|\overrightarrow{F_v}\right| = 15\,000 \cos 5^\circ$$
$$\doteq 14\,942.9$$

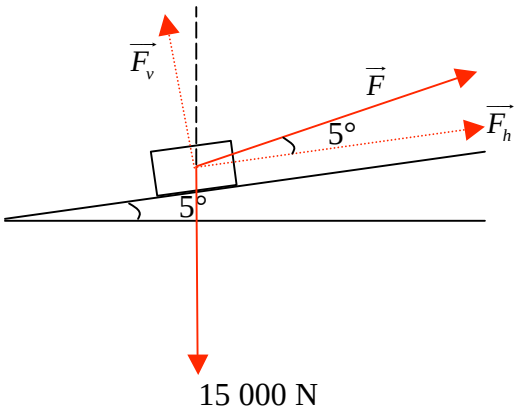
The perpendicular component is approximately 14 942.9 N.

The force needed to keep the car from rolling must have the same horizontal component (i.e. parallel to the hill) as the equilibrant vector.

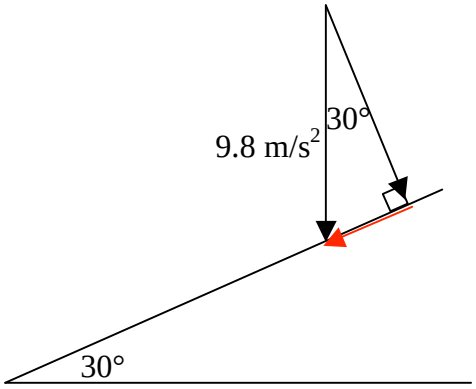
$$\left|\vec{F}\right| \cos 5^\circ = 1307.3$$
$$\left|\vec{F}\right| = \frac{1307.3}{\cos 5^\circ}$$
$$\doteq 1312.3$$

A force of 1312.3 N must be applied at angle of 5° above the hill to prevent the car from rolling.

The following is alternative diagram.



Chapter 6 Section 5 **Question 20 Page 351**



The ramp is frictionless. The box will have a vertical acceleration of 9.8 m/s^2 (acceleration due to gravity). However, the motion of the box will be constrained by the surface of the ramp. The acceleration will have two components, one parallel to the ramp and one perpendicular to the ramp. The acceleration parallel to the ramp will be given by 9.8sin 30° which equals 4.9 m/s^2.

Chapter 6 Section 5 **Question 21 Page 351**

a) In the formulas, the following constants and variables are used.

t is the time since the projectile started its motion.
$\vec{a}_x$ and $\vec{a}_y$ are the components of the acceleration vector in the *x*- and *y*-directions respectively.
$\vec{v}$ is the velocity of the projectile at time *t*.
$\vec{v}_{ix}$ and $\vec{v}_{iy}$ are the components of the velocity vector in the *x*- and *y*-directions respectively.
The formula $\vec{x} = \vec{v}_{ix}t + \frac{1}{2}\vec{a}_x t^2$ gives the horizontal displacement of the projectile at time *t*.
The formula $\vec{v}_x = \vec{v}_{ix} + \vec{a}_x t$ gives the horizontal component of the velocity of the projectile at time *t*.
The formula $\vec{y} = \vec{v}_{iy}t + \frac{1}{2}\vec{a}_y t^2$ gives the vertical displacement of the projectile at time *t*.
The formula $\vec{v}_y = \vec{v}_{iy} + \vec{a}_y t$ gives the vertical component of the velocity of the projectile at time *t*.

b) Answers may vary.

For example:
In calculus, the first derivative of horizontal displacement of the projectile is the horizontal component of the velocity of the projectile motion.

$$\begin{aligned}\frac{d\vec{x}}{dt} &= \frac{d}{dt}\left(\vec{v}_{ix}t + \frac{1}{2}\vec{a}_x t^2\right)\\ &= \vec{v}_{ix} + \vec{a}_x t\\ &= \vec{v}_x\end{aligned}$$

Similarly, the first derivative of vertical displacement of the projectile is the vertical component of the velocity of the projectile motion.

$$\begin{aligned}\frac{d\vec{y}}{dt} &= \frac{d}{dt}\left(\vec{v}_{iy}t + \frac{1}{2}\vec{a}_y t^2\right)\\ &= \vec{v}_{iy} + \vec{a}_y t\\ &= \vec{v}_y\end{aligned}$$

For example:
In calculus, the first derivative of the horizontal component of displacement is the horizontal component of velocity of the projectile motion.

$$\begin{aligned}\frac{d\vec{x}}{dt} &= \frac{d}{dt}\left(\vec{v}_{ix}t + \frac{1}{2}\vec{a}_x t^2\right)\\ &= \vec{v}_{ix} + \vec{a}_x t\\ &= \vec{v}_x\end{aligned}$$

Similarly, the first derivative of the vertical component of displacement is the vertical component of velocity of the projectile motion.

$$\begin{aligned}\frac{d\vec{y}}{dt} &= \frac{d}{dt}\left(\vec{v}_{iy}t + \frac{1}{2}\vec{a}_y t^2\right)\\ &= \vec{v}_{iy} + \vec{a}_y t\\ &= \vec{v}_y\end{aligned}$$

The second derivative of the horizontal component of displacement, or the first derivative of the horizontal component of velocity, is the horizontal component of acceleration of the projectile motion.

$$\begin{aligned}\frac{d\vec{v}_x}{dt} &= \frac{d}{dt}\left(\vec{v}_{ix} + \vec{a}_x t\right)\\ &= \vec{a}_x\end{aligned}$$

Similarly, the second derivative of the vertical component of displacement, or the first derivative of the vertical component of velocity, is the vertical component of acceleration of the projectile motion.

$$\begin{aligned}\frac{d\vec{v}_y}{dt} &= \frac{d}{dt}\left(\vec{v}_{iy} + \vec{a}_y t\right)\\ &= \vec{a}_y\end{aligned}$$

Chapter 6 Section 5 **Question 22 Page 351**

Determine the horizontal and vertical components of the velocity.

$$\vec{v}_{ix} = 30\cos 30^{\circ}$$
$$\doteq 26.0$$
$$\vec{v}_{iy} = 30\sin 30^{\circ}$$
$$\doteq 15$$

Assume there is no horizontal acceleration, but that the vertical acceleration is –9.8 m/s^2.
From the equation in question 21,

$$x = 26t \text{ and } v_x = 26$$
$$y = 15t - 4.9t^2 \text{ and } v_y = 15 - 9.8t$$

For the total time of flight, determine when $y = 0$.

$$15t - 4.9t^2 = 0$$
$$t(15 - 4.9t) = 0$$
$$t = 0, \frac{15}{4.9}$$
$$\doteq 0, 3.06$$

Since the ball is on the ground at $t = 0$ s and $t = 3.06$ s, the total time of the flight is 3.06 s.

To find the position when the ball lands,

$$x = 26(3.06)$$
$$\doteq 79.56$$

The ball lands about 79.6 m from its starting point.

To find the maximum height of the ball, the vertical velocity must be zero.

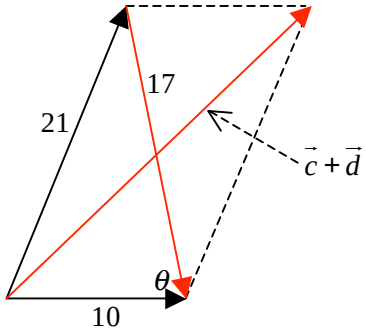
$$15 - 9.8t = 0$$
$$t = \frac{15}{9.8}$$
$$\doteq 1.53$$

When $t = 1.53$,

$$y = 15(1.53) - 4.9(1.53)^2$$
$$\doteq 11.48$$

The maximum height of the ball is about 11.5 m.

Chapter 6 Section 5 **Question 23 Page 351**



Use the cosine law to find θ.

$$21^2 = 10^2 + 17^2 - 2(10)(17)\cos\theta$$

$$\cos\theta = \frac{52}{340}$$

$$\theta = \cos^{-1}\left(\frac{52}{340}\right)$$

$$\doteq 81.2^\circ$$

Recall, the diagonals of a parallelogram bisect each other.
In the lower small triangle, use the cosine law again.

$$\left[0.5\left(\vec{c}+\vec{d}\right)\right]^2 = 10^2 + 8.5^2 - 2(10)(8.5)\cos 81.2^\circ$$

$$\doteq 146.24$$

$$\left[0.5\left(\vec{c}+\vec{d}\right)\right] \doteq 12.09$$

Therefore, the length of $\vec{c}+\vec{d}$ is about 24.2 m.

Chapter 6 Section 5 **Question 24 Page 351**

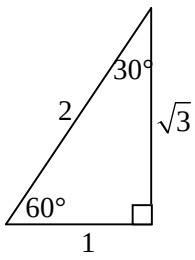
$$\log_{\sqrt{3}}(\sin x) - \log_{\sqrt{3}}(\cos x) = 1$$

$$\log_{\sqrt{3}}\left(\frac{\sin x}{\cos x}\right) = 1$$

$$\log_{\sqrt{3}}(\tan x) = 1$$

$$\tan x = \sqrt{3}$$

$$x = 60^\circ$$



Chapter 6 Review

Chapter 6 Review Question 1 Page 352

a) A vector would be a suitable model. The magnitude is 70 km/h, and the direction is northeast.

b) A vector would not be a suitable model. The magnitude is 5 km/h, but no direction is given. This is a scalar quantity.

c) A vector would be a suitable model. The magnitude is 800 km/h, and the direction is 80° above the horizontal.

d) A vector would not be a suitable model. The magnitude is 20 km, but no direction is given. This is a scalar quantity.

e) A vector would not be a suitable model. The magnitude is 180 cm, but no direction is given. This is a scalar quantity.

Chapter 6 Review Question 2 Page 352

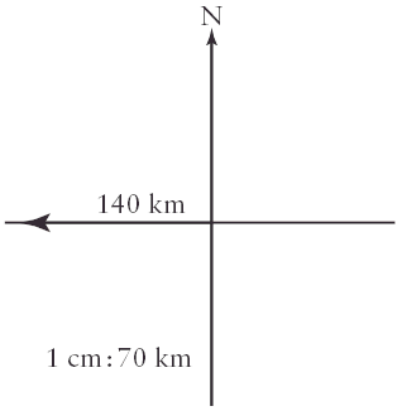
a) S50°E

b) N80°E

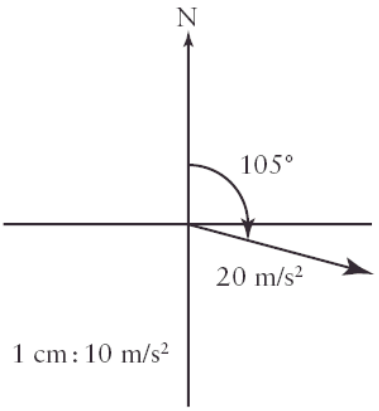
c) S70°W

Chapter 6 Review Question 3 Page 352

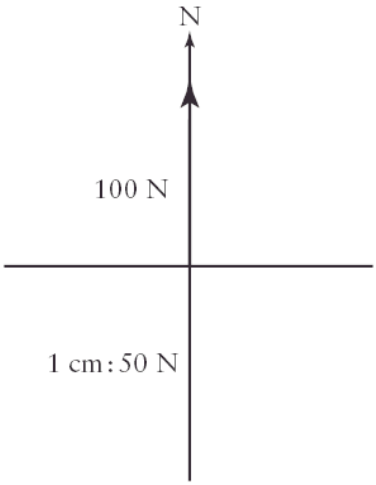
a)



b)



c)



Chapter 6 Review **Question 4 Page 352**

a) $\overrightarrow{HA} + \overrightarrow{AB} = \overrightarrow{HB}$

b) $\overrightarrow{GH} - \overrightarrow{GF} = \overrightarrow{GH} + \overrightarrow{FG}$

$= \overrightarrow{FG} + \overrightarrow{GH}$

$= \overrightarrow{FH}$

c) $\overrightarrow{FE} + \overrightarrow{BA} = \overrightarrow{AB} + \overrightarrow{BA}$

$= \vec{0}$

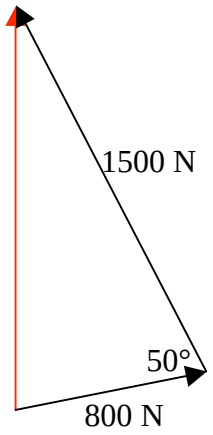
d) $\overrightarrow{GA} - \overrightarrow{EH} + \overrightarrow{DG} = \overrightarrow{GA} + \overrightarrow{HE} + \overrightarrow{DG}$

$= \overrightarrow{DG} + \overrightarrow{GA} + \overrightarrow{AD}$

$= \vec{0}$

Chapter 6 Review **Question 5 Page 352**

a) Draw a vector diagram of the situation.
The tension vectors will be in the same direction as the wires supporting the camera. The resultant vector must be vertical since the tensions are created as a result of the weight of the camera, a vector acting downward.



b) To find the magnitude of the resultant, use the cosine law.

$\left|\vec{R}\right|^2 = 800^2 + 15000^2 - 2(800)(1500)\cos 50^0$

$\doteq 1\,347\,309.7$

$\left|\vec{R}\right| \doteq 1160.7$

The resultant is about 1160.7 N in a vertical up direction.

Chapter 6 Review **Question 6 Page 352**

a) Let $\vec{u}$ be the weight of an apple (1 N down) and $\vec{v}$ be the weight of the small car (10 000 N down).
Then $\vec{v} = 10\,000\vec{u}$.

b) Let $\vec{u}$ be the velocity of the boat (25 km/h northbound) on its initial trip and $\vec{v}$ be its velocity on its return trip (5 km/h southbound).
Then $\vec{v} = -\frac{1}{5}\vec{u}$.

c) Let $\overrightarrow{a_E}$ be the acceleration due to gravity on Earth (9.8 m/s^2 down) and $\overrightarrow{a_M}$ be the acceleration due to gravity on the Moon (1.63 m/s^2 down).
Then $\overrightarrow{a_E} = \frac{9.8}{1.63}\overrightarrow{a_M}$ or $\overrightarrow{a_M} = 0.166\overrightarrow{a_E}$.

Chapter 6 Review Question 7 Page 352

a) $\overrightarrow{EC} = \overrightarrow{ED} + \overrightarrow{DC}$

$= \frac{1}{2}\overrightarrow{AC} + \overrightarrow{AB}$

b) $\overrightarrow{CE} = -\overrightarrow{CE}$

$= -\frac{1}{2}\overrightarrow{AC} - \overrightarrow{AB}$

c) $\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB}$

$= -\overrightarrow{AC} + \overrightarrow{AB}$

d) $\overrightarrow{AE} = \overrightarrow{AC} + \overrightarrow{CD} + \overrightarrow{DE}$

$= \overrightarrow{AC} - \overrightarrow{AB} - \frac{1}{2}\overrightarrow{AC}$

$= -\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC}$

Chapter 6 Review Question 8 Page 353

a) Use the Pythagorean theorem to find the magnitude of the resultant force.

$\left|\vec{R}\right|^2 = 300^2 + 500^2$

$= 340\ 000$

$\left|\vec{R}\right| \doteq 583.095$

To find the direction of the resultant force, use trigonometry.

Let θ represent the angle of $\vec{R}$ to the horizontal vector.

$\tan\theta = \frac{300}{500}$

$\theta = \tan^{-1}\left(\frac{300}{500}\right)$

$\doteq 30.96^\circ$

The resultant vector has a magnitude of about 583.1 N and a direction of about 31.0° above the horizontal.

b) Use the Pythagorean theorem to find the magnitude of the resultant force.

$\left|\vec{R}\right|^2 = 300^2 + 240^2$

$= 147\ 600$

$\left|\vec{R}\right| \doteq 384.19$

To find the direction of the resultant force, use trigonometry.
Let θ represent the angle of $\vec{R}$ to the horizontal vector.

$$\tan\theta = \frac{240}{300}$$

$$\theta = \tan^{-1}\left(\frac{240}{300}\right)$$

$$\doteq 38.66^{\circ}$$

To find the bearing, 90° + 38.7° = 128.7°.

The resultant vector has a magnitude of about 384.2 km/h at a bearing of about 128.7°.

c) The vectors are at right angles to each other since 295° – 25° = 270°.
Use the Pythagorean theorem to find the magnitude of the resultant force.

$$\left|\vec{R}\right|^2 = 16.1^2 + 12.7^2$$

$$= 420.5$$

$$\left|\vec{R}\right| \doteq 20.51$$

To find the direction of the resultant force, use trigonometry.
Let θ represent the angle of $\vec{R}$ to the 16.1 km vector.

$$\tan\theta = \frac{12.7}{16.7}$$

$$\theta = \tan^{-1}\left(\frac{12.7}{16.1}\right)$$

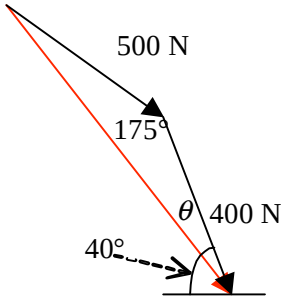
$$\doteq 38.27^{\circ}$$

To find the bearing, 295° + 38.3° = 333.3°.

The resultant vector has a magnitude of about 20.5 km at a bearing of about 333.3°.

Chapter 6 Review **Question 9 Page 353**

Draw a vector diagram.



To find the magnitude of the resultant, use the cosine law.

$$\left|\vec{R}\right|^2 = 400^2 + 500^2 - 2(400)(500)\cos 175^0$$
$$\doteq 808\ 477.9$$
$$\left|\vec{R}\right| \doteq 899.2$$

To fine the angle of the resultant, use the sine law.

$$\frac{\sin\theta}{500} = \frac{\sin 175^o}{899.2}$$
$$\sin\theta = \frac{500\sin 175^o}{899.2}$$
$$\theta = \sin^{-1}\left(\frac{500\sin 175^o}{899.2}\right)$$
$$\doteq 2.78^o$$

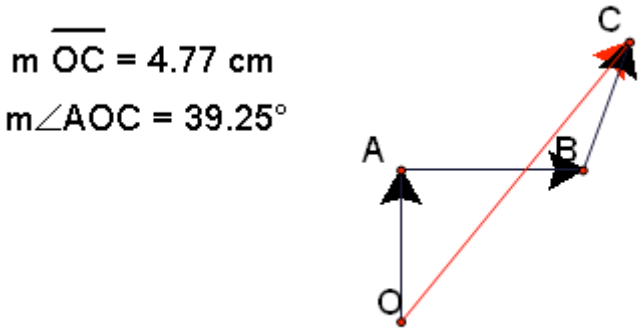
To find the angle for the resultant, 40° – 2.8° = 37.2°.

The resultant force is about 899.2 N at an angle of about 37.2° above the horizontal.

Chapter 6 Review **Question 10 Page 353**

a) Different methods are possible.

Method 1:
Use *The Geometer's Sketchpad®*
Draw the three vectors using the Translate tool and a scale of 2 cm = 100 N.



The resultant force vector is about 238.5 N at a bearing of 39.3° (N39.3°E).

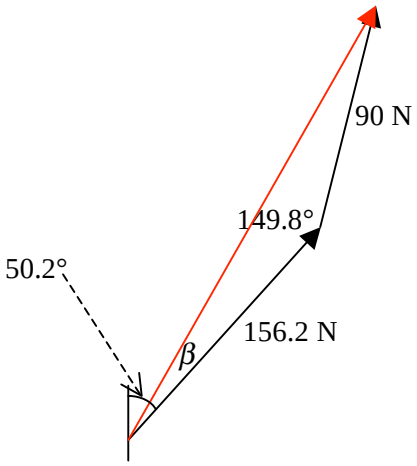
Method 2:
Use trigonometry.

Add the first two forces. Use the Pythagorean theorem to find the magnitude.

$$\left|\overrightarrow{R_1}\right|^2 = 100^2 + 120^2$$
$$= 24\ 400$$
$$\left|\overrightarrow{R_1}\right| \doteq 156.2$$

To find the direction of the resultant force, use trigonometry.
Let θ represent the angle of $\vec{R}_1$ to north.

$$\tan\theta = \frac{120}{100}$$
$$\theta = \tan^{-1}\left(\frac{120}{100}\right)$$
$$\doteq 50.2^\circ$$



To find the magnitude of the resultant, use the cosine law.

$$\left|\overrightarrow{R}\right|^2 = 156.2^2 + 90^2 - 2(156.2)(90)\cos 149.8^\circ$$
$$\doteq 56\ 798.4$$
$$\left|\overrightarrow{R}\right| \doteq 238.3$$

To fine the angle of the resultant, use the sine law.

$$\frac{\sin\theta}{90} = \frac{\sin 149.8^\circ}{238.3}$$

$$\sin\theta = \frac{90\sin 149.8^\circ}{238.3}$$

$$\theta = \sin^{-1}\left(\frac{90\sin 149.8^\circ}{238.3}\right)$$

$$\doteq 10.95^\circ$$

To find the angle for the resultant, 50.2° – 11.0° = 38.2°.

The resultant force vector is about 238.3 N at a bearing of 039.2° (N39.2°E).

Note the slight difference in the accuracy of the two methods. *The Geometer's Sketchpad*® solution could be made more accurate by increasing the number of decimal places for measurements and calculations.

Chapter 6 Review **Question 11 Page 353**

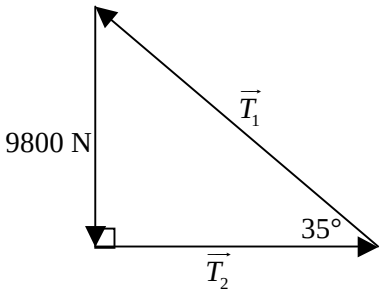
Answers may vary.

For example:
The "hang time" depends on the basketball player's vertical speed at the time he or she jumps in the air. The basketball player's speed will be greatest when he jumps in the air, at the moment his feet leave the ground, and will rapidly decrease to zero as he rises to the maximum point of his jump. Just before and just after he reaches the maximum point of his jump, the basketball player will have very little vertical speed and his distance above the ground will not change much. He will continue to move forward with very little up or down movement and this effect will cause the illusion that he is "hanging" in the air.

Chapter 6 Review **Question 12 Page 353**

a) The weight of the load is the mass times the force of gravity: 1000(9.8) or 9800 N.

b) There are three forces present: the weight, tension in the boom, and tension in the cable. These three forces form a right-angled triangle.



To find the magnitude of the tension in the cable, use trigonometry.

$$\frac{\left|\vec{T_1}\right|}{9800} = \csc 35^\circ$$

$$\left|\vec{T_1}\right| = 9800 \csc 35^\circ$$

$$= \frac{9800}{\sin 35^\circ}$$

$$\doteq 17\ 085.8$$

The tension in the cable has a magnitude of 17 085.8 N and makes a 35° angle with the horizontal boom.

c) To fine the horizontal force on the boom use trigonometry.

$$\frac{\left|\vec{T_2}\right|}{9800} = \cot 35^\circ$$

$$\left|\vec{T_2}\right| = 9800 \cot 35^\circ$$

$$= \frac{9800}{\tan 35^\circ}$$

$$\doteq 13\ 995.9$$

The horizontal force on the boom is 13 995.9 N directed towards the building.

d) The vertical equilibrant component of the tension in the cable is –9800 N, the opposite of the weight of the sign.

Chapter 6 Review Chapter Problem Wrap-Up Page 353

Solutions for the Chapter Problem Wrap-Up are shown in the Teacher's Resource.

Chapter 6 Practice Test

Chapter 6 Practice Test Question 1 Page 354

C is the correct answer.

Velocity has both magnitude and direction associated with it. Speed only has magnitude.
Answers A, B, and D are incorrect.

Chapter 6 Practice Test Question 2 Page 354

C is the correct answer.

Your velocity has constant magnitude but your direction is constantly changing.
A is incorrect. Velocity requires that a direction be specified.
B is incorrect. Your speed is always 70 km/h.
D is incorrect.

Chapter 6 Practice Test Question 3 Page 354

D is the correct answer.

A and B are the same vectors. Two answers cannot be correct.
C has the sum of two vectors pointing right. The resultant would have to point right as well.

Chapter 6 Practice Test Question 4 Page 354

D is the correct answer.

A is incorrect since the package will have momentum; i.e., it will continue to travel in the direction of the plane.
B is partially correct. The motion of the package is dependent on the force of gravity and its initial velocity as determined by the velocity of the plane.
C is incorrect as it ignores the force of gravity.
D is most correct as the motion will depend on the force of gravity and its initial velocity as determined by the velocity of the plane but also the velocity of any wind present during its descent.

Chapter 6 Practice Test Question 5 Page 354

True.

Gravity only acts in a downward direction. For the horizontal velocity to change there needs to be an unbalanced horizontal force acting on the projectile.

Chapter 6 Practice Test Question 6 Page 354

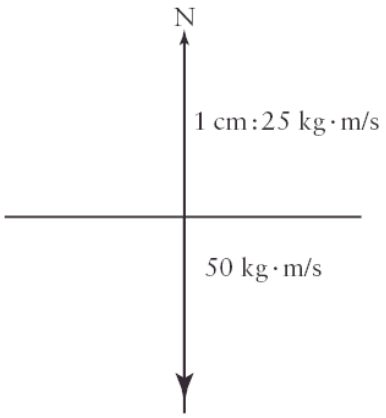
a) 310° (360 – 50 = 310°)

b) 010°

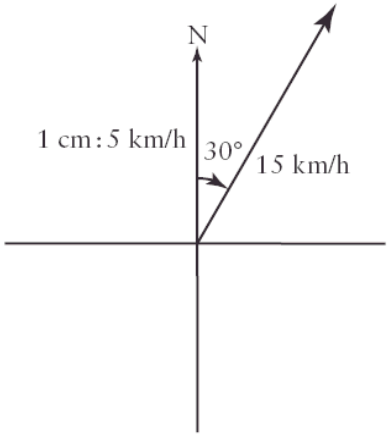
c) 140° (180° – 40° = 140°)

Chapter 6 Practice Test **Question 7 Page 354**

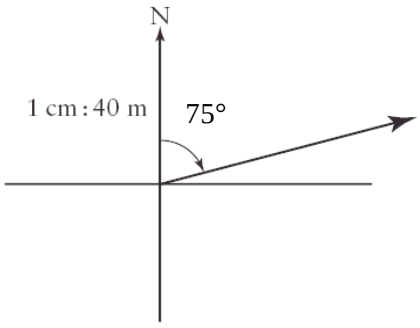
a)



b)



c)



Chapter 6 Practice Test **Question 8 Page 354**

a) $\overrightarrow{AE} + \overrightarrow{EB} = \overrightarrow{AB}$

b) $\overrightarrow{AC} - \overrightarrow{BC} = \overrightarrow{AC} + \overrightarrow{CB}$
$= \overrightarrow{AB}$

c) $\overrightarrow{CE} + \overrightarrow{DB} + \overrightarrow{AD} = \overrightarrow{CE} + \overrightarrow{EA} + \overrightarrow{AD}$
$= \overrightarrow{CD}$

d) $\overrightarrow{DB} - \overrightarrow{EA} - \overrightarrow{DE} = \overrightarrow{DB} + \overrightarrow{AE} + \overrightarrow{ED}$
$= \left(\overrightarrow{DB} + \overrightarrow{BD}\right) + \overrightarrow{ED}$
$= \vec{0} + \overrightarrow{ED}$
$= \overrightarrow{ED}$

Chapter 6 Practice Test **Question 9 Page 355**

a) $\overrightarrow{DE}$

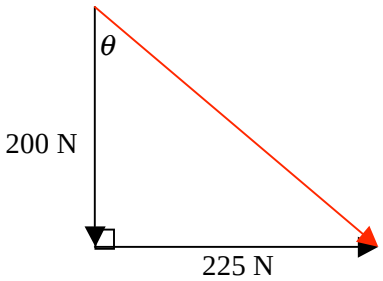
b) $\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$

c) $\overrightarrow{AB} - \overrightarrow{AD} = \overrightarrow{DA} + \overrightarrow{AB}$
$= \overrightarrow{DB}$

d) $\overrightarrow{AE} - \overrightarrow{CD} + \overrightarrow{BD} - \overrightarrow{AD} = \overrightarrow{AE} + \overrightarrow{DC} + \overrightarrow{BD} + \overrightarrow{DA}$
$= \left(\overrightarrow{DA} + \overrightarrow{AE}\right) + \left(\overrightarrow{BD} + \overrightarrow{DC}\right)$
$= \overrightarrow{DE} + \overrightarrow{BC}$
$= \overrightarrow{DE} + \overrightarrow{ED}$
$= \vec{0}$

Chapter 6 Practice Test **Question 10 Page 355**

Draw a diagram.



Use the Pythagorean theorem to find the magnitude of the resultant.

$$\left|\vec{R}\right|^2 = 200^2 + 225^2$$
$$= 90\,625$$
$$\left|\vec{R}\right| \doteq 301.04$$

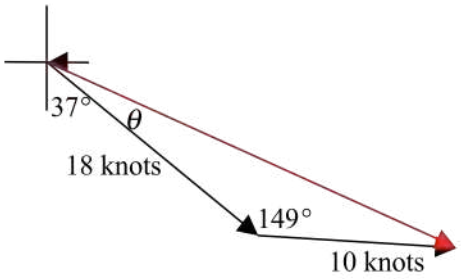
To find the direction of the resultant force, use trigonometry.
Let θ represent the angle of $\vec{R}$ to the sideline as shown.

$$\tan\theta = \frac{225}{200}$$
$$\theta = \tan^{-1}\left(\frac{225}{200}\right)$$
$$\doteq 48.4^\circ$$

The resultant force is about 301.0 N and its direction is about 48.4° away from the sideline.

Chapter 6 Practice Test **Question 10 Page 355**

Draw a diagram.
The angle between the velocity vectors is 143° – 112° = 31° (when they are tail to tail).
Thus, the angle between them in the diagram is 180° – 31° = 149°.
Let θ be the angle between the ship's velocity vector and the resultant.



To find the magnitude of the resultant, use the cosine law.

$$\left|\vec{R}\right|^2 = 18^2 + 10^2 - 2(18)(10)\cos 149^0$$
$$\doteq 732.58$$
$$\left|\vec{R}\right| \doteq 27.07$$

To find the direction of the resultant, use the sine law.

$$\frac{\sin\theta}{10} = \frac{\sin 149^\circ}{27.07}$$
$$\sin\theta = \frac{10\sin 149^\circ}{27.07}$$
$$\theta = \sin^{-1}\left(\frac{10\sin 149^\circ}{27.07}\right)$$
$$\doteq 10.97^\circ$$

For the bearing, 153° – 11° = 132°.

The ground velocity of the ship is about 27.1 knots at a bearing of 132°.

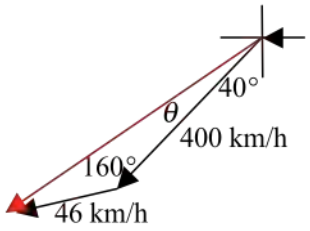
Chapter 6 Practice Test **Question 12 Page 355**

a) The force parallel to the ramp is 150cos 10° = 147.7 N.
The force perpendicular to the ramp is 150 sin 10°= 26.0 N.

b) Answers may vary. For example: The crate is kept at rest by a force of 147.7 N parallel to the ramp that stops the crate from sinking into the surface of the ramp and by the friction force of 26.0 N perpendicular to the surface of the ramp, pointing upward that prevents the crate from sliding down the ramp.

Chapter 6 Practice Test **Question 13 Page 355**

Draw a diagram.
The angle between the velocity vectors is 240° – 220° = 20° (when they are tail to tail).
Thus, the angle between them in the diagram is 180° – 20° = 160°.
Let θ be the angle between the airplane's velocity vector and the resultant.



To find the magnitude of the resultant, use the cosine law.

$$\left|\vec{R}\right|^2 = 400^2 + 46^2 - 2(400)(46)\cos 160^0$$
$$\doteq 196\,696.7$$
$$\left|\vec{R}\right| \doteq 443.5$$

To find the direction of the resultant, use the sine law.

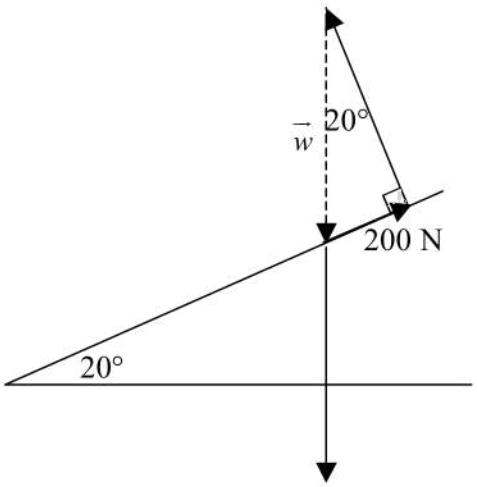
$$\frac{\sin\theta}{46} = \frac{\sin 160^o}{443.5}$$
$$\sin\theta = \frac{46\sin 160^o}{443.5}$$
$$\theta = \sin^{-1}\left(\frac{46\sin 160^o}{443.5}\right)$$
$$\doteq 2.03^o$$

For the bearing, 220° + 2° = 222°.

The ground velocity of the airplane is about 443.5 km/h at a bearing of 222°.

Chapter 6 Practice Test **Question 14 Page 355**

Draw a diagram.



Use trigonometry to find the weight of the cart, $\left|\vec{w}\right|$.

$$\frac{\left|\vec{w}\right|}{200} = \csc 20^\circ$$

$$\left|\vec{w}\right| = 200\csc 20^\circ$$

$$= \frac{200}{\sin 20^\circ}$$

$$\doteq 584.76$$

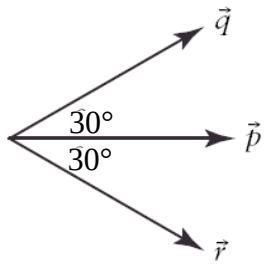
Convert to kg: $\frac{584.76}{9.8} \doteq 59.7$

The weight of Devon's father and his chair is 59.7 kg.

The component of the weight that is perpendicular to the surface of the ramp is $584.8\cos 20^\circ \doteq 549.5$ N.

Chapter 6 Practice Test **Question 15 Page 355**

Draw a diagram.



Consider scalar multiples $k\vec{q}$ and $k\vec{r}$. These vectors are parallel to $\vec{q}$ and $\vec{r}$ and have lengths 3k each.

The vertical components of $k\vec{q}$ and $k\vec{r}$ will be equal but in opposite directions.

The horizontal components will each be $3k\cos 30^\circ$.

To solve for k,

$$3k\cos 30^\circ + 3k\cos 30^\circ = 3$$

$$k\cos 30^\circ + k\cos 30^\circ = 1$$

$$k = \frac{1}{2\cos 30^\circ}$$

$$= \frac{1}{2\left(\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{1}{\sqrt{3}}$$

Thus $\vec{p} = \frac{1}{\sqrt{3}}\vec{q} + \frac{1}{\sqrt{3}}\vec{r}$.

Chapter 6 Practice Test **Question 16 Page 355**

a) Consider only the vertical motion of the firework shell.

The function for this motion is $\vec{h} = \vec{v}_0 t + \frac{1}{2}\vec{g}t^2$.

Our given information (where all vectors are vertical up) is: $\vec{g} = -9.8$

$$t = 4.8$$

$$\vec{h} = 100$$

Solve for $\vec{v}_0$.

$$100 = \vec{v}_0(4.8) + (0.5)(-9.8)(4.8)^2$$

$$\vec{v}_0 = \frac{100 + (4.9)(4.8)^2}{4.8}$$

$$\doteq 44.35$$

The initial velocity is 44.4 m/s upward.

However, this is only the vertical component. To find the resultant velocity,

$$\left|\vec{v}\right| = \frac{44.4}{\sin 82^\circ}$$

$$\doteq 44.8$$

The actual initial velocity is 44.8 m/s at an angle of 82° with the horizontal.

b) The distance from the launch site to a point directly below the place where the fireworks explode can be found by considering the uniform motion in the horizontal direction.

$t = 4.8$ s

$$\vec{v}_h = 44.8\cos 82°$$

$$\doteq 6.2$$

$$d = 6.2 \times 4.8$$
$$\doteq 29.9$$

The barriers should be placed about 129.9 m from the launch site.

c) Assume that the fireworks are travelling towards the spectators. Also assume that there is no wind that could force the fireworks or debris to travel off course.

Chapter 6 Practice Test **Question 17 Page 355**

Since the two velocities are at right angles to each other, use the Pythagorean theorem to find the magnitude of the resultant velocity.

$$\left|\vec{R}\right|^2 = 0.5^2 + 9.3^2$$
$$= 86.74$$
$$\left|\vec{R}\right| \doteq 9.3$$

To find the direction of the resultant force, use trigonometry.
Let θ represent the angle of $\vec{R}$ to the horizontal.

$$\tan\theta = \frac{0.5}{9.3}$$
$$\theta = \tan^{-1}\left(\frac{0.5}{9.3}\right)$$
$$\doteq 3.1^\circ$$

The glider is travelling at 9.3 m/s at angle of 3.1° below the horizontal.
The low angle of descent indicates that the glider will be airborne for a long time.

Chapter 6 Practice Test **Question 18 Page 355**

a) Since the forces are perpendicular, use the Pythagorean theorem to find the magnitude of the resultant force.

$$\left|\vec{R}\right|^2 = 20\,000^2 + 12\,000^2$$
$$= 544\,000\,000$$
$$\left|\vec{R}\right| \doteq 23\,323.8$$

To find the direction of the resultant force, use trigonometry.

Let θ represent the angle of $\vec{R}$ to the horizontal.

$$\tan\theta = \frac{12\ 000}{20\ 000}$$

$$\theta = \tan^{-1}\left(\frac{12\ 000}{20\ 000}\right)$$

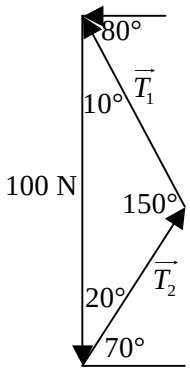
$$\doteq 30.96^\circ$$

The resultant force on the car is 23 323.8 N at 31.0° above the horizontal.

b) Answers may vary. For example:
It is important to know these components in order to determine the maximum weight that can be towed by the tow truck and the angle at which the car should be towed.

Chapter 6 Practice Test **Question 19 Page 355**

Draw a diagram of the situation.



b) To find $\vec{T}_1$, use the sine law.

$$\frac{\left|\vec{T}_1\right|}{\sin 20^\circ} = \frac{100}{\sin 150^\circ}$$

$$\left|\vec{T}_1\right| = \frac{100\sin 20^\circ}{\sin 150^\circ}$$

$$\doteq 68.4$$

To find $\vec{T}_2$, use the sine law a second time.

$$\frac{\left|\vec{T}_2\right|}{\sin 10^\circ} = \frac{100}{\sin 150^\circ}$$

$$\left|\vec{T}_2\right| = \frac{100\sin 10^\circ}{\sin 150^\circ}$$

$$\doteq 34.7$$

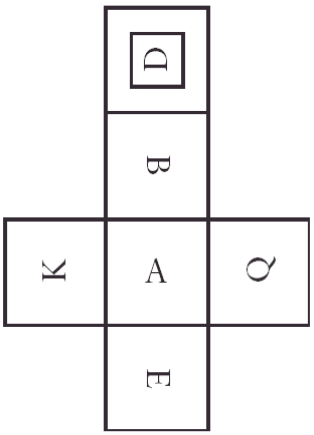
The tensions are about 68.4 N and 34.7 N in the directions of the two cables (80° and 70° below the horizontal respectively).

c) Answers may vary. For example:
If the 70°-angle cable were lengthened, then the other cable would become more vertical and have its tension increased. The lengthened cable would be pulled a bit less vertical and have its tension reduced.

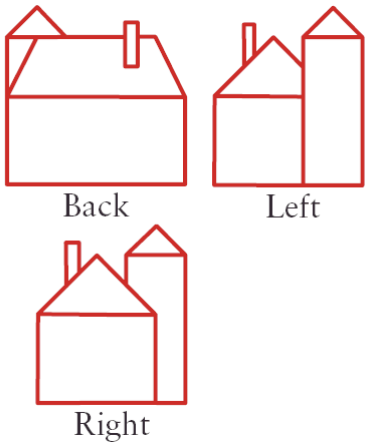
Chapter 7 **Cartesian Vectors**

Chapter 7 Prerequisite Skills

Chapter 7 Prerequisite Skills **Question 1 Page 358**

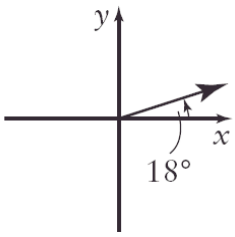


Chapter 7 Prerequisite Skills **Question 2 Page 358**

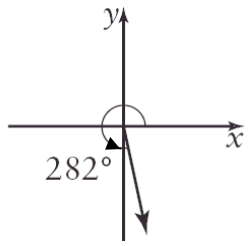


Chapter 7 Prerequisite Skills **Question 3 Page 358**

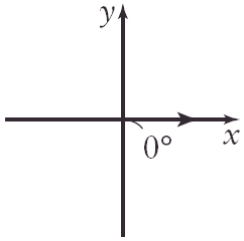
a) 18°



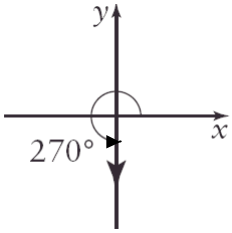
b) 282°



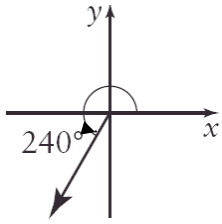
c) 0°



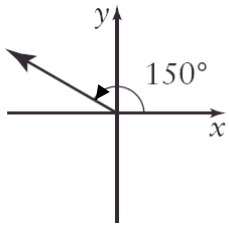
d) 270°



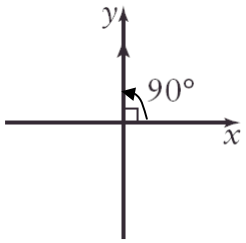
e) 240°



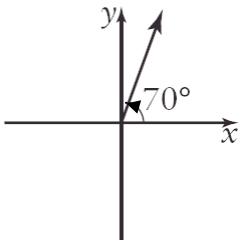
f) 150°



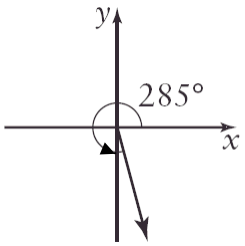
g) 90°



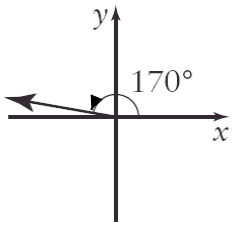
h) 70°



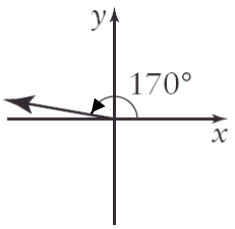
i) 285°



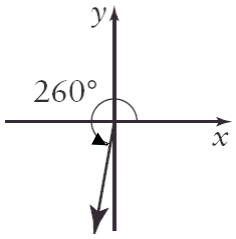
j) 220°



k) 170°



l) 260°



Chapter 7 Prerequisite Skills **Question 4 Page 358**

a) $\sqrt{(3-5)^2+(1-6)^2} = \sqrt{4+25}$
$= \sqrt{29}$

The distance is $\sqrt{29}$ units.

b) $\sqrt{(6-(-4))^2+(7-3)^2} = \sqrt{10^2+4^2}$
$= \sqrt{116}$
$= 2\sqrt{29}$

The distance is $2\sqrt{29}$ units.

c) $\sqrt{(-5-(-1))^2+(8-0)^2} = \sqrt{(-4)^2+8^2}$
$= \sqrt{80}$
$= 4\sqrt{5}$

The distance is $4\sqrt{5}$ units.

d) $\sqrt{(-3-5)^2+(-9-(-2))^2} = \sqrt{(-8)^2+(-7)^2}$
$= \sqrt{113}$

The distance is $\sqrt{113}$ units.

Chapter 7 Prerequisite Skills **Question 5 Page 358**

Answers may vary. For example: Let $a = 4$, $x = 3$, $y = 7$, and $z = 5$.

a) Verify: $x + y = y + x$.

L.S. $= x + y$
$= 3 + 7$
$= 10$

R.S. $= y + x$
$= 7 + 3$
$= 10$

Therefore, L.S. = R.S.

In words, when adding two numbers, the order of the operation does not matter.

b) Verify: $x \times y = y \times x$.

L.S. $= x \times y$
$= 3 \times 7$
$= 21$

R.S. $= y \times x$
$= 7 \times 3$
$= 21$

Therefore, L.S. = R.S.

In words, when multiplying two numbers, the order of the operation does not matter.

c) Verify: $(x + y) + z = x + (y + z)$.

L.S. $= (x + y) + z$
$= (3 + 7) + 5$
$= 10 + 5$
$= 15$

R.S. $= x + (y + z)$
$= 7 + (3 + 5)$
$= 7 + 8$
$= 15$

Therefore, L.S. = R.S.

In words, when adding three numbers at a time, the grouping of the operations does not matter.

d) Verify: $(x \times y) \times z = x \times (y \times z)$.

L.S. $= (x \times y) \times z$
$= (3 \times 7) \times 5$
$= 21 \times 5$
$= 105$

R.S. $= x \times (y \times z)$
$= 7 \times (3 \times 5)$
$= 7 \times 15$
$= 105$

Therefore, L.S. = R.S.

In words, when multiplying three numbers at a time, the grouping of the operations does not matter.

e) Verify: $a(x + y) = ax + ay$.

L.S. $= a(x + y)$
$= 4(3 + 7)$
$= 4(10)$
$= 40$

R.S. $= ax + ay$
$= 4(3) + 4(7)$
$= 12 + 28$
$= 40$

Therefore, L.S. = R.S.

In words, when multiplying a binomial by a factor, you can multiply each of the terms in the binomial separately and then add the partial products together.

f) Verify: $x - y \neq y - x$.

L.S. $= x - y$
$= 3 - 7$
$= -4$

R.S. $= y - x$
$= 7 - 3$
$= 4$

Therefore, L.S. $\neq$ R.S.

In words, when subtracting two numbers, the order of the operation does matter. When the order of operation is reversed the answer is of the opposite sign.

Chapter 7 Prerequisite Skills **Question 6 Page 358**

a)

$5x + 3y = 11$	①
$2x + y = 4$	②
$-5x - 3y = -11$	$-1$①
$6x + 3y = 12$	3②
$x = 1$	$-1$① + 3②

Substitute $x = 1$ into equation ②.

$2(1) + y = 4$

$y = 2$

Therefore, $(x, y) = (1, 2)$.

b)

$2x + 6y = 14$ ①

$x - 4y = -14$ ②

$2x + 6y = 14$ ①

$-2x + 8y = 28$ $-2$②

$14y = 42$ ① $- 2$②

$y = 3$

Substitute $y = 3$ into equation ②.

$x - 4(3) = -14$

$x = -2$

Therefore, $(x, y) = (-2, 3)$.

c)

$3x - 5y = -5$ ①

$-6x + 2y = 2$ ②

$6x - 10y = -10$ 2①

$-6x + 2y = 2$ ②

$-8y = -8$ 2① + ②

$y = 1$

Substitute $y = 1$ into equation ②.

$-6x + 2(1) = 2$

$-6x = 0$

$x = 0$

Therefore, $(x, y) = (0, 1)$.

d)

$-1.5x + 3.2y = 10$ ①

$0.5x + 0.4y = 4$ ②

$-1.5x + 3.2y = 10$ ①

$1.5x + 1.2y = 12$ 3②

$4.4y = 22$ ① + 3②

$y = 5$

Substitute $y = 5$ into equation ②.

$0.5x + 0.4(5) = 4$

$0.5x = 2$

$x = 4$

Therefore, $(x, y) = (4, 5)$.

Chapter 7 Prerequisite Skills **Question 7 Page 358**

B is the correct response. It is not equivalent.

A The equation can be multiplied by $\frac{1}{3}$ to get this result.

B If you rearrange the equation into slope *y*-intercept form, the result is

$$9x - 6y = 18$$
$$-6y = -9x + 18$$
$$y = \frac{-9}{-6}x + \frac{18}{-6}$$
$$y = \frac{3}{2}x - 3$$

This is not the same as B.

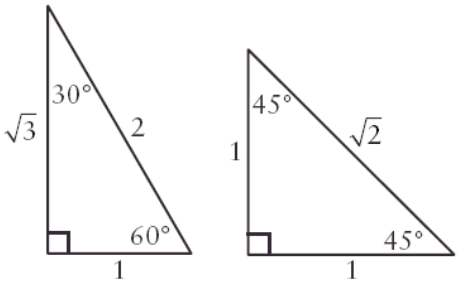
C The equation can be multiplied by $-\frac{1}{3}$ and rearranged to get this result.

D The equation can be multiplied by $\frac{1}{9}$ to get this result.

$$9x - 6y = 18$$
$$\frac{9}{9}x - \frac{6}{9}y = \frac{18}{9}$$
$$x - \frac{2}{3}y = 2$$

Chapter 7 Prerequisite Skills **Question 8 Page 359**

Use the special triangles.



a) $\frac{1}{2}$ **b)** $\frac{1}{2}$ **c)** $\frac{\sqrt{3}}{2}$ **d)** $\frac{1}{\sqrt{2}}$

e) $\frac{1}{\sqrt{2}}$ **f)** 0 **g)** 1 **h)** 0

i) $\cos 120^\circ = -\cos 60^\circ = -\frac{1}{2}$ **j)** $\sin 300^\circ = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$ **k)** 1 **l)** –1

Chapter 7 Prerequisite Skills **Question 9 Page 359**

a) 0.3 **b)** 0.7 **c)** −0.6

d) −0.9 **e)** 0.8 **f)** 0.8

Chapter 7 Prerequisite Skills **Question 10 Page 359**

a) Use the Pythagorean theorem

$$x^2 + 5^2 = 13^2$$
$$x^2 = 169 - 25$$
$$x^2 = 144$$
$$x = 12$$

Therefore, x = 12 cm.

b) Use the Pythagorean theorem.

$$y^2 = 4.5^2 + 8.9^2$$
$$y^2 = 20.25 + 79.21$$
$$y^2 = 99.46$$
$$y \doteq 9.97$$
$$y \doteq 10.0$$

Therefore, $y \doteq$ 10.0 cm.

Chapter 7 Prerequisite Skills **Question 11 Page 359**

a) $\sin 38^\circ = \dfrac{h}{5}$

$$h = 5\sin 38°$$

b) $3^2 + 4^2 = 5^2$

The missing side of the base triangle has a length of 4 units.

$$\sin 30^\circ = \frac{4}{h}$$
$$h = \frac{4}{\sin 30^\circ}$$

Chapter 7 Prerequisite Skills **Question 12 Page 359**

a) $(a_1 + b_1)^2 = (a_1 + b_1)(a_1 + b_1)$

$$= a_1{}^2 + a_1b_1 + a_1b_1 + b_1{}^2$$
$$= a_1{}^2 + 2a_1b_1 + b_1{}^2$$

b) $(a_1 + b_1)(a_1 - b_1) = a_1^2 - a_1b_1 + a_1b_1 - b_1^2$

$= a_1^2 - b_1^2$

c) $(a_1^2 + a_2^2)(b_1^2 + b_2^2) = a_1^2b_1^2 + a_1^2b_2^2 + a_2^2b_1^2 + a_2^2b_2^2$

d) $(a_1b_1 + a_2b_2 + a_3b_3)^3$

$= [(a_1b_1 + a_2b_2 + a_3b_3)(a_1b_1 + a_2b_2 + a_3b_3)](a_1b_1 + a_2b_2 + a_3b_3)$

$= [a_1^2b_1^2 + a_1b_1a_2b_2 + a_1b_1a_3b_3 + a_2b_2a_1b_1 + a_2^2b_2^2 + a_2b_2a_3b_3 + a_3b_3a_1b_1 + a_3b_3a_2b_2 + a_3^2b_3^2](a_1b_1 + a_2b_2 + a_3b_3)$

$= a_1^3b_1^3 + a_1^2a_2b_1^2b_2 + a_1^2a_3b_1^2b_3 + a_1^2a_2b_1^2b_2 + a_1a_2^2b_1b_2^2 + a_1a_2a_3b_1b_2b_3 + a_1^2a_3b_1^2b_3 + a_1a_2a_3b_1b_2b_3 + a_1a_3^2b_1b_3^2$

$+ a_1^2a_2b_1^2b_2 + a_1a_2^2b_1b_2^2 + a_1a_2a_3b_1b_2b_3 + a_1a_2^2b_1b_2^2 + a_2^3b_2^3 + a_2^2a_3b_2^2b_3 + a_1a_2a_3b_1b_2b_3 + a_2^2a_3b_2^2b_3 + a_2a_3^2b_2b_3^2$

$+ a_1^2a_3b_1^2b_3 + a_1a_2a_3b_1b_2b_3 + a_1a_3^2b_1b_3^2 + a_1a_2a_3b_1b_2b_3 + a_2^2a_3b_2^2b_3 + a_2a_3^2b_2b_3^2 + a_1a_3^2b_1b_3^2 + a_2a_3^2b_2b_3^2 + a_3^3b_3^3$

$= a_1^3b_1^3 + a_2^3b_2^3 + a_3^3b_3^3 + 3a_1^2a_2b_1^2b_2 + 3a_1a_2^2b_1b_2^2 + 3a_1^2a_3b_1^2b_3 + 3a_1a_3^2b_1b_3^2 + 3a_2^2a_3b_2^2b_3 + 3a_2a_3^2b_2b_3^2 + 6a_1a_2a_3b_1b_2b_3$

Chapter 7 Prerequisite Skills **Question 13 Page 359**

a) $A = \frac{1}{2}\left|0(2) - 0(0) + 0(2) - 2(2) + 2(1) - 3(2) + 3(-1) - 3(1) + 3(0) - 0(0)\right|$

$= \frac{1}{2}\left|-14\right|$

$= 7$

The area is 7 square units.

b) $A = \frac{1}{2}\left|1(-2) - 3(1) + 3(-4) - 0(-2) + 0(0) - (-5)(-4) + (-5)(4) - (-3)(0) + (-3)(1) - 1(4)\right|$

$= \frac{1}{2}\left|-64\right|$

$= 32$

The area is 32 square units.

c) $A = \frac{1}{2}\left|4(-7) - 5(-10) + 5(2) - 3(-7) + 3(5) - 0(2) + 0(3) - (-2)(5) + (-2)(-10) - 4(3)\right|$

$= \frac{1}{2}\left|86\right|$

$= 43$

The area is 43 square units.

Chapter 7 Prerequisite Skills **Question 13 Page 359**

Yes. There are similar formulas for triangles and hexagons. In fact, there are similar formulas for any polygon that does not intersect itself.

For a triangle with vertices (x_1,y_1), (x_2,y_2), and (x_3,y_3), the area is

$$A = \frac{1}{2}\left|x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3\right|$$

Answers may vary. For example, Consider a triangle with vertices (1, 1), (2, 7), and (–3, –4), the area is

$$A = \frac{1}{2}\left|1(7) - 2(1) + 2(-4) - (-3)(7) + (-3)(1) - 1(-4)\right|$$
$$= \frac{1}{2}\left|19\right|$$
$$= 9.5$$

The area is 9.5 square units.

For a hexagon with vertices (x_1,y_1), (x_2,y_2), (x_3,y_3), (x_4,y_4), (x_5,y_5), and (x_6,y_6), the area is

$$A = \frac{1}{2}\left|x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_4 - x_4y_3 + x_4y_5 - x_5y_4 + x_5y_6 - x_6y_5 + x_6y_1 - x_1y_6\right|$$

For a hexagon with vertices (0, 0), (3, 2), (2, 4), (0, 5), (–2, 4), and (–3, 2), the area is

$$A = \frac{1}{2}\left|0(2) - 3(0) + 3(4) - 2(2) + 2(5) - 0(4) + 0(4) - (-2)(5) + (-2)(2) - (-3)(4) + (-3)(0) - 0(2)\right|$$
$$= \frac{1}{2}\left|36\right|$$
$$= 18$$

The area is 18 square units.

Chapter 7 Section 1 **Cartesian Vectors**

Chapter 7 Section 1 **Question 1 Page 367**

a) $2\vec{i} + \vec{j}$ **b)** $3\vec{i} - 5\vec{j}$

c) $-3\vec{i} - 6\vec{j}$ **d)** $5\vec{i}$

e) $9\vec{i} - 7\vec{j}$ **f)** $-8\vec{j}$

g) $-6\vec{i}$ **i)** $-5.2\vec{i} - 6.1\vec{j}$

Chapter 7 Section 1 **Question 2 Page 367**

a) [1, 1] **b)** [–4, 0] **c)** [0, 2] **d)** [3, 8]

e) [–5, –2] **f)** [7, –4] **g)** [0, –8.2] **h)** [–2.5, 3.3]

Chapter 7 Section 1 **Question 3 Page 367**

a) $\overrightarrow{AB} = [2, -5]$ **b)** $\overrightarrow{CD} = [5, 1]$

c) $\overrightarrow{EF} = [0, 7]$ **d)** $\overrightarrow{GH} = [-2, 9]$

Chapter 7 Section 1 **Question 4 Page 367**

a) $\overrightarrow{AB} = \sqrt{2^2 + (-5)^2}$
$= \sqrt{29}$

The magnitude of the vector is $\sqrt{29}$ units.

b) $\overrightarrow{CD} = \sqrt{5^2 + 1^2}$
$= \sqrt{26}$

The magnitude of the vector is $\sqrt{26}$ units.

c) $\overrightarrow{EF} = \sqrt{0^2 + 7^2}$
$= 7$

The magnitude of the vector is 7 units.

d) $\overrightarrow{GH} = \sqrt{(-2)^2 + 9^2}$
$= \sqrt{85}$

The magnitude of the vector is $\sqrt{85}$ units.

Chapter 7 Section 1 **Question 5 Page 368**

a) $\vec{v}_h = [5,0]$
$\vec{v}_v = [0,-1]$

b) The magnitude of $\vec{v}$, $\left|\vec{v}\right|$, is $\sqrt{26}$ units.

Unit vectors collinear with $\vec{v}$ have the form $\frac{1}{\left|\vec{v}\right|}\vec{v}$ and $-\frac{1}{\left|\vec{v}\right|}\vec{v}$.

Therefore, the required unit vectors are $\left[\frac{5}{\sqrt{26}}, -\frac{1}{\sqrt{26}}\right]$ and $\left[-\frac{5}{\sqrt{26}}, \frac{1}{\sqrt{26}}\right]$.

c) Let Q(x, y) be the point.

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$\left[5, -1\right] = [x, y] - \left[-2, -7\right]$$
$$\left[5, -1\right] = \left[x + 2, y + 7\right]$$

$\therefore x + 2 = 5$ and $y + 7 = -1$

$x = 3$ $\quad$ $y = -8$

The point is Q$(3, -8)$.

d) Let L(x, y) be the point.

$$\overrightarrow{LM} = \overrightarrow{OM} - \overrightarrow{OL}$$
$$\left[5, -1\right] = [5, 8] - \left[x, y\right]$$
$$\left[5, -1\right] = \left[5 - x, 8 - y\right]$$

$\therefore 5 - x = 5$ and $8 - y = -1$

$x = 0$ $\quad$ $y = 9$

The point is L$(0, 9)$.

Chapter 7 Section 1 **Question 6 Page 368**

a) $\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ}$

$= \left[-6, 1\right] - \left[-2, -1\right]$

$= \left[-4, 2\right]$

b) $\overrightarrow{RP} = [-6,1] - [-3,4]$

$= [-3,-3]$

$\left|\overrightarrow{RP}\right| = \sqrt{(-3)^2 + (-3)^2}$

$= \sqrt{18}$

$= 3\sqrt{2}$

The vector is $3\sqrt{2}$ units.

c) $\left|\overrightarrow{QP}\right| = \sqrt{(-4)^2 + 2^2}$

$= \sqrt{20}$

$\left|\overrightarrow{QR}\right| = \sqrt{((-3) - (-2))^2 + (4 - (-1))^2}$

$= \sqrt{26}$

$\left|\overrightarrow{RP}\right| = \sqrt{18}$

The perimeter is $\sqrt{18}$ units $+ \sqrt{26}$ units $+ \sqrt{20}$ units $\doteq 13.8$ units.

Chapter 7 Section 1 **Question 7 Page 368**

a) $8\vec{u} = 8[4,-1]$

$= [32,-8]$

b) $-8\vec{u} = -[32,-8]$

$= [-32,8]$

c) $\vec{u} + \vec{v} = [4,-1] + [2,7]$

$= [6,6]$

d) $\vec{v} - \vec{u} = \vec{v} + \left(-\vec{u}\right)$

$= [2,7] + [-4,1]$

$= [-2,8]$

e) $5\vec{u} - 3\vec{v} = 5[4,-1] - 3[2,7]$

$= [20,-5] + [-6,-21]$

$= [14,-26]$

f) $-4\vec{u}+7\vec{v}=-4\left[4,-1\right]+7\left[2,7\right]$

$=\left[-16,4\right]+\left[14,49\right]$

$=\left[-2,53\right]$

Chapter 7 Section 1 **Question 8 Page 368**

B is the correct response.

$\vec{a}=2\vec{b}$

$=-\vec{d}$

$=-\frac{2}{3}\vec{e}$

Therefore, $\vec{a}$, $\vec{b}$, $\vec{d}$, and $\vec{e}$ are collinear vectors.

Chapter 7 Section 1 **Question 9 Page 368**

a) $\vec{v}=\left[500\cos 30^{\circ}, 500\sin 30^{\circ}\right]$

$=\left[433.0, 250\right]$

b) $\vec{v}=\left[1000\cos 18^{\circ}, 1000\sin 18^{\circ}\right]$

$=\left[951.1, 309.0\right]$

c) $\vec{v}=\left[0, 125\right]$

d) $\vec{v}=\left[230, 0\right]$

e) $\vec{v}=\left[0, -25\right]$

f) $\vec{v}=\left[-650, 0\right]$

Chapter 7 Section 1 **Question 10 Page 368**

The ship's vector is $\left[30\cos(-102^{\circ}), 30\sin(-102^{\circ})\right]=\left[-6.237, -29.344\right]$.

The current's vector is $[14\cos 158^{\circ}, 14\sin 158^{\circ}]=\left[-12.981, 5.244\right]$.

The resultant vector is [–19.219, –24.100].

The magnitude of the resultant is $\sqrt{(-19.219)^2+(-24.100)^2}\doteq 30.825$.

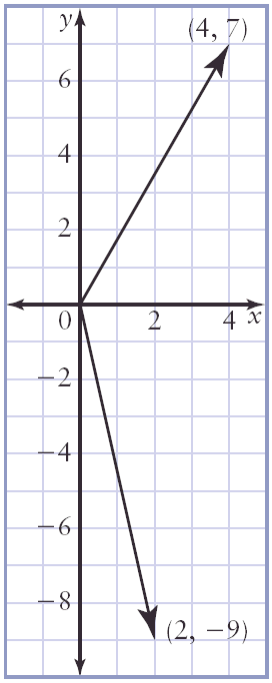
The angle of the resultant with the *west* direction is $-\tan^{-1}\left(\frac{24.100}{19.219}\right)\doteq -51.429^{\circ}$. (The resultant is pointing SW.)

The bearing is 270º – 51.4º = 218.6º.

The resultant is about 30.8 km/h on a bearing of 218.6°.

Chapter 7 Section 1 **Question 11 Page 368**

11. a)



b) $|\vec{a}| + |\vec{b}| = \sqrt{4^2 + 7^2} + \sqrt{2^2 + (-9)^2}$

$= \sqrt{57} + \sqrt{85}$

$\doteq 17.3$

$|\vec{a} + \vec{b}| = |[6, -2]|$

$= \sqrt{6^2 + 2^2}$

$= \sqrt{40}$

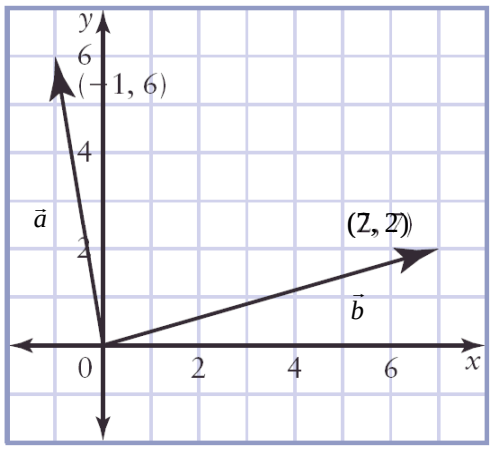
$\doteq 6.3$

$|\vec{a}| + |\vec{b}|$ is greater.

c) Answers may vary. For example: Yes, this will be true for all pairs of vectors, unless the two vectors are collinear. Then $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$.

Chapter 7 Section 1 **Question 12 Page 368**

a)



b) $\left|\vec{a}+\vec{b}\right| \doteq 10$

$$\begin{aligned}\left|\vec{a}-\vec{b}\right| &= \left|\left[-8, 4\right]\right| \\ &= \sqrt{(-8)^2+4^2} \\ &= \sqrt{80} \\ &\doteq 8.9\end{aligned}$$

$\left|\vec{a}+\vec{b}\right|$ is greater.

c) Answers may vary. For example: Yes this will be true for all pairs of vectors separated by an acute angle.

If the angle is 90º, $\left|\vec{a}+\vec{b}\right| = \left|\vec{a}-\vec{b}\right|$.

If the angle is obtuse, then $\left|\vec{a}+\vec{b}\right| < \left|\vec{a}-\vec{b}\right|$.

Chapter 7 Section 1 **Question 13 Page 368**

a) L.S. $= \left(\vec{u} + \vec{v}\right) + \vec{w}$

$= \left(\left[u_1, u_2\right] + \left[v_1, v_2\right]\right) + \left[w_1, w_2\right]$

$= \left(\left[u_1 + v_1,\ u_2 + v_2\right]\right) + \left[w_1, w_2\right]$

$= \left[u_1 + v_1 + w_1,\ u_2 + v_2 + w_2\right]$

R.S. $= \vec{u} + \left(\vec{v} + \vec{w}\right)$

$= \left[u_1, u_2\right] + \left(\left[v_1, v_2\right] + \left[w_1, w_2\right]\right)$

$= \left[u_1, u_2\right] + \left[v_1 + w_1,\ v_2 + w_2\right]$

$= \left[u_1 + v_1 + w_1,\ u_2 + v_2 + w_2\right]$

L.S. = R.S.

Therefore, $\left(\vec{u} + \vec{v}\right) + \vec{w} = \vec{u} + \left(\vec{v} + \vec{w}\right)$.

b) L.S. $= k\left(\vec{u} + \vec{v}\right)$

$= k\left(\left[u_1,\ u_2\right] + \left[v_1,\ v_2\right]\right)$

$= k\left[u_1 + v_1,\ u_2 + v_2\right]$

$= \left[k\left(u_1 + v_1\right),\ k\left(u_2 + v_2\right)\right]$

$= \left[ku_1 + kv_1,\ ku_2 + kv_2\right]$

R.S. $= k\vec{u} + k\vec{v}$

$= k\left[u_1,\ u_2\right] + k\left[v_1,\ v_2\right]$

$= \left[ku_1,\ ku_2\right] + \left[kv_1,\ kv_2\right]$

$= \left[ku_1 + kv_1,\ ku_2 + kv_2\right]$

L.S. = R.S.

Therefore, $k\left(\vec{u} + \vec{v}\right) = k\vec{u} + k\vec{v}$.

c) L.S. $= \vec{u} + \vec{v}$

$= \left[u_1, u_2\right] + \left[v_1, v_2\right]$

$= \left[u_1 + v_1, u_2 + v_2\right]$

R.S. $= \vec{v} + \vec{u}$

$= \left[v_1, v_2\right] + \left[u_1, u_2\right]$

$= \left[v_1 + u_1, v_2 + u_2\right]$

$= \left[u_1 + v_1, u_2 + v_2\right]$

Therefore, $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

d) L.S. $= \left(k + m\right)\vec{u}$

$= \left(k + m\right)\left[u_1, u_2\right]$

$= \left[\left(k + m\right)u_1, \left(k + m\right)u_2\right]$

$= \left[ku_1 + mu_1, ku_2 + mu_2\right]$

R.S. $= k\vec{u} + m\vec{u}$

$= k\left[u_1, u_2\right] + m\left[u_1, u_2\right]$

$= \left[ku_1, ku_2\right] + \left[mu_1, mu_2\right]$

$= \left[ku_1 + mu_1, ku_2 + mu_2\right]$

Therefore, $\left(k + m\right)\vec{u} = k\vec{u} + m\vec{u}$.

Chapter 7 Section 1 **Question 14 Page 368**

$\left|\vec{F}\right| = \left[180\cos 30^\circ, 180\sin 30^\circ\right]$

$\doteq \left[155.9, 90\right]$

Chapter 7 Section 1 **Question 15 Page 368**

$\left|\vec{F}\right| = \left[250\cos(-35^\circ), 250\sin(-35^\circ)\right]$

$\doteq \left[204.8, -143.4\right]$

Chapter 7 Section 1 **Question 16 Page 368**

The airplane's vector is $\left[550\cos 10^{\circ}, 550\sin 10^{\circ}\right] = \left[541.644, 95.506\right]$.

The wind's vector is $[60\cos 150^{\circ}, 60\sin 150^{\circ}] = \left[-51.962, 30\right]$.

The resultant vector is [489.682, 125.506].

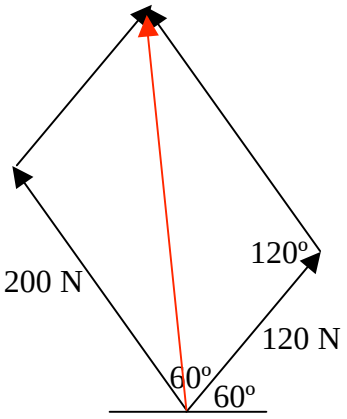
The magnitude of the resultant is $\sqrt{(489.682)^2 + (125.506)^2} \doteq 505.51$.

The angle of the resultant with the *east* direction is $\tan^{-1}\left(\frac{125.506}{489.682}\right) \doteq 14.376^{\circ}$. (The resultant is pointing NE.)

The bearing is $90^{\circ} - 14.4^{\circ} = 75.6^{\circ}$.

The resultant is about 505.5 km/h on a bearing of 075.6°.

Chapter 7 Section 1 **Question 17 Page 369**



The vector for Emily's kick is $[120\cos 60^{\circ}, 120\sin 60^{\circ}] = \left[60, 103.92\right]$.

The vector for Claire's kick is $[200\cos 120^{\circ}, 200\sin 120^{\circ}] = \left[-100, 173.21\right]$.

The resultant vector is [–40, 277.13].

The magnitude of the resultant is $\sqrt{\left(-40\right)^2 + \left(277.13\right)^2} \doteq 280.0$.

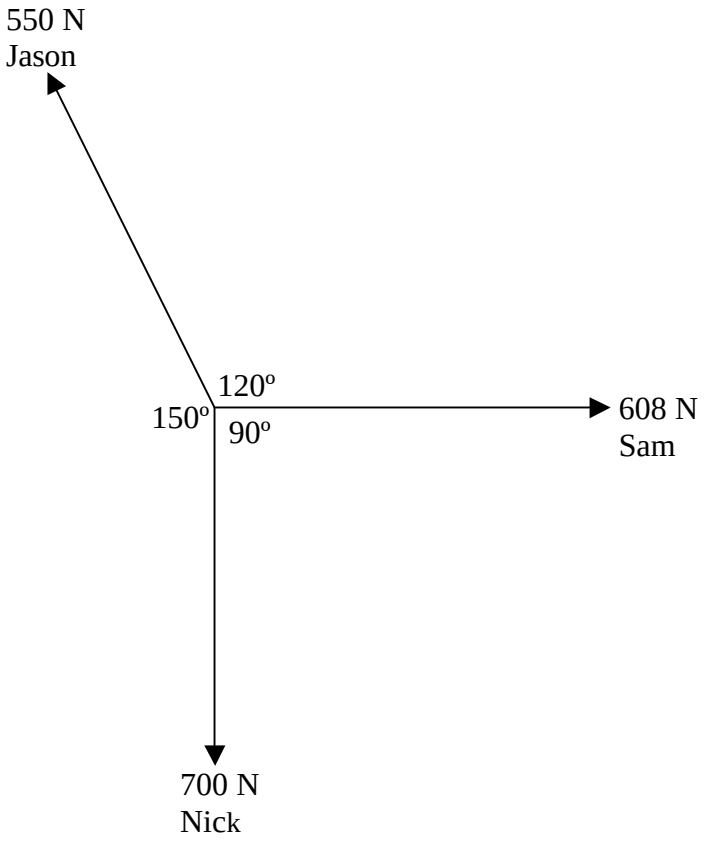
The angle of the resultant is $\tan^{-1}\left(\frac{277.13}{-40}\right) \doteq -81.79^{\circ}$.

The correct angle is (the opposite direction) $180^{\circ} - 81.8^{\circ} = 98.2^{\circ}$.
The resultant is about 280.0 N at an angle of 98.2° to the line between the centres of the goals.

This is the same answer as for question 11 in section 6.4.

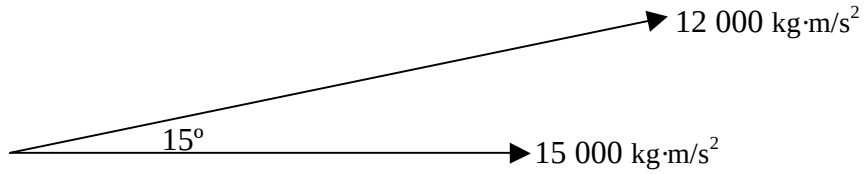
Chapter 7 Section 1 **Question 18 Page 369**

Draw a diagram.



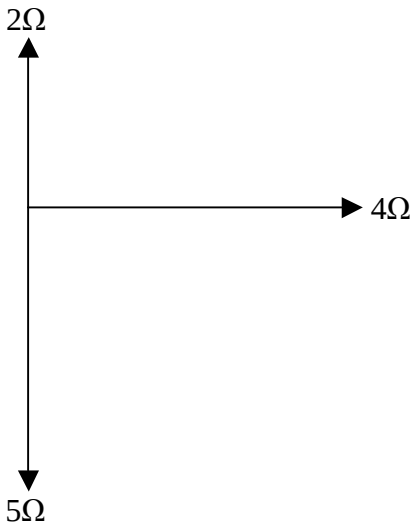
Use east as the reference direction.
The three force vectors are [608, 0], [550cos 120º, 550sin 120º], and [0, –700].
The resultant vector is [333, –223.7].
The magnitude of the resultant is $\sqrt{333^2 + (-223.7)^2} \doteq 401.2$.
The resultant angle is $\tan^{-1}\left(\frac{-223.7}{333}\right) \doteq -33.9^\circ$.
The resultant force on the basketball is about 401.2 N in a direction 33.9º below Sam's force (towards Nick's force).

Chapter 7 Section 1 **Question 19 Page 369**



Place the 15 000 kg·m/s^2 vector along the positive x–axis.
The two vectors are [15 000, 0] and [12 000cos 15°, 12 000sin 15°].
The sum of the vectors is [26 591.1, 3105.8].
The magnitude of the resultant is $\sqrt{(26\ 591.1)^2 + (3105.8)^2} \doteq 26\ 771.9$.
The resultant angle is $\tan^{-1}\left(\frac{3105.8}{26\ 591.1}\right) \doteq 6.66^\circ$.
The resultant momentum is about 26 771.9 kg·m/s^2 at an angle of 6.7° to the 15 000 kg·m/s^2 vector.

Chapter 7 Section 1 **Question 20 Page 369**



The three vectors are [4, 0], [0, 2], and [0, –5].
The vector sum is [4, –3].

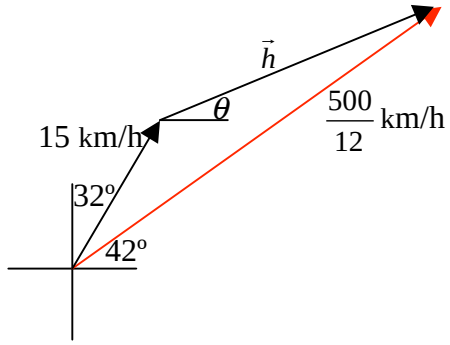
a) The magnitude of the resultant is $\sqrt{4^2 + (-3)^2} = 5\Omega$.

b) The angle of the resultant is $\tan^{-1}\left(\frac{-3}{4}\right) \doteq -36.9^\circ$.

The resultant makes a 36.9° below the positive x–axis.

Chapter 7 Section 1 **Question 21 Page 369**

Draw a diagram.



The resultant vector must be $\left[\frac{500}{12}\cos 42^\circ, \frac{500}{12}\sin 42^\circ\right]$.

The current vector is [15cos 58°, 15sin 58°].

The heading vector is the difference between the resultant vector and the current vector.

$$\vec{h} = \left[\frac{500}{12}\cos 42^\circ, \frac{500}{12}\sin 42^\circ\right] - [15\cos 58^\circ, 15\sin 58^\circ]$$

$$\doteq [23.016, 15.160]$$

$$\left|\vec{h}\right| = \sqrt{(23.016)^2 + (15.160)^2}$$

$$\doteq 27.560$$

$$\theta = \tan^{-1}\left(\frac{15.160}{23.016}\right)$$

$$\doteq 33.37$$

The bearing is 90° – 33.37° = 56.63°.

The captain should set a speed of 27.6 km/h at a bearing of 056.6°.

Chapter 7 Section 1 **Question 22 Page 369**

a) Solve for x.

$$\sqrt{x^2+(3x)^2}=9$$
$$10x^2=81 \qquad \text{Square both sides.}$$
$$x^2=\frac{81}{10}$$
$$x=\pm\frac{9}{\sqrt{10}}$$
$$x=\pm\frac{9\sqrt{10}}{10}$$

b) $\left|\left[2x,\ x\right]+\left[x,\ 2x\right]\right|=6$

$$\left|\left[3x,\ 3x\right]\right|=6$$
$$\sqrt{(3x)^2+(3x)^2}=6$$
$$\sqrt{18x^2}=6$$
$$18x^2=36$$
$$x^2=2$$
$$x=\pm\sqrt{2}$$

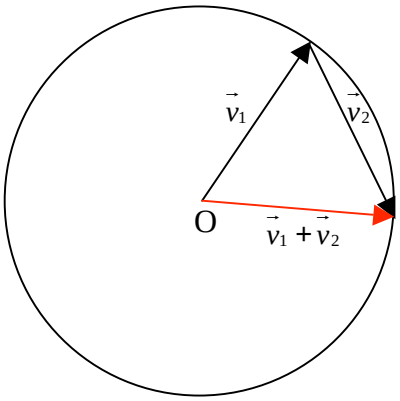
c) $\left|\left[3x,\ 7\right]+\left[5x,\ x\right]\right|=10x$

$$\left|\left[8x,\ x+7\right]\right|=10x$$
$$\sqrt{(8x)^2+(x+7)^2}=10x$$
$$\sqrt{64x^2+x^2+14x+49}=10x$$
$$\sqrt{65x^2+14x+49}=10x$$
$$65x^2+14x+49=100x^2$$
$$35x^2-14x-49=0$$
$$5x^2-2x-7=0$$
$$(5x-7)(x+1)=0$$

$x=\frac{7}{5}$ or $x=-1$

Chapter 7 Section 1 **Question 23 Page 369**

Yes. Consider one unit vector starting at the origin. Its tip will lie somewhere on the unit circle (circle with centre the origin and radius one unit). Draw a second unit vector starting at the tip of the first vector and having its tip also on the unit circle. The resulting sum vector will start at the origin and have its tip on the unit circle. Clearly this is a unit vector.



Chapter 7 Section 1 **Question 24 Page 369**

Find the x-intercept of the circle.

$$(x-7)^2 + (0-4)^2 = 25$$
$$(x-7)^2 = 9$$
$$x-7 = \pm 3$$

$x = 10$ or $x = 4$

The centre of the circle is (7, 4).

Solve for k. Set $y = 0$.

$$0 = 3(10)^2 - 42(10) + k$$
$$k = 120$$

Find the vertex of the parabola by completing the square.

$$y = 3x^2 - 42x + 120$$
$$y = 3(x^2 - 14x) + 120$$
$$y = 3(x^2 - 14x + 49 - 49) + 120$$
$$y = 3(x-7)^2 - 3(49) + 120$$
$$y = 3(x-7)^2 - 27$$

The vertex of the parabola is (7, –27).

The distance from the centre of the circle to the vertex of the parabola is 4 – (–27) or 31 units.

Chapter 7 Section 1 **Question 25 Page 369**

Method 1:

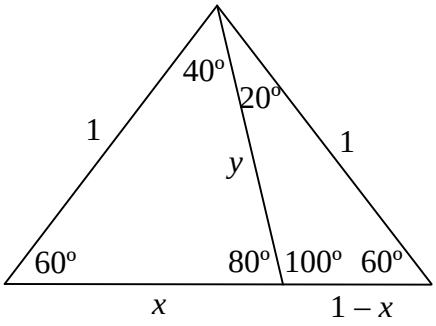
Use a trigonometric identity. Let $\angle A = 40^\circ$ and $\angle B = 20^\circ$.

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\begin{aligned}\sin 40^\circ + \sin 20^\circ &= 2\sin 30^\circ \cos 10^\circ \\ &= 2\left(\frac{1}{2}\right)\sin\left(90^\circ - 10^\circ\right) \\ &= \sin 80^\circ\end{aligned}$$

Method 2:

Start with an equilateral triangle.



Using the sine law in the two smaller triangles, you get

$$\begin{aligned}\frac{y}{\sin 60^\circ} &= \frac{1}{\sin 80^\circ} \\ &= \frac{1-x}{\sin 20^\circ} \\ &= \frac{x}{\sin 40^\circ}\end{aligned}$$

From the second and last fractions,

$$x = \frac{\sin 40^\circ}{\sin 80^\circ}$$

From the third and last actions,

$$1 - x = \frac{\sin 20^\circ}{\sin 40^\circ}$$

Adding these two equation together,

$$\begin{aligned}1 &= \frac{\sin 40^\circ}{\sin 80^\circ} + \frac{\sin 20^\circ}{\sin 80^\circ} \\ \sin 80^\circ &= \sin 40^\circ + \sin 20^\circ\end{aligned}$$

Chapter 7 Section 2 **Dot Product**

Chapter 7 Section 2 **Question 1 Page 375**

a) $\vec{a} \cdot \vec{b} = 70(115)\cos 70^\circ$
$\doteq 2753.3$

b) $\vec{c} \cdot \vec{d} = 8(12)\cos 150^\circ$
$\doteq -83.1$

c) $\vec{e} \cdot \vec{f} = 200(150)\cos 90^\circ$
$= 0$

d) $\vec{g} \cdot \vec{h} = 5000(4500)\cos 180^\circ$
$\doteq -22\ 500\ 000$

Chapter 7 Section 2 **Question 2 Page 376**

a) $\vec{u} \cdot \vec{v} = 6(10)\cos 30^\circ$
$\doteq 52.0$

b) $\vec{s} \cdot \vec{t} = 30(15)\cos 120^\circ$
$= -225$

c) $\vec{f} \cdot \vec{g} = 5.8(13.4)\cos 180^\circ$
$\doteq -77.7$

d) $\vec{q} \cdot \vec{r} = 4.0(6.1)\cos 90^\circ$
$= 0$

e) $\vec{a} \cdot \vec{b} = 850(400)\cos 58^\circ$
$\doteq 180\ 172.5$

f) $\vec{m} \cdot \vec{p} = 16(2)\cos 153^\circ$
$\doteq -28.5$

Chapter 7 Section 2 **Question 3 Page 376**

Answers may vary. For example:

a) $\vec{u} \cdot \left(k\vec{u} + \vec{v}\right) = k\left(\vec{u} \cdot \vec{u}\right) + \vec{u} \cdot \vec{v}$

$= k\left|\vec{u}\right|^2 + \vec{u} \cdot \vec{v}$

b) $\left(k\vec{u} - \vec{v}\right) \cdot \left(l\vec{v}\right) = kl\left(\vec{u} \cdot \vec{v}\right) - l\left(\vec{v} \cdot \vec{v}\right)$

$= kl\left(\vec{u} \cdot \vec{v}\right) - l\left|\vec{v}\right|^2$

c) $\left(\vec{u} - \vec{v}\right) \cdot \left(\vec{u} - \vec{v}\right) = \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v}$

$= \left|\vec{u}\right|^2 - 2\vec{u} \cdot \vec{v} + \left|\vec{v}\right|^2$

d) $\left(\vec{u} + \vec{v}\right) \cdot \left(\vec{w} + \vec{x}\right) = \vec{u} \cdot \vec{w} + \vec{u} \cdot \vec{x} + \vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{x}$

Chapter 7 Section 2 **Question 4 Page 376**

a) $\vec{u} \cdot \vec{v} = 2(3) + 4(-1)$

$= 2$

b) $\vec{m} \cdot \vec{n} = -5(0) + (-7)(7)$

$= -49$

c) $\vec{s} \cdot \vec{t} = 9(3) + (-3)(-3)$

$= 36$

d) $\vec{p} \cdot \vec{q} = -6(9) + 2(1)$

$= -52$

e) $\vec{a} \cdot \vec{b} = 2(9) + 3(-7)$

$= -3$

f) $\vec{s} \cdot \vec{t} = 4(-1) + 1(-1)$

$= -5$

Chapter 7 Section 2 **Question 5 Page 376**

a) $\vec{v} \cdot \vec{w}$ is a scalar; a vector dotted with a scalar is not defined.

b) This expression has meaning. It is the absolute value of a scalar (the dot product).

c) This expression has meaning. $\vec{v} \cdot \vec{w}$ is a scalar. This is a vector multiplied by a scalar.

d) This expression has meaning. This is the magnitude of a vector, squared.

e) This expression does not have meaning. Multiplying two vectors has not been defined.

f) This expression has meaning. $\vec{u} \cdot \vec{v}$ is a scalar which can be squared.

Chapter 7 Section 2 **Question 6 Page 376**

a) Using geometric vectors:

$$\begin{aligned}\vec{i} \cdot \vec{j} &= \left|\vec{i}\right|\left|\vec{j}\right| \cos\theta \\ &= 1(1)\cos 90° \\ &= 1(0) \\ &= 0\end{aligned}$$

Using Cartesian vectors:

$$\begin{aligned}\vec{i} \cdot \vec{j} &= [1, 0] \cdot [0, 1] \\ &= 1(0) + 0(1) \\ &= 0 + 0 \\ &= 0\end{aligned}$$

b) The dot product of the two unit vectors $\vec{i}$ and $\vec{j}$ is 0 using either method. This will be true for any two vectors.

Chapter 7 Section 2 **Question 7 Page 376**

a) $$\begin{aligned}\vec{u} \cdot \left(\vec{v} + \vec{w}\right) &= [3, -5] \cdot \left([-6, 1] + [4, 7]\right) \\ &= [3, -5] \cdot [-2, 8] \\ &= 3(-2) + (-5)(8) \\ &= -46\end{aligned}$$

b) $$\begin{aligned}\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{w} &= [3, -5] \cdot [-6, 1] + [-6, 1] \cdot [4, 7] \\ &= 3(-6) + (-5)(1) + (-6)(4) + 1(7) \\ &= -40\end{aligned}$$

c) $$\begin{aligned}\left(\vec{u} + \vec{v}\right) \cdot \left(\vec{u} - \vec{v}\right) &= \left([3, -5] + [-6, 1]\right) \cdot \left([3, -5] - [-6, 1]\right) \\ &= [-3, -4] \cdot [9, -6] \\ &= (-3)(9) + (-4)(-6) \\ &= -3\end{aligned}$$

d) This is not possible. It is the sum of a vector and a scalar, which is not defined.

e) $-3\vec{v} \cdot 2\vec{w} = \left(-3\left[-6, 1\right]\right) \cdot \left(2[4, 7]\right)$
$= \left[18, -3\right] \cdot \left[8, 14\right]$
$= 18(8) + (-3)(14)$
$= 102$

f) $5\vec{u} \cdot \left(2\vec{v} - \vec{w}\right) = 5\left[3, -5\right] \cdot \left(2\left[-6, 1\right] - [4, 7]\right)$
$= \left[15, -25\right] \cdot \left[-16, -5\right]$
$= 15(-16) + (-25)(-5)$
$= -115$

g) This is not possible. The dot product of three vectors has not been defined, nor has the dot product of a scalar and a vector.

h) $\left(\vec{u} + 2\vec{v}\right) \cdot \left(3\vec{w} - \vec{u}\right) = \left(\left[3, -5\right] + 2\left[-6, 1\right]\right) \cdot \left(3\left[4, 7\right] - [3, -5]\right)$
$= \left[-9, -3\right] \cdot \left[9, 26\right]$
$= (-9)(9) + (-3)(26)$
$= -159$

i) $\vec{u} \cdot \vec{u} = \left[3, -5\right] \cdot \left[3, -5\right]$
$= 3(3) + (-5)(-5)$
$= 34$

j) $\vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} = \left[-6, 1\right] \cdot \left[-6, 1\right] + \left[4, 7\right] \cdot \left[4, 7\right]$
$= (-6)(-6) + 1(1) + 4(4) + 7(7)$
$= 102$

Chapter 7 Section 2 **Question 8 Page 376**

a) The vectors for the three sides of ΔABC are $\overrightarrow{AB} = \left[-5, 2\right]$, $\overrightarrow{BC} = \left[7, 3\right]$, and $\overrightarrow{AC} = \left[2, 5\right]$.

$\overrightarrow{AB} \cdot \overrightarrow{AC} = (-5)(2) + 2(5)$
$= 0$

Since the dot product is zero, ΔABC is a right-angle triangle where $\angle BAC$ is the right angle.

The vectors for the three sides of ΔSTU are $\overrightarrow{ST} = \left[-7, 1\right]$, $\overrightarrow{TU} = \left[-2, -11\right]$, and $\overrightarrow{SU} = \left[-9, -10\right]$.
Since none of the pairs of vectors have a dot product of zero, ΔSTU is not a right-angled triangle.

b) The problem could be solved by plotting the points on a sheet of graph paper, then calculating the slopes of each of the sides. If the slopes of any of the two sides in the triangle are negative reciprocals, then the two sides would meet at a 90° angle and therefore the triangle would be a right-angled triangle.

Another solution would involve calculating the lengths of the three side vectors and determining if these three numbers satisfy the Pythagorean theorem relationship.

Chapter 7 Section 2 Question 9 Page 376

a) The two vectors have the same components in reverse order and the value of x in the second vector is opposite to the value of y in the first vector.

b) The two vectors are perpendicular.

c) Using algebra,

$$\begin{aligned}[7, -3]\cdot[3, 7] &= 7(3) + (-3)(7)\\ &= 0\end{aligned}$$

Therefore, the vectors are perpendicular.

Chapter 7 Section 2 Question 10 Page 376

Many vectors are possible. One possibility is [4, –18].

$$\begin{aligned}[9, 2]\cdot[4, -18] &= 9(4) + 2(-18)\\ &= 0\end{aligned}$$

Since the dot product is zero, [4, –18] is perpendicular to [9, 2].

Chapter 7 Section 2 Question 11 Page 376

If $\vec{u}$ and $\vec{v}$ are perpendicular, then $\vec{u}\cdot\vec{v} = 0$.

$$\begin{aligned}[2, 5]\cdot[k, 4] &= 0\\ 2k + 20 &= 0\\ k &= -10\end{aligned}$$

$\vec{u}$ and $\vec{v}$ are perpendicular if $k = -10$.

Chapter 7 Section 2 Question 12 Page 376

If $\vec{u}$ and $\vec{v}$ are perpendicular, then $\vec{u}\cdot\vec{v} = 0$.

$$\begin{aligned}[k, 3]\cdot[k, 2k] &= 0\\ k^2 + 6k &= 0\\ k(k + 6) &= 0\end{aligned}$$

$k = 0$ or $k = 6$

$\vec{u}$ and $\vec{v}$ are perpendicular if $k = -6$.

If $k = 0$, the vectors are [0, 3] and [0, 0]. These two vectors are not usually considered to be perpendicular, even though their dot product is zero.

Chapter 7 Section 2 **Question 13 Page 376**

a) Use vectors [2, 3] and [3, –2].
If these non-zero vectors are plotted on grid paper, tail to tail, they will form a 90º angle.
Also,

$$\begin{aligned}[2, 3]\cdot[3, -2] &= 2(3) + 3(-2)\\ &= 0\end{aligned}$$

b) Use vectors $\vec{a} = [3, -5]$ and $\vec{b} = [2, 1]$.

$$\begin{aligned}\text{L.S.} &= \vec{a}\cdot\vec{b}\\ &= [3, -5]\cdot[2, 1]\\ &= 3(2) + (-5)(1)\\ &= 1\\ \text{R.S.} &= \vec{b}\cdot\vec{a}\\ &= [2, 1]\cdot[3, -5]\\ &= 2(3) + 1(-5)\\ &= 1\end{aligned}$$

Therefore, L.S. = R.S. and $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.

c) Let $\vec{u} = \left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]$ and $\vec{a} = [1, \sqrt{3}]$.
Since $\vec{a} = 2\vec{u}$, the vectors are collinear.

$$\begin{aligned}|\vec{u}| &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\\ &= \sqrt{\frac{1}{4} + \frac{3}{4}}\\ &= 1\end{aligned}$$

Since $|\vec{u}| = 1$, $\vec{u}$ is a unit vector.

$$\text{L.S.} = \vec{a} \cdot \vec{u}$$
$$= \left[1, \sqrt{3}\right] \cdot \left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]$$
$$= 1\left(\frac{1}{2}\right) + \sqrt{3}\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{1}{2} + \frac{3}{2}$$
$$= 2$$
$$\text{R.S.} = \left|\vec{a}\right|$$
$$= \sqrt{1^2 + \left(\sqrt{3}\right)^2}$$
$$= \sqrt{1+3}$$
$$2$$

Therefore, L.S. = R.S. and $\vec{a} \cdot \vec{u} = \left|\vec{a}\right|$.

Chapter 7 Section 2 Question 14 Page 376

a) Since this is an "if and only if" statement, two proofs are required.

Proof 1 ($\Rightarrow$)
Let $\vec{a} = [a_1, a_2]$ and $\vec{b} = [b_1, b_2]$ so that $\vec{a} \cdot \vec{b} = 0$.
Show $\vec{a} \perp \vec{b}$.
Since $\vec{a} \cdot \vec{b} = 0$,
$$a_1 b_1 + a_2 b_2 = 0$$
$$a_1 b_1 = -a_2 b_2$$
$$\frac{b_2}{b_1} = -\frac{a_1}{a_2}$$

This statement says that the slope of $\vec{b}$ is the negative reciprocal of the slope of $\vec{a}$.
Therefore, $\vec{a} \perp \vec{b}$.

Proof 2 ($\Leftarrow$)
Let $\vec{a} = [a_1, a_2]$ and $\vec{b} = [b_1, b_2]$ so that $\vec{a} \perp \vec{b}$.
Show that $\vec{a} \cdot \vec{b} = 0$.
Since $\vec{a} \perp \vec{b}$, consider the slopes of these two vectors; the slopes must be negative reciprocals of each other.

$$\frac{b_2}{b_1} = -\frac{a_1}{a_2}$$
$$a_1 b_1 = -a_2 b_2$$
$$a_1 b_1 + a_2 b_2 = 0$$

Therefore, $\vec{a} \cdot \vec{b} = 0$.

b) Let $\vec{a} = [a_1, a_2]$ and $\vec{b} = [b_1, b_2]$ be any two vectors.

$$\text{L.S.} = \vec{a} \cdot \vec{b}$$
$$= [a_1, a_2] \cdot [b_1, b_2]$$
$$= a_1 b_1 + a_2 b_2$$
$$\text{R.S.} = \vec{b} \cdot \vec{a}$$
$$= [b_1, b_2] \cdot [a_1, a_2]$$
$$= b_1 a_1 + b_2 a_2$$
$$= a_1 b_1 + a_2 b_2 \qquad \text{commutative property for multiplication in } \mathbb{R}$$

Therefore, L.S. = R.S. and $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

c) Let $\vec{u} = [u_1, u_2]$ where $u_1^2 + u_2^2 = 1$.

Let $\vec{a} = k\vec{u}$
$$= [ku_1, ku_2]$$
$$\text{L.S.} = \vec{a} \cdot \vec{u}$$
$$= [ku_1, ku_2] \cdot [u_1, u_2]$$
$$= ku_1^2 + ku_2^2$$
$$\text{R.S.} = |\vec{a}|$$
$$= [ku_1, ku_2]$$
$$= \sqrt{(ku_1)^2 + (ku_2)^2}$$
$$= \sqrt{k^2 u_1^2 + k^2 u_2^2}$$
$$= \sqrt{k^2 (u_1^2 + u_2^2)}$$
$$= \sqrt{k^2 (1)}$$
$$= k$$

Therefore, L.S. = R.S. and $\vec{a} \cdot \vec{u} = |\vec{a}|$.

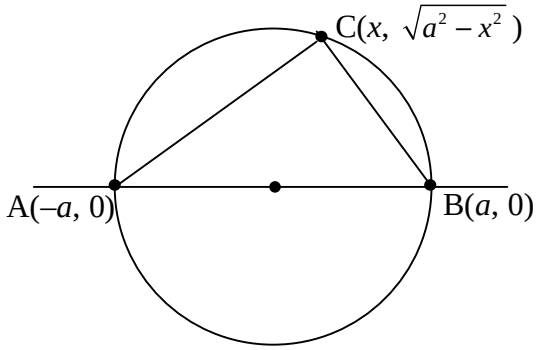
Chapter 7 Section 2 Question 15 Page 377

Let three vectors be $\vec{a} = [3, 0]$, $\vec{b} = [0, 5]$, and $\vec{c} = [0, 7]$.
In this case, $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \cdot \vec{c} = 0$, but $\vec{b} \neq \vec{c}$.

Therefore, if $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$, it is not always true that $\vec{b} = \vec{c}$.

Chapter 7 Section 2 Question 16 Page 377

Use Cartesian vectors.
Consider a circle with centre (0, 0) and radius *a*. Its equation will be $x^2 + y^2 = a^2$.



$\overrightarrow{AC} = [x + a, \sqrt{a^2 - x^2}]$ and $\overrightarrow{BC} = \left[x - a, \sqrt{a^2 - x^2}\right]$.

$$\begin{aligned}\overrightarrow{AC} \cdot \overrightarrow{BC} &= (x + a)(x - a) + \sqrt{a^2 - x^2}\sqrt{a^2 - x^2} \\ &= x^2 - a^2 + a^2 - x^2 \\ &= 0\end{aligned}$$

Therefore, $\overrightarrow{AC} \perp \overrightarrow{BC}$ and $\angle ACB$ is a right angle.

Alternative proof using geometric vectors.

$$\begin{aligned}\overrightarrow{CA} \cdot \overrightarrow{CB} &= \left(\overrightarrow{CO} + \overrightarrow{OA}\right) \cdot \left(\overrightarrow{CO} + \overrightarrow{OB}\right) \\ &= \left(\overrightarrow{CO} + \overrightarrow{OA}\right) \cdot \left(\overrightarrow{CO} - \overrightarrow{OA}\right) \\ &= \overrightarrow{CO} \cdot \overrightarrow{CO} - \overrightarrow{CO} \cdot \overrightarrow{OA} + \overrightarrow{CO} \cdot \overrightarrow{OA} - \overrightarrow{OA} \cdot \overrightarrow{OA} \\ &= \left|\overrightarrow{CO}\right|^2 - \left|\overrightarrow{OA}\right|^2 \\ &= 0 \qquad \text{Since } \left|\overrightarrow{CO}\right| = \left|\overrightarrow{OA}\right| \text{ because they are radii.}\end{aligned}$$

Since $\overrightarrow{CA} \cdot \overrightarrow{CB} = 0$, therefore the angle between $\overrightarrow{CA}$ and $\overrightarrow{CB}$ is 90°.

Therefore, $\angle ACB$ is a right angle.

Chapter 7 Section 2 **Question 17 Page 377**

Let $\vec{u} = [1, 2]$, $\vec{v} = [3, 4]$, $\vec{w} = [-1, -4]$, and $k = 2$.

a) L.S. $= (k\vec{u}) \cdot \vec{v}$

$= (2[1, 2]) \cdot [3, 4]$

$= [2, 4] \cdot [3, 4]$

$= 2(3) + 4(4)$

$= 22$

M.S. $= k(\vec{u} \cdot \vec{v})$

$= 2([1, 2] \cdot [3, 4])$

$= 2(1(3) + 2(4))$

$= 2(11)$

$= 22$

R.S. $= \vec{u} \cdot (k\vec{v})$

$= [1, 2] \cdot (2[3, 4])$

$= [1, 2] \cdot [6, 8]$

$= 1(6) + 2(8)$

$= 22$

Clearly all three sides are equal.

$(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v})$

$= \vec{u} \cdot (k\vec{v})$

b) L.S. $= \vec{u} \cdot (\vec{v} + \vec{w})$

$= [1, 2] \cdot ([3, 4] + [-1, -4])$

$= [1, 2] \cdot [2, 0]$

$= 1(2) + 2(0)$

$= 2$

R.S. $= \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

$= [1, 2] \cdot [3, 4] + [1, 2] \cdot [-1, -4]$

$= 1(3) + 2(4) + 1(-1) + 2(-4)$

$= 2$

Therefore, $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.

c) L.S. $= \left(\vec{u} + \vec{v}\right) \cdot \vec{w}$

$= \left([1, 2] + [3, 4]\right) \cdot [-1, -4]$

$= [4, 6] \cdot [-1, -4]$

$= 4(-1) + 6(-4)$

$= -28$

R.S. $= \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

$= [1, 2] \cdot [-1, -4] + [3, 4] \cdot [-1, -4]$

$= 1(-1) + 2(-4) + 3(-1) + 4(-4)$

$= -28$

Therefore, $\left(\vec{u} + \vec{v}\right) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$.

Chapter 7 Section 2 **Question 18 Page 377**

Let $\vec{u} = [u_1, u_2]$, $\vec{v} = [v_1, v_2]$, and $\vec{w} = [w_1, w_2]$.

a) L.S. $= \left(k\vec{u}\right) \cdot \vec{v}$

$= \left(k[u_1, u_2]\right) \cdot [v_1, v_2]$

$= [ku_1, ku_2] \cdot [v_1, v_2]$

$= ku_1v_1 + ku_2v_2$

M.S. $= k\left(\vec{u} \cdot \vec{v}\right)$

$= k\left([u_1, u_2] \cdot [v_1, v_2]\right)$

$= k\left(u_1v_1 + u_2v_2\right)$

$= ku_1v_1 + ku_2v_2$

R.S. $= \vec{u} \cdot \left(k\vec{v}\right)$

$= [u_1, u_2] \cdot \left(k[v_1, v_2]\right)$

$= [u_1, u_2] \cdot [kv_1, kv_2]$

$= ku_1v_1 + ku_2v_2$

Clearly all three sides are equal and $\left(k\vec{u}\right) \cdot \vec{v} = k\left(\vec{u} \cdot \vec{v}\right) = \vec{u} \cdot \left(k\vec{v}\right)$.

b) L.S. $= \vec{u} \cdot \left(\vec{v} + \vec{w} \right)$

$= \left[u_1, u_2 \right] \cdot \left(\left[v_1, v_2 \right] + \left[w_1, w_2 \right] \right)$

$= \left[u_1, u_2 \right] \cdot \left[v_1 + w_1, v_2 + w_2 \right]$

$= u_1(v_1 + w_1) + u_2(v_2 + w_2)$

$= u_1 v_1 + u_1 w_1 + u_2 v_2 + u_2 w_2$

R.S. $= \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

$= \left[u_1, u_2 \right] \cdot \left[v_1, v_2 \right] + \left[u_1, u_2 \right] \cdot \left[w_1, w_2 \right]$

$= u_1 v_1 + u_2 v_2 + u_1 w_1 + u_2 w_2$

$= u_1 v_1 + u_1 w_1 + u_2 v_2 + u_2 w_2$

Therefore, $\vec{u} \cdot \left(\vec{v} + \vec{w} \right) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.

c) L.S. $= \left(\vec{u} + \vec{v} \right) \cdot \vec{w}$

$= \left(\left[u_1, u_2 \right] + \left[v_1, v_2 \right] \right) \cdot \left[w_1, w_2 \right]$

$= \left[u_1 + v_1, u_2 + v_2 \right] \cdot \left[w_1, w_2 \right]$

$= (u_1 + v_1) w_1 + (u_2 + v_2) w_2$

$= u_1 w_1 + v_1 w_1 + u_2 w_2 + v_2 w_2$

R.S. $= \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

$= \left[u_1, u_2 \right] \cdot \left[w_1, w_2 \right] + \left[v_1, v_2 \right] \cdot \left[w_1, w_2 \right]$

$= u_1 w_1 + u_2 w_2 + v_1 w_1 + v_2 w_2$

$= u_1 w_1 + v_1 w_1 + u_2 w_2 + v_2 w_2$

Therefore, $\left(\vec{u} + \vec{v} \right) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$.

Chapter 7 Section 2 **Question 19 Page 377**

Solutions for Achievement Checks are shown in the Teacher's Resource.

Chapter 7 Section 2 **Question 20 Page 377**

a) $P = \vec{V} \cdot \vec{I}$

b) $P = \vec{V} \cdot \vec{I}$

$= 120(5) \cos 15^\circ$

$\doteq 579.6$

The power is approximately 579.6 W.

Chapter 7 Section 2 **Question 21 Page 377**

a) Let Q(x, y) be any point on the locus.

$\overrightarrow{OQ} = [x,\ y]$ and $\overrightarrow{OP} = [5, 5]$.

$$\overrightarrow{OP} \cdot \overrightarrow{OQ} = 0$$
$$5x + 5y = 0$$
$$y = -x$$

The set of points Q will form the line with equation $y = -x$.

b) Let Q(x, y) be any point on the locus.

$\overrightarrow{OQ} = [x,\ y]$ and $\overrightarrow{OP} = [5, 5]$.

$$\overrightarrow{OP} \cdot \overrightarrow{OQ} = -\frac{\sqrt{2}}{2}$$
$$5x + 5y = -\frac{\sqrt{2}}{2}$$
$$y = -x - \frac{\sqrt{2}}{10}$$

The set of points Q form a line perpendicular to OP and have a y-intercept of $-\frac{\sqrt{2}}{10}$.

c) Let Q(x, y) be any point on the locus.

$\overrightarrow{OQ} = [x,\ y]$ and $\overrightarrow{OP} = [5, 5]$.

$$\overrightarrow{OP} \cdot \overrightarrow{OQ} = -\frac{\sqrt{3}}{2}$$
$$5x + 5y = -\frac{\sqrt{3}}{2}$$
$$y = -x - \frac{\sqrt{3}}{10}$$

The set of points Q form a line perpendicular to OP and have a y-intercept of $-\frac{\sqrt{3}}{10}$.

Chapter 7 Section 2 Question 22 Page 377

There are 57 impossible totals less than $5.00.
This problem is best done with a spreadsheet, taking care to round off answers correctly.

The impossible totals are:

0.04	1.00	2.04	3.00	4.04
0.13	1.09	2.13	3.09	4.13
0.22	1.17	2.22	3.17	4.22
0.30	1.26	2.30	3.26	4.30
0.39	1.35	2.39	3.35	4.39
0.48	1.43	2.48	3.43	4.48
0.56	1.52	2.56	3.52	4.56
0.65	1.61	2.65	3.61	4.65
0.74	1.69	2.74	3.69	4.74
0.83	1.78	2.82	3.78	4.82
0.91	1.87	2.91	3.87	4.91
	1.96		3.95	(5.00)

Note that this is not what really happens in practice. The PST and GST are calculated and rounded off separately. Then they are added together. In the "real" situation 0.07 and 0.11 are impossible totals, while 0.04 and 0.13 are possible. The "real" answer to this problem is left as a challenge.

Chapter 7 Section 2 Question 23 Page 377

No. The truck cannot make it under the bridge.

Set up axes so that the vertex of the parabola is at (0, 5) and the x-intercepts are 3 and –3.
The equation of the parabola is of the form $y = ax^2 + 5;$ where $a < 0$.
Substitute (3, 0) to determine a.

$0 = a(3)^2 + 5$

$a = -\frac{5}{9}$

The equation of the parabola is $y = -\frac{5}{9}x^2 + 5$.

For the truck to pass safely, (1.5, 4) must be below the curve.
Let x = 1.5, then y = 3.75.

The truck needs at least 0.25 m of extra clearance to make it under the bridge.

Chapter 7 Section 3 **Applications of the Dot Product**

Chapter 7 Section 3 **Question 1 Page 384**

a) $\vec{F} \cdot \vec{s} = [5, 2] \cdot [7, 4]$

$= 5(7) + 2(4)$

$= 43$

The work done is 43 N • m or 43 J.

b) $\vec{F} \cdot \vec{s} = [100, 400] \cdot [12, 27]$

$= 100(12) + 400(27)$

$= 12\ 000$

The work done is 12 000 N • m or 12 000 J.

c) $\vec{F} \cdot \vec{s} = [67.8, 3.9] \cdot [4.7, 3.2]$

$= 67.8(4.7) + 3.9(3.2)$

$\doteq 331.1$

The work done is approximately 331.1 N • m or 331.1 J.

Chapter 7 Section 3 **Question 2 Page 384**

a) $\vec{F} \cdot \vec{s} = 50(12)\cos 10^\circ$

$\doteq 590.9$

The work done is approximately 590.0 N • m or 590.0 J.

b) $\vec{F} \cdot \vec{s} = 350(42)\cos 30^\circ$

$\doteq 12\ 730.6$

The work done is approximately 12 730.6 N • m or 12 730.6 J.

c) $\vec{F} \cdot \vec{s} = 241(45.2)\cos 80^\circ$

$\doteq 1891.6$

The work done is approximately 1891.6 N • m or 1891.6 J.

d) $\vec{F} \cdot \vec{s} = 1000(7)\cos 20^\circ$

$\doteq 6577.8$

The work done is approximately 6577.8 N • m or 6577.8 J.

Chapter 7 Section 3 **Question 3 Page 384**

a) $\cos\theta = \dfrac{\vec{p}\cdot\vec{q}}{\left|\vec{p}\right|\left|\vec{q}\right|}$

$= \dfrac{\left[7, 8\right]\cdot\left[4, 3\right]}{\left|\left[7, 8\right]\right|\left|\left[4, 3\right]\right|}$

$= \dfrac{7(4)+8(3)}{\sqrt{7^2+8^2}\sqrt{4^2+3^2}}$

$= \dfrac{52}{53.1507}$

$\theta = \cos^{-1}\left(\dfrac{52}{53.1507}\right)$

$\doteq 11.9^\circ$

b) $\cos\theta = \dfrac{\vec{r}\cdot\vec{s}}{\left|\vec{r}\right|\left|\vec{s}\right|}$

$= \dfrac{\left[-2, -8\right]\cdot\left[6, -1\right]}{\left|\left[-2, -8\right]\right|\left|\left[6, -1\right]\right|}$

$= \dfrac{-2(6)+(-8)(-1)}{\sqrt{(-2)^2+(-8)^2}\sqrt{6^2+(-1)^2}}$

$= \dfrac{-4}{50.1597}$

$\theta = \cos^{-1}\left(\dfrac{-4}{50.1597}\right)$

$\doteq 94.6^\circ$

c) $\cos\theta = \dfrac{\vec{t}\cdot\vec{u}}{\left|\vec{t}\right|\left|\vec{u}\right|}$

$= \dfrac{\left[-7, 2\right]\cdot\left[6, 11\right]}{\left|\left[-7, 2\right]\right|\left|\left[6, 11\right]\right|}$

$= \dfrac{-7(6)+2(11)}{\sqrt{(-7)^2+2^2}\sqrt{6^2+11^2}}$

$= \dfrac{-20}{91.2195}$

$\theta = \cos^{-1}\left(\dfrac{-20}{91.2195}\right)$

$\doteq 102.7^\circ$

d) $\cos\theta = \dfrac{\vec{e}\cdot\vec{f}}{\left|\vec{e}\right|\left|\vec{f}\right|}$

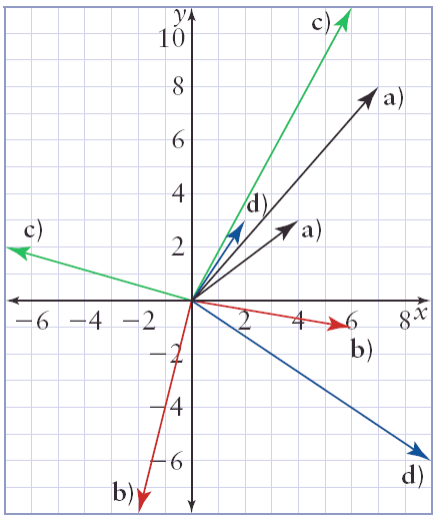
$= \dfrac{[2, 3]\cdot[9, -6]}{\left|[2, 3]\right|\left|[9, -6]\right|}$

$= \dfrac{2(9) + 3(-6)}{\sqrt{2^2 + 3^2}\sqrt{9^2 + (-6)^2}}$

$= \dfrac{0}{39}$

$\theta = \cos^{-1}(0)$

$= 90°$



Chapter 7 Section 3 **Question 4 Page 384**

a) $\text{proj}_{\vec{v}}\vec{u} = \left|\vec{u}\right|\cos\theta\left(\dfrac{1}{\left|\vec{v}\right|}\vec{v}\right)$

$= 10\cos 25°\left(\dfrac{1}{18}\vec{v}\right)$

$\doteq 0.50\vec{v}$ or $9.1\left(\dfrac{1}{\left|\vec{v}\right|}\vec{v}\right)$

The projection has magnitude 9.1 and has the same direction as $\vec{v}$.

b) $\left|\text{proj}_{\vec{v}}\vec{u}\right| = \left|\vec{u}\right|\cos\theta$

$= 7\cos 110^\circ$

$\doteq -2.4$

The projection has magnitude 2.4 and has direction opposite to $\vec{v}$.

c) $\left|\text{proj}_{\vec{v}}\vec{u}\right| = \left|\vec{u}\right|\cos\theta$

$= 20\cos 90^\circ$

$= 0$

The projection has zero magnitude.

Chapter 7 Section 3 **Question 5 Page 384**

a) $\text{proj}_{\vec{b}}\vec{a} = \left(\dfrac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)\vec{b}$

$= \left(\dfrac{[6, -1]\cdot[11, 5]}{[11, 5]\cdot[11, 5]}\right)[11, 5]$

$= \left(\dfrac{61}{146}\right)[11, 5]$

$\doteq [4.6, 2.1]$

b) $\text{proj}_{\vec{d}}\vec{c} = \left(\dfrac{\vec{c}\cdot\vec{d}}{\vec{c}\cdot\vec{d}}\right)\vec{d}$

$= \left(\dfrac{[2, 7]\cdot[-4, 3]}{[-4, 3]\cdot[-4, 3]}\right)[-4, 3]$

$= \left(\dfrac{13}{25}\right)[-4, 3]$

$= [-2.08, 1.56]$

c) $\text{proj}_{\vec{f}}\vec{e} = \left(\dfrac{\vec{e}\cdot\vec{f}}{\vec{f}\cdot\vec{f}}\right)\vec{f}$

$= \left(\dfrac{[-2, -5]\cdot[-5, 1]}{[-5, 1]\cdot[-5, 1]}\right)[-5, 1]$

$= \left(\dfrac{5}{26}\right)[-5, 1]$

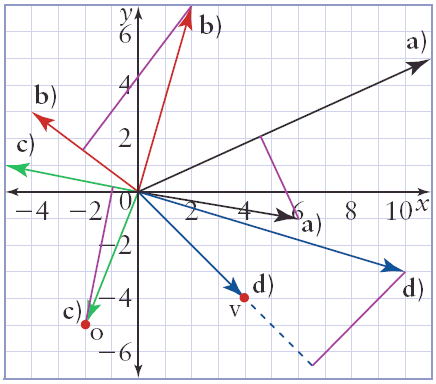
$\doteq [-1.0, 0.2]$

d) $\text{proj}_{\vec{h}}\vec{g} = \left(\dfrac{\vec{g}\cdot\vec{h}}{\vec{h}\cdot\vec{h}}\right)\vec{h}$

$= \left(\dfrac{[10,-3]\cdot[4,-4]}{[4,-4]\cdot[4,-4]}\right)[4,-4]$

$= \left(\dfrac{52}{32}\right)[4,-4]$

$= [6.5,-6.5]$



Chapter 7 Section 3 **Question 6 Page 385**

$\vec{F}\cdot\vec{s} = \left|\vec{F}\right|\left|\vec{s}\right|\cos\theta$

$= 50(8)\cos 30^\circ$

$\doteq 346.4$

The work done is approximately 346.4 J.

Chapter 7 Section 3 **Question 7 Page 385**

a) $\vec{u} = \dfrac{1}{\left|[6,1]\right|}[6,1]$

$= \dfrac{1}{\sqrt{37}}[6,1]$

$= \left[\dfrac{6}{\sqrt{37}}, \dfrac{1}{\sqrt{37}}\right]$

b) $\vec{f} = 25\left[\dfrac{6}{\sqrt{37}}, \dfrac{1}{\sqrt{37}}\right]$

$= \left[\dfrac{150}{\sqrt{37}}, \dfrac{25}{\sqrt{37}}\right]$

c) $\vec{s} = \left[11, 0\right]$

$$W = \vec{F} \cdot \vec{s}$$
$$= \left[\frac{150}{\sqrt{37}}, \frac{25}{\sqrt{37}}\right] \cdot \left[11, 0\right]$$
$$= \left(\frac{150}{\sqrt{37}}\right)(11) + \left(\frac{25}{\sqrt{37}}\right)(0)$$
$$\doteq 271.3 \text{ J}$$

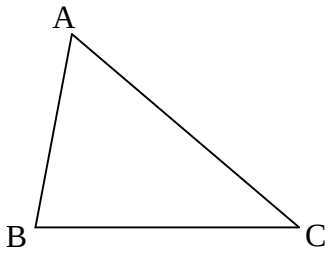
Chapter 7 Section 3 **Question 8 Page 385**

$$W = \left|\vec{F}\right|\left|\vec{s}\right|\cos\theta$$
$$150 = \left|\vec{F}\right|(8)\cos 20^\circ$$
$$\left|\vec{F}\right| = \frac{150}{8\cos 20^\circ}$$
$$\doteq 20.0$$

The force is approximately 20.0 N.

Chapter 7 Section 3 **Question 9 Page 385**

The vectors representing the sides of the triangle are $\overrightarrow{AB} = \left[-1, -8\right]$, $\overrightarrow{BC} = \left[-5, -1\right]$, and $\overrightarrow{AC} = \left[-6, -9\right]$.



Choose vectors carefully to find the interior angles of the triangle.

$$\cos B = \frac{\overrightarrow{AB} \cdot \overrightarrow{CB}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{CB}\right|}$$
$$= \frac{[-1, -8] \cdot [5, 1]}{\left|[-1, -8]\right|\left|[5, 1]\right|}$$
$$= \frac{-1(5) + (-8)(1)}{\sqrt{(-1)^2 + (-8)^2}\sqrt{5^2 + 1^2}}$$
$$\doteq \frac{-13}{41.1096}$$
$$\angle B = \cos^{-1}\left(\frac{-13}{41.1096}\right)$$
$$\doteq 108.4$$

Angle B is approximately 108.4 º.

$$\cos C = \frac{\overrightarrow{AC} \cdot \overrightarrow{BC}}{\left|\overrightarrow{AC}\right|\left|\overrightarrow{BC}\right|}$$
$$= \frac{[-6, -9] \cdot [-5, -1]}{\left|[-6, -9]\right|\left|[-5, -1]\right|}$$
$$= \frac{-6(-5) + (-9)(-1)}{\sqrt{(-6)^2 + (-9)^2}\sqrt{(-5)^2 + (-1)^2}}$$
$$\doteq \frac{39}{55.1543}$$
$$\angle C = \cos^{-1}\left(\frac{39}{55.1543}\right)$$
$$\doteq 45.0$$

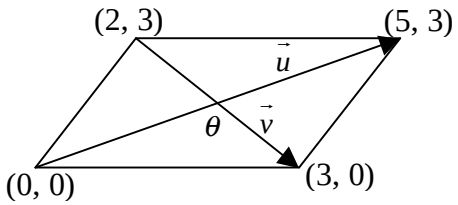
Angle C is approximately 45.0º.

$\angle$ A = 180º – 108.4º – 45.0º
= 26.6º

Therefore, $\angle$ ABC = 108.4°, $\angle$ BCA = 45.0°, and $\angle$ CAB = 26.6°.

Chapter 7 Section 3 **Question 10 Page 385**

The vectors representing the diagonals are $\vec{u} = [5-0, 3-0] = [5, 3]$ and $\vec{v} = [3-2, 0-3] = [1, -3]$.

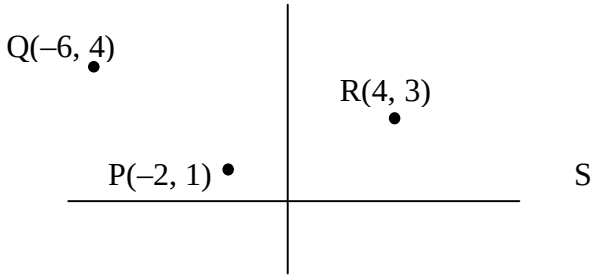


$$\cos\theta = \frac{[5, 3]\cdot[1, -3]}{\|[5, 3]\|\|[1, -3]\|}$$
$$= \frac{5(1) + 3(-3)}{\sqrt{5^2 + 3^2}\sqrt{1^2 + (-3)^2}}$$
$$\doteq \frac{-4}{18.4391}$$
$$\theta = \cos^{-1}\left(\frac{-4}{18.4391}\right)$$
$$\doteq 102.5$$

The measure of θ is approximately 102.5°.

The angles of intersection are 102.5° and 180° – 102.5° = 77.5°.

Chapter 7 Section 3 **Question 11 Page 385**



a) *Method 1:*

$$\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{PS}$$
$$= \overrightarrow{OP} + \overrightarrow{QR}$$
$$= [-2, 1] + [10, -1]$$
$$= [8, 0]$$

S has coordinates (8, 0).

Method 2:

Let S have coordinates (x, y).

Since RS and PQ are parallel, their slopes are the same.

$$\frac{y-3}{x-4} = \frac{4-1}{-6-(-2)}$$

$$= -\frac{3}{4}$$

$3x + 4y = 24$ ①

Since PS and QR are parallel, their slopes are the same.

$$\frac{y-1}{x+2} = \frac{3-4}{4-(-6)}$$

$$= -\frac{1}{10}$$

$x + 10y = 8$ ②

① – 3② gives:

$-26y = 0$

$y = 0$

Substitute $y = 0$ into ②.

$x + 10(0) = 8$

$x = 8$

S has coordinates (8, 0).

b) $\cos\angle PQR = \dfrac{\overrightarrow{QP} \cdot \overrightarrow{QR}}{\left|\overrightarrow{QP}\right|\left|\overrightarrow{QR}\right|}$

$$= \frac{[4, -3] \cdot [10, -1]}{\left|[4, -3]\right|\left|[10, -1]\right|}$$

$$= \frac{4(10) + (-3)(-1)}{\sqrt{4^2 + (-3)^2}\sqrt{10^2 + (-1)^2}}$$

$$\doteq \frac{44}{50.2494}$$

$$\angle PQR = \cos^{-1}\left(\frac{44}{50.2494}\right)$$
$$\doteq 28.9$$

The interior angles are 29° and 180° – 29° = 151°.

c) Let θ be the angle between the diagonals PR and QS.

$$\cos\theta = \frac{\overrightarrow{PR}\cdot\overrightarrow{QS}}{\left|\overrightarrow{PR}\right|\left|\overrightarrow{QS}\right|}$$
$$= \frac{[6, 2]\cdot[14, -4]}{\left|[6, 2]\right|\left|[14, -4]\right|}$$
$$= \frac{6(14)+2(-4)}{\sqrt{6^2+2^2}\sqrt{14^2+(-4)^2}}$$
$$\doteq \frac{76}{92.0869}$$
$$\theta = \cos^{-1}\left(\frac{76}{92.0869}\right)$$
$$\doteq 34.4$$

Angle θ is approximately 34°.

The angles between the diagonals are 34° and 180° – 34° = 146°.

Chapter 7 Section 3 **Question 12 Page 385**

$$\overrightarrow{PQ} = [8-4, 3-7]$$
$$= [4, -4]$$

Use $\vec{i} = [1, 0]$ as a vector along the *x*-axis.

$$\cos\theta = \frac{\overrightarrow{PQ}\cdot\vec{i}}{\left|\overrightarrow{PQ}\right|\left|\vec{i}\right|}$$
$$= \frac{[4, -4]\cdot[1, 0]}{\left|[4, -4]\right|(1)}$$
$$= \frac{4(1)+(-4)(0)}{\sqrt{4^2+(-4)^2}}$$
$$\doteq \frac{4}{5.6569}$$

$$\theta = \cos^{-1}\left(\frac{4}{5.6569}\right)$$
$$\doteq 45$$

The angle between $\overrightarrow{PQ}$ and the positive x-axis is 45°. (This angle could be considered –45° since the vector is below the positive x-axis.)

Chapter 7 Section 3 **Question 13 Page 385**

$$\left|\text{proj}_{\vec{u}}\vec{v}\right| = \left|\vec{v}\right|\cos\theta$$
$$= \left|\vec{v}\right|\cos\theta \times \frac{\left|\vec{u}\right|}{\left|\vec{u}\right|}$$
$$= \frac{\left|\vec{v}\right|\left|\vec{u}\right|\cos\theta}{\left|\vec{u}\right|}$$
$$= \frac{\vec{v}\cdot\vec{u}}{\left|\vec{u}\right|}$$

Chapter 7 Section 3 **Question 14 Page 385**

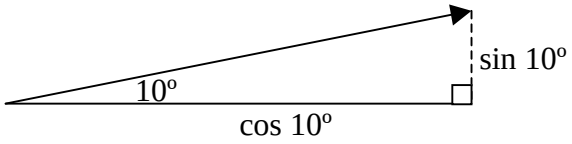
$$\text{proj}_{\vec{u}}\vec{v} = \left|\vec{v}\right|\cos\theta\left(\frac{1}{\left|\vec{u}\right|}\vec{u}\right)$$
$$= \left|\vec{v}\right|\cos\theta\left(\frac{1}{\left|\vec{u}\right|}\vec{u}\right)\times\frac{\left|\vec{u}\right|}{\left|\vec{u}\right|}$$
$$= \frac{\left|\vec{v}\right|\left|\vec{u}\right|\cos\theta}{\left|\vec{u}\right|\left|\vec{u}\right|}\vec{u}$$
$$= \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$$

Chapter 7 Section 3 **Question 15 Page 385**

$$\vec{a}\cdot\vec{b} = [42, 23]\cdot[115, 95]$$
$$= 42(115) + 23(95)$$
$$= 7015$$

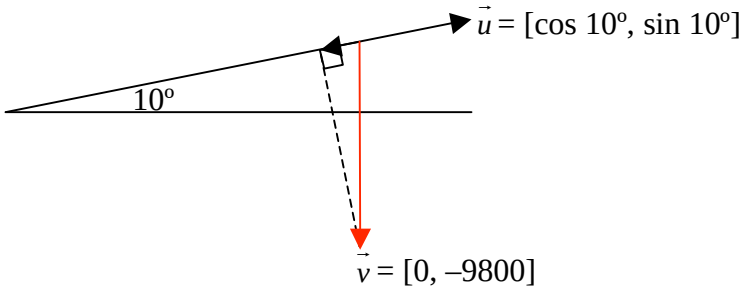
The dot product is $7015**.** It represents the total revenue from the sales of the digital music players and the DVD players.

Chapter 7 Section 3 **Question 16 Page 385**



Using the diagram it is clear that [cos 10°, sin 10°] represents a vector parallel to the road surface.

The force due to gravity is $1000 \times 9.8 = 9800$ N down.



$$\left|\text{proj}_{\vec{u}}\vec{v}\right| = \left|\frac{\vec{v}\cdot\vec{u}}{\left|\vec{u}\right|}\right|$$

$$= \left|\frac{[0, -9800]\cdot[\cos 10^\circ, \sin 10^\circ]}{\left|[\cos 10^\circ, \sin 10^\circ]\right|}\right|$$

$$= \left|\frac{-9800\sin 10^\circ}{\sqrt{\cos^2(10^\circ)+\sin^2(10^\circ)}}\right|$$

$$\doteq \left|\frac{-1701.8}{1}\right|$$

$$= 1701.8$$

The component of the force of gravity along the road vector is 1701.8 N.

Chapter 7 Section 3 **Question 17 Page 385**

a) The hill (displacement) vector is $\vec{u}$ = [2500, 84].

The towing (force) vector is $\vec{v}$ = [30 000, 18 000].

$$\left|\text{proj}_{\vec{u}}\vec{v}\right| = \left|\frac{\vec{v}\cdot\vec{u}}{\left|\vec{u}\right|}\right|$$

$$= \left|\frac{[30\,000, 18\,000]\cdot[2500, 84]}{\left|[2500, 84]\right|}\right|$$

$$= \left|\frac{76\,512\,000}{\sqrt{2500^2 + 84^2}}\right|$$

$$\doteq 30\,587.5$$

The magnitude of the force drawing the car up the hill is about 30 587.5 N

Use the Pythagorean theorem to find the force perpendicular to the hill.

$$x^2 + (30\,587.5)^2 = \left(\sqrt{30\,000^2 + 18\,000^2}\right)^2$$

$$x^2 = 288\,404\,843.8$$

$$x \doteq 16\,982.5$$

The magnitude of the force perpendicular to the hill, tending to lift the car, is about 16 982.5 N.

b) $W = \vec{F}\cdot\vec{s}$

$$= [30\,000, 18\,000]\cdot[2500, 84]$$

$$= 30\,000(2500) + 18\,000(84)$$

$$= 76\,512\,000$$

The work done is 76 512 000 N · m (or J).

c) If only considering the raisin the altitude of the car, the displacement vector is [0, 84]. In this case,

$W = \vec{F}\cdot\vec{s}$

$$= [30\,000, 18\,000]\cdot[0, 84]$$

$$= 30\,000(0) + 18\,000(84)$$

$$= 1512\,000$$

The work done is 1512 000 N · m (or J)

d) Answers will vary. Work is determined by a force vector and a displacement vector. A change in the displacement vector will lead to a change in the work done. Note that the answer in part c) is not realistic since it is not possible to raise the car directly upward by 84 m.

Chapter 7 Section 3 **Question 18 Page 385**

The force in this case is 25(9.8) = 245 N.

$$\begin{aligned} W &= \vec{F} \cdot \vec{s} \\ &= \left|\vec{F}\right|\left|\vec{s}\right|\cos\theta \\ &= 245(6)\cos 30^\circ \\ &\doteq 1273.1 \end{aligned}$$

The work done is about 1273.1 J.

Chapter 7 Section 3 **Question 19 Page 385**

$$\begin{aligned} W &= \vec{F} \cdot \vec{s} \\ &= \left|\vec{F}\right|\left|\vec{s}\right|\cos\theta \\ &= 234(15)\cos 12^\circ \\ &\doteq 3433.3 \end{aligned}$$

The work done is about 3433.3 J.

Chapter 7 Section 3 **Question 20 Page 385**

$$\begin{aligned} W &= \vec{F} \cdot \vec{s} \\ &= \left|\vec{F}\right|\left|\vec{s}\right|\cos\theta \\ &= 3(30)\cos 25^\circ + 4(30)\cos 5^\circ + 5(30)\cos 25^\circ \\ &\doteq 337.1 \end{aligned}$$

The work done is about 337.1 J.

Chapter 7 Section 3 **Question 21 Page 385**

The diagonals of the square are $\vec{d}_1 = [1, 1]$ and $\vec{d}_2 = [1, -1]$.

$$\begin{aligned} \text{proj}_{\vec{d}_1}\vec{i} &= \left(\frac{\vec{i} \cdot \vec{d}_1}{\vec{d}_1 \cdot \vec{d}_1}\right)\vec{d}_1 \\ &= \left(\frac{[1, 0] \cdot [1, 1]}{[1, 1] \cdot [1, 1]}\right)[1, 1] \\ &= \left(\frac{1}{2}\right)[1, 1] \\ &= \left[\frac{1}{2}, \frac{1}{2}\right] \end{aligned}$$

$$\begin{aligned} \text{proj}_{\vec{d}_1}\vec{j} &= \left(\frac{\vec{j} \cdot \vec{d}_1}{\vec{d}_1 \cdot \vec{d}_1}\right)\vec{d}_1 \\ &= \left(\frac{[0, 1] \cdot [1, 1]}{[1, 1] \cdot [1, 1]}\right)[1, 1] \\ &= \left(\frac{1}{2}\right)[1, 1] \\ &= \left[\frac{1}{2}, \frac{1}{2}\right] \end{aligned}$$

$$\text{proj}_{\vec{d}_2}\vec{i} = \left(\frac{\vec{i}\cdot\vec{d}_2}{\vec{d}_2\cdot\vec{d}_2}\right)\vec{d}_2$$
$$= \left(\frac{[1,0]\cdot[1,-1]}{[1,-1]\cdot[1,-1]}\right)[1,-1]$$
$$= \left(\frac{1}{2}\right)[1,-1]$$
$$= \left[\frac{1}{2},-\frac{1}{2}\right]$$

$$\text{proj}_{\vec{d}_2}\vec{j} = \left(\frac{\vec{j}\cdot\vec{d}_2}{\vec{d}_2\cdot\vec{d}_2}\right)\vec{d}_2$$
$$= \left(\frac{[0,1]\cdot[1,-1]}{[1,-1]\cdot[1,-1]}\right)[1,-1]$$
$$= \left(\frac{-1}{2}\right)[1,-1]$$
$$= \left[-\frac{1}{2},\frac{1}{2}\right]$$

Chapter 7 Section 3 **Question 22 Page 385**

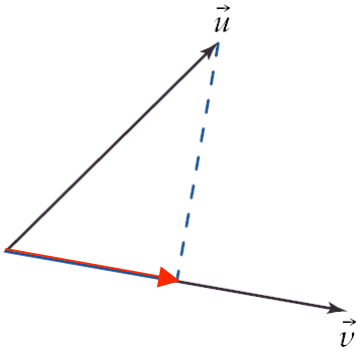
a) The following information is needed: magnitude of the force; distance along the ramp; angle between the force and the ramp.

b) $W = \vec{F}\cdot\vec{s}$
$= \left|\vec{F}\right|\left|\vec{s}\right|\cos\theta$
$= 5000(5)\cos 15^\circ$
$\doteq 24\,148.1$

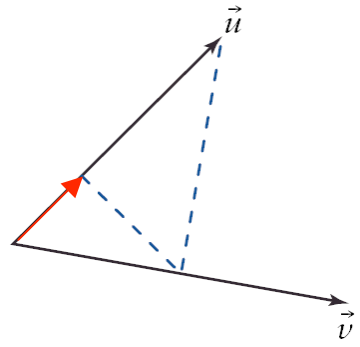
The work done is about 24 148.1 J.

Chapter 7 Section 3 **Question 23 Page 385**

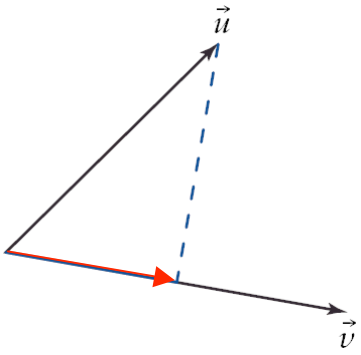
a) The red vector indicates the answer.



b) The red vector indicates the answer.



c) The red vector indicates the answer.



Chapter 7 Section 3 **Question 24 Page 385**

Use components to represent the vectors.

$$\vec{u} = [\cos\theta, -\sin\theta]$$
$$\vec{v} = \left[\cos\theta, \sin\theta\right]$$
$$\vec{w} = \left[0, 1\right]$$

$$\text{L.S.} = \vec{v}$$
$$\begin{aligned}\text{R.S.} &= \vec{u} - 2\left(\vec{u}\cdot\vec{w}\right)\vec{w} \\ &= [\cos\theta, -\sin\theta] - 2\left([\cos\theta, -\sin\theta]\cdot\left[0, 1\right]\right)\left[0, 1\right] \\ &= [\cos\theta, -\sin\theta] - 2(-\sin\theta)\left[0, 1\right] \\ &= [\cos\theta, -\sin\theta] + \left[0, 2\sin\theta\right] \\ &= \left[\cos\theta, \sin\theta\right] \\ &= \vec{v}\end{aligned}$$

Therefore, L.S. = R.S.

Chapter 7 Section 3 **Question 25 Page 385**

a) $\text{proj}_{\vec{b}}\vec{a} = \left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)\vec{b}$

$= \left(\frac{[6, 5]\cdot[1, 3]}{[1, 3]\cdot[1, 3]}\right)[1, 3]$

$= \left(\frac{21}{10}\right)[1, 3]$

$= [2.1, 6.3]$

b) The component vector for the $\vec{b}$ direction is [2.1, 6.3].
The component vector in the perpendicular direction can be found by vector subtraction.
[6, 5] – [[2.1, 6.3] = [3.9, –1.3]

Check perpendicularity using the dot product.
[2.1, 6.3] · [3.9, –1.3] = 2.1(3.9) + 6.3(–1.3)
= 0
Therefore, the two perpendicular components are [2.1, 6.3] and [3.9, –1.3].

Chapter 7 Section 3 **Question 26 Page 385**

$\text{proj}_{\vec{u}}\vec{F} = \left(\frac{\vec{F}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$

$= \left(\frac{[25, 18]\cdot[2, 5]}{[2, 5]\cdot[2, 5]}\right)[2, 5]$

$= \left(\frac{140}{29}\right)[2, 5]$

$= \left[\frac{280}{29}, \frac{700}{29}\right]$

$\doteq [9.7, 24.1]$

As in question 25, find the perpendicular component vector by subtraction.

$[25,18] - \left[\frac{280}{29}, \frac{700}{29}\right] = \left[\frac{445}{29}, \frac{-178}{29}\right]$

$\doteq [15.3, -6.1]$

Use the dot product to check perpendicularity.

$\left[\frac{280}{29}, \frac{700}{29}\right]\cdot\left[\frac{445}{29}, -\frac{178}{29}\right] = 0$

The perpendicular component vectors are $\left[\frac{280}{29}, \frac{700}{29}\right]$ and $\left[\frac{445}{29}, -\frac{178}{29}\right]$.

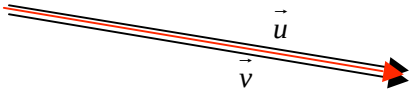
Chapter 7 Section 3 **Question 27 Page 385**

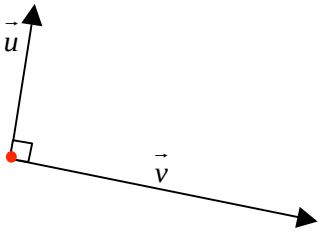
a) These projection vectors are in different directions unless $\vec{u} = k\vec{v}$.

The projections will be equal only when $k = 1$; that is, when $\vec{u} = \vec{v}$.

The projections will also be equal if $\vec{u} \cdot \vec{v} = 0$; the two projection vectors are $\vec{0}$ if the vectors $\vec{u}$ and $\vec{v}$ are perpendicular.

These two situations are demonstrated in the diagrams below.



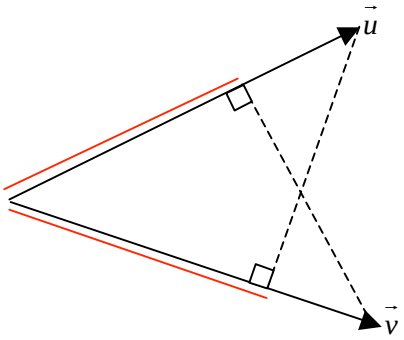


b) $\left|\text{proj}_{\vec{u}}\vec{v}\right| = \left|\text{proj}_{\vec{v}}\vec{u}\right|$

$$\left|\vec{v}\right|\cos\theta = \left|\vec{u}\right|\cos\theta$$

$$\left|\vec{v}\right| = \left|\vec{u}\right| \quad \text{or} \quad \cos\theta = 0$$

This is true whenever $\vec{u}$ and $\vec{v}$ have the same magnitude or if the vectors $\vec{u}$ and $\vec{v}$ are perpendicular. This first situation is demonstrated in the diagram below.



Chapter 7 Section 3 **Question 28 Page 385**

There are three situations to consider.

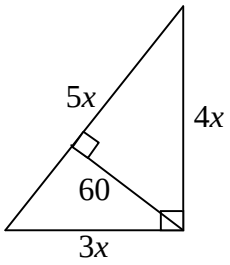
Case 1: Suppose the "3" side has length 60.

Then the "4" side has length 80 and the area is $\frac{1}{2}(60 \text{ cm})(80 \text{ cm}) = 2400 \text{ cm}^2$.

Case 2: Suppose the "4" side has length 60.
Then the "3" side has length 45 and the area is clearly less than 2400 cm^2 as in Case 1.

Case 3: Suppose the altitude to the hypotenuse has length 60.
The sides of the large triangle can be labelled as $3x$, $4x$, and $5x$.

Using similar triangles,



$$\frac{4x}{60} = \frac{5x}{3x}$$
$$12x^2 - 300x = 0$$
$$x(x - 25) = 0$$
$$x = 0,\ 25$$

Therefore, $x = 25$ and the area is $\frac{1}{2}(125 \text{ cm})(60 \text{ cm}) = 3750 \text{ cm}^2$.
The maximum possible area is Case 3 which is 3750 cm^2.

Chapter 7 Section 3 Question 29 Page 385

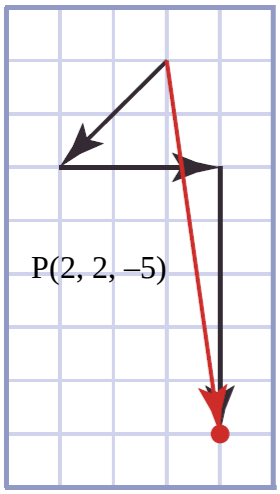
$$f(u^n) = \frac{1471}{n}\log_u(u^n) + \frac{538(u^n)}{u^n}$$
$$= \frac{1471}{n}(n) + 538$$
$$= 1471 + 538$$
$$= 2009$$

Chapter 7 Section 4 **Vectors in Three-Space**

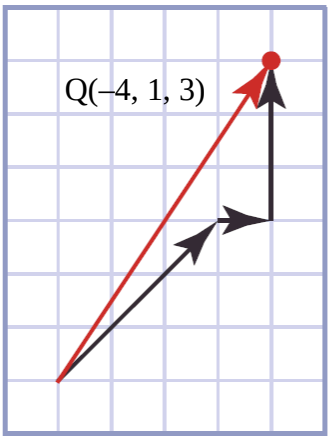
Chapter 7 Section 4 **Question 1 Page 399**

a) Answers may vary. For example:

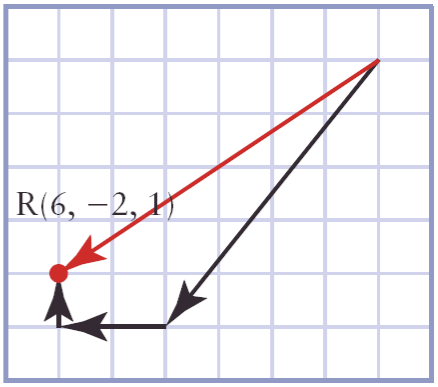
i)



ii)



iii)



b) Since only the *z*-coordinate is negative, P(2, 3, −5) is in the octant at the front right bottom of the 3-D grid.
Since only the *x*-coordinate is negative, Q(−4, 1, 3) is in the octant at the back right top of the 3-D grid.
Since only the *y*-coordinate is negative, R(6, −2, 1) is in the octant at the front left top of the 3-D grid.

c) $\left|\overrightarrow{PS}\right| = \sqrt{(1-2)^2 + (1-3)^2 + (-1+5)^2} = \sqrt{21}$

$\left|\overrightarrow{QS}\right| = \sqrt{(1+4)^2 + (1-1)^2 + (-1-3)^2} = \sqrt{41}$

$\left|\overrightarrow{RS}\right| = \sqrt{(1-6)^2 + (1+2)^2 + (-1-1)^2} = \sqrt{38}$

P(2, 3, –5) is the closest point to S.

d) The *z*-coordinate gives the distance to the *xy*-plane.
R(6, −2, 1) is the closest point to the *xy*-plane.

e) The distance to the *z*-axis is determined by $\sqrt{x^2 + y^2}$.

For P, $d = \sqrt{2^2 + 3^2} = \sqrt{13}$.

For Q, $d = \sqrt{(-4)^2 + 1^2} = \sqrt{17}$.

For R, $d = \sqrt{6^2 + (-2)^2} = \sqrt{40}$.

P(2, 3, –5) is the closest point to the *z*-axis..

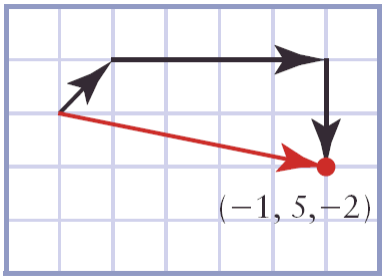
Chapter 7 Section 4 **Question 2 Page 399**

a) The set of points P(x, y, z) where the x and y coordinates are equal.
b) The set of points P(x, y, z) where the absolute values of the x, y, and z values are equal. (e.g., (2, 2, 2) or (5, –5, –5))

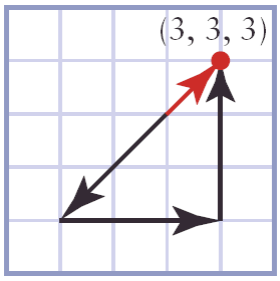
Chapter 7 Section 4 **Question 3 Page 399**

Answers may vary. For example:

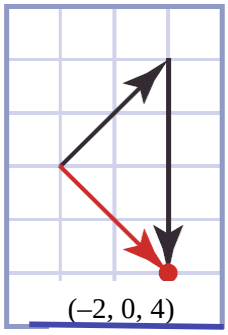
a)



b)



c)



Chapter 7 Section 4 **Question 4 Page 399**

a) $|\vec{u}| = \sqrt{(-1)^2 + 5^2 + (-2)^2}$

$= \sqrt{30}$

b) $\left|\vec{u}\right| = \sqrt{3^2 + 3^2 + 3^2}$

$= \sqrt{27}$

$= 3\sqrt{3}$

c) $\left|\vec{u}\right| = \sqrt{0^2 + (-2)^2 + (-4)^2}$

$= \sqrt{20}$

$= 2\sqrt{5}$

Chapter 7 Section 4 **Question 5 Page 399**

a) $3\vec{i} - 5\vec{j} + 2\vec{k}$

b) $-3\vec{i} - 6\vec{j} + 9\vec{k}$

c) $5\vec{i} - 7\vec{k}$

Chapter 7 Section 4 **Question 6 Page 399**

a) [3, 8, 0]

b) [−5, 0, −2]

c) [7, −4, 9]

Chapter 7 Section 4 **Question 7 Page 399**

Yes, since $\vec{v} = -2\vec{u}$.

Chapter 7 Section 4 **Question 8 Page 399**

$\left|\left[4,1,-7\right]\right| = \sqrt{4^2 + 1^2 + (-7)^2}$

$= \sqrt{66}$

Unit vectors are of the form $\pm\frac{1}{\left|\vec{v}\right|}\vec{v}$.

Therefore, the unit vectors are $\left[\frac{4}{\sqrt{66}}, \frac{1}{\sqrt{66}}, -\frac{7}{\sqrt{66}}\right]$ and $\left[-\frac{4}{\sqrt{66}}, -\frac{1}{\sqrt{66}}, \frac{7}{\sqrt{66}}\right]$.

Chapter 7 Section 4 **Question 9 Page 399**

$\vec{u}$ and $\vec{v}$ are collinear if and only if $\vec{u} = k\vec{v}$ for some $k \in \mathbb{R}$.

a) $[a, 3, 6] = k[-8, 12, b]$

Comparing the y-coordinates gives $k = \frac{1}{4}$.

So, $a = \frac{1}{4}(-8)$ $\qquad 6 = \frac{1}{4}b$

$a = -2$ $\qquad b = 24$

b) $[a, 2, 0] = k[-3, -6, -b]$

Comparing the y-coordinates gives $k = -\frac{1}{3}$.

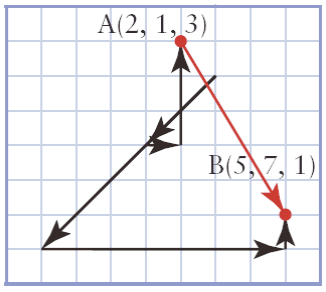
So, $a = -\frac{1}{3}(-3)$ $\qquad 0 = -\frac{1}{3}(-b)$

$a = 1$ $\qquad b = 0$

Chapter 7 Section 4 **Question 10 Page 399**

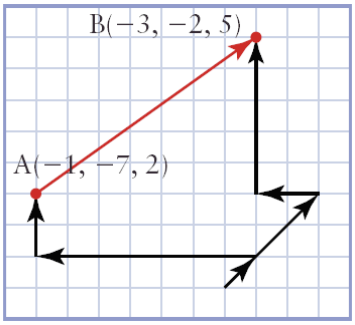
Answers may vary. For example:

a)



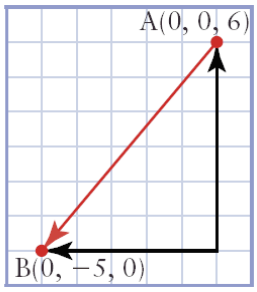
$\overrightarrow{AB} = [3, 6, -2]$

b)



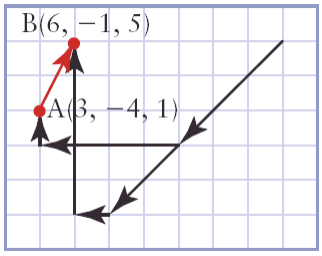
$\overrightarrow{AB} = [-2, 5, 3]$

c)



$\overrightarrow{AB} = [0, -5, -6]$

d)



$\overrightarrow{AB} = [3, 3, 4]$

Chapter 7 Section 4 **Question 11 Page 399**

a) $\left|\overrightarrow{AB}\right| = \sqrt{3^2 + 6^2 + (-2)^2}$

$= \sqrt{49}$

$= 7$

b) $\left|\overrightarrow{AB}\right| = \sqrt{(-2)^2 + 5^2 + 3^2}$

$= \sqrt{38}$

c) $\left|\overrightarrow{AB}\right| = \sqrt{0^2 + (-5)^2 + (-6)^2}$

$= \sqrt{61}$

d) $\left|\overrightarrow{AB}\right| = \sqrt{3^2 + 3^2 + 4^2}$

$= \sqrt{34}$

Chapter 7 Section 4 **Question 12 Page 399**

$$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM}$$
$$[2, 4, -7] = \overrightarrow{ON} - [-5, 0, 3]$$
$$\overrightarrow{ON} = [2, 4, -7] + [-5, 0, 3]$$
$$\overrightarrow{ON} = [-3, 4, -4]$$

Chapter 7 Section 4 **Question 13 Page 399**

$$\overrightarrow{DE} = \overrightarrow{OE} - \overrightarrow{OD}$$
$$[-4, 2, 6] = [3, 3, 1] - \overrightarrow{OD}$$
$$\overrightarrow{OD} = [3, 3, 1] - [-4, 2, 6]$$
$$\overrightarrow{OD} = [7, 1, -5]$$

Chapter 7 Section 4 **Question 14 Page 399**

a) $\overrightarrow{AB} = [0 - 0, 4 - (-3), -4 - 2]$
$= [0, 7, -6]$

b) $\overrightarrow{CD} = [-3 - 4, -3 - 5, 5 - 0]$
$= [-7, -8, 5]$

Chapter 7 Section 4 **Question 15 Page 400**

a) $7\vec{a} = 7[-4, 1, 7]$
$= [-28, 7, 49]$

b) $\vec{a} + \vec{b} + \vec{c} = [-4, 1, 7] + [2, 0, -3] + [1, -1, 5]$
$= [-1, 0, 9]$

c) $\vec{b} + \vec{c} + \vec{a} = [2, 0, -3] + [1, -1, 5] + [-4, 1, 7]$
$= [-1, 0, 9]$

d) $\vec{b} - \vec{c} = [2, 0, -3] - [1, -1, 5]$
$= [1, 1, -8]$

e) $3\vec{a} - 2\vec{b} + 4\vec{c} = 3[-4, 1, 7] - 2[2, 0, -3] + 4[1, -1, 5]$

$= [-12, 3, 21] + [-4, 0, +6] + [4, -4, 20]$

$= [-12, -1, 47]$

f) $\vec{a} \cdot \vec{c} = [-4, 1, 7] \cdot [1, -1, 5]$

$= -4(1) + 1(-1) + 7(5)$

$= 30$

g) $\vec{b} \cdot (\vec{a} + \vec{c}) = [2, 0, -3] \cdot ([-4, 1, 7] + [1, -1, 5])$

$= [2, 0, -3] \cdot [-3, 0, 12]$

$= 2(-3) + 0(0) + -3(12)$

$= -42$

h) $\vec{b} \cdot \vec{c} - \vec{a} \cdot \vec{c} = [2, 0, -3] \cdot [1, -1, 5] - [-4, 1, 7] \cdot [1, -1, 5]$

$= 2(1) + 0(-1) + -3(5) - (-4(1) + 1(-1) + 7(5))$

$= -43$

i) $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = ([-4, 1, 7] + [2, 0, -3]) \cdot ([-4, 1, 7] - [2, 0, -3])$

$= [-2, 1, 4] \cdot [-6, 1, 10]$

$= -2(-6) + 1(1) + 4(10)$

$= 53$

Chapter 7 Section 4 **Question 16 Page 400**

$$\cos\theta = \frac{\vec{g} \cdot \vec{h}}{|\vec{g}||\vec{h}|}$$

$$= \frac{[6, 1, 2] \cdot [-5, 3, 6]}{\sqrt{6^2 + 1^2 + 2^2}\sqrt{(-5)^2 + 3^2 + 6^2}}$$

$$= \frac{-15}{\sqrt{41}\sqrt{70}}$$

$$\doteq -0.2800$$

$$\theta = \cos^{-1}(-0.2800)$$

$$\doteq 106.3$$

The angle has a measure of approximately 106.3º.

Chapter 7 Section 4 **Question 17 Page 400**

a) Let $[x, y, z]$ be a vector orthogonal to $[3, -1, 4]$.

$$[x, y, z] \cdot [3, -1, 4] = 0$$

$$3x - y + 4z = 0$$

Select any values for x and z and solve for y.

Let $x = 2$ and $y = 1$, then $z = \frac{-5}{4}$.

Let $x = -3$ and $y = 5$, then $z = \frac{7}{2}$.

Two orthogonal vectors are $\left[2, 1, -\frac{5}{4}\right]$ and $\left[-3, 5, \frac{7}{2}\right]$.

b) Let $[x, y, z]$ be a vector orthogonal to $[-4, -9, 3]$.

$$[x, y, z] \cdot [-4, -9, 3] = 0$$

$$-4x - 9y + 3z = 0$$

Select any values for x and y and solve for z.

Let $x = 1$ and $y = 2$.

$$-4(1) - 9(2) + 3z = 0$$

$$3z = 22$$

$$z = \frac{22}{3}$$

Let $x = -3$ and $y = 2$.

$$-4(-3) - 9(2) + 3z = 0$$

$$3z = 6$$

$$z = 2$$

Two orthogonal vectors are $\left[1, 2, \frac{22}{3}\right]$ and $[-3, 2, 2]$.

Chapter 7 Section 4 **Question 18 Page 400**

a) Since the x- and y-coordinates are zero, each of the vectors is on the y-axis.

b) Since the z-coordinates are zero, each of the vectors is in the x-y plane.

c) Since the x-coordinates are zero, each of the vectors is in the y-z plane.

Chapter 7 Section 4 Question 19 Page 400

Unit vectors parallel to $\vec{u}$ have the form $\pm\frac{1}{|\vec{u}|}\vec{u}$.

a) $|\vec{a}| = \sqrt{5^2 + (-3)^2 + 2^2}$

$= \sqrt{38}$

The unit vectors parallel to $\vec{a}$ are $\left[\frac{5}{\sqrt{38}}, -\frac{3}{\sqrt{38}}, \frac{2}{\sqrt{38}}\right]$ and $\left[-\frac{5}{\sqrt{38}}, \frac{3}{\sqrt{38}}, -\frac{2}{\sqrt{38}}\right]$.

b) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= [5+7, -2-8, -2-3]$

$= [12, -10, -5]$

$|\overrightarrow{PQ}| = \sqrt{12^2 + (-10)^2 + (-5)^2}$

$= \sqrt{269}$

The unit vectors parallel to $\overrightarrow{PQ}$ are $\left[\frac{12}{\sqrt{269}}, -\frac{10}{\sqrt{269}}, -\frac{5}{\sqrt{269}}\right]$ and $\left[-\frac{12}{\sqrt{269}}, \frac{10}{\sqrt{269}}, \frac{5}{\sqrt{269}}\right]$.

c) $\vec{u} = [5, 6, -3]$

$|\vec{u}| = \sqrt{5^2 + 6^2 + (-3)^2}$

$= \sqrt{70}$

The unit vectors parallel to $\vec{u}$ are $\vec{u}_1 = \frac{5}{\sqrt{70}}\vec{i} + \frac{6}{\sqrt{70}}\vec{j} - \frac{3}{\sqrt{70}}\vec{k}$ and $\vec{u}_2 = -\frac{5}{\sqrt{70}}\vec{i} - \frac{6}{\sqrt{70}}\vec{j} + \frac{3}{\sqrt{70}}\vec{k}$.

d) $\vec{f} = [-3, -2, -9]$

$|\vec{f}| = \sqrt{(-3)^2 + (-2)^2 + (-9)^2}$

$= \sqrt{94}$

The unit vectors parallel to $\vec{f}$ are $\vec{f}_1 = -\frac{3}{\sqrt{94}}\vec{i} - \frac{2}{\sqrt{94}}\vec{j} - \frac{9}{\sqrt{94}}\vec{k}$ and $\vec{f}_2 = \frac{3}{\sqrt{94}}\vec{i} + \frac{2}{\sqrt{94}}\vec{j} + \frac{9}{\sqrt{94}}\vec{k}$.

Chapter 7 Section 4 **Question 20 Page 400**

$$\begin{aligned}\text{L.S.} &= k\vec{u} \\ &= k\left[u_1, u_2, u_3\right] \\ &= k\left(\left[u_1, 0, 0\right]+\left[0, u_2, 0\right]+\left[0, 0, u_3\right]\right) \\ &= k\left[u_1\vec{i}+u_2\vec{j}+u_3\vec{k}\right] \\ &= ku_1\vec{i}+ku_2\vec{j}+ku_3\vec{k} \\ &= k\vec{u}_1\left[1, 0, 0\right]+k\vec{u}_2\left[0, 1, 0\right]+k\vec{u}_3\left[0, 0, 1\right] \\ &= k\vec{u}_1+k\vec{u}_2+k\vec{u}_3\end{aligned}$$

$$\text{R.S.} = k\vec{u}_1+k\vec{u}_2+k\vec{u}_3$$

Therefore, L.S. = R.S.

Chapter 7 Section 4 **Question 21 Page 400**

$$\begin{aligned}\text{L.S.} &= \vec{u}+\vec{v} \\ &= \left[u_1, u_2, u_3\right]+\left[v_1, v_2, v_3\right] \\ &= \left[u_1, 0, 0\right]+\left[0, u_2, 0\right]+\left[0, 0, u_3\right]+\left[v_1, 0, 0\right]+\left[0, v_2, 0\right]+\left[0, 0, v_3\right] \\ &= u_1\vec{i}+u_2\vec{j}+u_3\vec{k}+v_1\vec{i}+v_2\vec{j}+v_3\vec{k} \\ &= (u_1+v_1)\vec{i}+(u_2+v_2)\vec{j}+(u_3+v_3)\vec{k} \\ &= (u_1+v_1)\left[1, 0, 0\right]+(u_2+v_2)\left[0, 1, 0\right]+(u_3+v_3)\left[0, 0, 1\right] \\ &= \left[u_1+v_1, 0, 0\right]+\left[0, u_2+v_2, 0\right]+\left[0, 0, u_3+v_3\right] \\ &= \left[u_1+v_1, u_2+v_2, u_3+v_3\right] \\ \text{R.S.} &= \left[u_1+v_1, u_2+v_2, u_3+v_3\right]\end{aligned}$$

Therefore, L.S. = R.S.

Chapter 7 Section 4 **Question 22 Page 400**

$$\begin{aligned}\overrightarrow{P_1P_2} &= \overrightarrow{P_1O}+\overrightarrow{OP_2} \\ &= \overrightarrow{OP_2}+\overrightarrow{P_1O} \\ &= \overrightarrow{OP_2}-\overrightarrow{OP_1} \\ &= \left[x_2, y_2, z_2\right]-\left[x_1, y_1, z_1\right] \\ &= \left[x_2-x_1, y_2-y_1, z_2-z_1\right]\end{aligned}$$

Chapter 7 Section 4 **Question 23 Page 400**

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA}\\ &= \left[-4-2, 8-3, 1-(-5)\right]\\ &= \left[-6, 5, 6\right]\\ \left|\overrightarrow{AB}\right| &= \sqrt{(-6)^2+5^2+6^2}\\ &= \sqrt{97}\\ \overrightarrow{BC} &= \left[10, -12, -1\right]\\ \left|\overrightarrow{BC}\right| &= \sqrt{245}\\ \overrightarrow{AC} &= \left[4, -7, 5\right]\\ \left|\overrightarrow{AC}\right| &= \sqrt{90}\end{aligned}$$

Since none of the sides have equal length, this is a scalene triangle.

Chapter 7 Section 4 **Question 24 Page 400**

Proof 1 ($\Rightarrow$)
Suppose you are given $\vec{0}$.

$$\vec{0} = \left[0,0,0\right]$$

$$\begin{aligned}\left|\vec{0}\right| &= \sqrt{0^2+0^2+0^2}\\ &= \sqrt{0+0+0}\\ &= 0\end{aligned}$$

Proof 2 ($\Leftarrow$)
Now suppose you are given a vector [*x*, *y*, *z*] whose length is zero.

$$\begin{aligned}\sqrt{x^2+y^2+z^2} &= 0\\ x^2+y^2+z^2 &= 0\end{aligned}$$

The sum of three positive numbers is zero. Therefore each of the numbers must be zero.

$$\begin{aligned}x^2 = y^2 = z^2 &= 0\\ x = y = z &= 0\end{aligned}$$

Therefore, [*x*, *y*, *z*] is the zero vector.

The magnitude of a vector is zero if and only if ($\Leftrightarrow$) the vector is the zero vector.

Chapter 7 Section 4 **Question 25 Page 400**

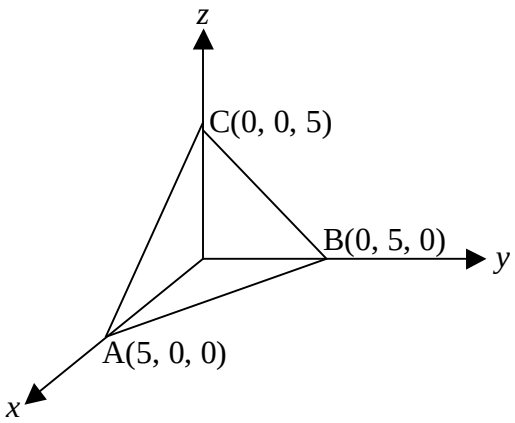
a) $\overrightarrow{AB} = [-5, 5, 0]$ $\overrightarrow{AC} = [-5, 0, 5]$ $\overrightarrow{BC} = [0, -5, 5]$

$\left|\overrightarrow{AB}\right| = \sqrt{(-5)^2 + 5^2 + 0^2}$ $\left|\overrightarrow{AC}\right| = \sqrt{(-5)^2 + 0^2 + 5^2}$ $\left|\overrightarrow{BC}\right| = \sqrt{0^2 + (-5)^2 + 5^2}$

$= \sqrt{50}$ $= \sqrt{50}$ $= \sqrt{50}$

Since $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{BC}\right| = \left|\overrightarrow{AC}\right|$, ΔABC is an equilateral triangle.

b) Draw a diagram. It appears that the closest point to the origin is the centroid of the triangle $\left[\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right]$.



In chapter 6, it was shown that the centroid of ΔABC is defined by the position vector $\overrightarrow{OX} = \frac{1}{3}\left(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}\right)$.

The point on the interior of the triangle closest to the origin is the point $\left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right)$.

Chapter 7 Section 4 **Question 26 Page 400**

a) Assume the vectors are placed on a coordinate system as in the diagram in the question.

$\vec{R} = [25, 0, 0] + [0, 35, 0] + [0, 0, 40]$

$= [25, 35, 40]$

$\left|\vec{R}\right| = \sqrt{25^2 + 35^2 + 40^2}$

$= \sqrt{3450}$

$\doteq 58.7$

The magnitude of the resultant is about 58.7 N.

b) $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\left|\vec{u}\right|\left|\vec{v}\right|}$

$$= \frac{[25, 35, 40]\cdot[0, 35, 0]}{\sqrt{25^2 + 35^2 + 40^2}\sqrt{0^2 + 35^2 + 0^2}}$$

$$= \frac{1225}{\sqrt{3450}(35)}$$

$$\doteq 0.5959$$

$\theta = \cos^{-1}(0.5959)$

$= 53.4$

The resultant makes a 53.4° angle with the 35-N force.

Chapter 7 Section 4 **Question 27 Page 401**

$\vec{a} = [2, -3, 5]$

$\vec{b} = [6, 3, -1]$

$\vec{c} = [-5, -1, 7]$

a) $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = [8, 0, 4]\cdot[-4, -6, 6]$

$= 8(-4) + 0(-6) + 4(6)$

$= -8$

b) $\vec{a}\cdot(\vec{b}+\vec{c}) = [2, -3, 5]\cdot[1, 2, 6]$

$= 2(1) + (-3)(2) + 5(6)$

$= 26$

c) $(\vec{b}+\vec{c})\cdot(\vec{b}+\vec{a}) = [1, 2, 6]\cdot[8, 0, 4]$

$= 1(8) + 2(0) + 6(4)$

$= 32$

d) $2\vec{c}\cdot(3\vec{a}-2\vec{b}) = [-10, -2, 14]\cdot[-6, -15, 17]$

$= -10 - 6 + (-2)(-15) + 14(17)$

$= 328$

Chapter 7 Section 4 Question 28 Page 401

$$\begin{aligned}
\text{L.S.} &= \vec{u}\cdot\vec{v} \\
&= \left[u_1, u_2, u_3\right]\cdot\left[v_1, v_2, v_3\right] \\
&= \left(u_1\vec{i} + u_2\vec{j} + u_3\vec{k}\right)\cdot\left(v_1\vec{i} + v_2\vec{j} + v_3\vec{k}\right) \\
&= u_1v_1\vec{i}\cdot\vec{i} + u_1v_2\vec{i}\cdot\vec{j} + u_1v_3\vec{i}\cdot\vec{k} + u_2v_1\vec{j}\cdot\vec{i} + + u_2v_2\vec{j}\cdot\vec{j} + + u_2v_3\vec{j}\cdot\vec{k} + + u_3v_1\vec{k}\cdot\vec{i} + + u_3v_2\vec{k}\cdot\vec{j} + u_3v_3\vec{k}\cdot\vec{k} \quad \text{Step 1} \\
&= u_1v_1\vec{i}\cdot\vec{i} + 0 + 0 + 0 + u_2v_2\vec{j}\cdot\vec{j} + 0 + 0 + 0 + u_3v_3\vec{k}\cdot\vec{k} \quad \text{Step 2} \\
&= u_1v_1 + u_2v_2 + u_3v_3 \\
\text{R.S.} &= u_1v_1 + u_2v_2 + u_3v_3
\end{aligned}$$

Therefore, L.S. = R.S.

In the above proof, Step 1 is possible using the distributive property for geometric vectors.
Step 2 is possible because $\vec{i}\cdot\vec{j} = \vec{i}\cdot\vec{k} = \vec{j}\cdot\vec{i} = \vec{j}\cdot\vec{k} = \vec{k}\cdot\vec{i} = \vec{k}\cdot\vec{j} = 0$ (perpendicular vectors)
and $\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1$ (parallel unit vectors) .

Chapter 7 Section 4 Question 29 Page 401

$\vec{u} \perp \vec{v}$ if and only if $\vec{u}\cdot\vec{v} = 0$.

a) $\left[4, 1, 3\right]\cdot\left[-1, 5, k\right] = 0$

$$\begin{aligned}
-4 + 5 + 3k &= 0 \\
3k &= -1 \\
k &= -\frac{1}{3}
\end{aligned}$$

b) $\left[k, 3, 6\right]\cdot\left[5, k, 8\right] = 0$

$$\begin{aligned}
5k + 3k + 48 &= 0 \\
8k &= -48 \\
k &= -6
\end{aligned}$$

c) $\left[11, 3, 2k\right]\cdot\left[k, 4, k\right] = 0$

$$\begin{aligned}
11k + 12 + 2k^2 &= 0 \\
2k^2 + 11k + 12 &= 0 \\
(2k + 3)(k + 4) &= 0
\end{aligned}$$

$k = -\frac{3}{2}$ or $k = -4$

Chapter 7 Section 4 **Question 30 Page 401**

$\vec{i} \cdot \vec{j} = [1, 0, 0] \cdot [0, 1, 0]$
$= 1(0) + 0(1) + 0(0)$
$= 0$

$\vec{j} \cdot \vec{k} = [0, 1, 0] \cdot [0, 0, 1]$
$= 0(0) + 1(0) + 0(1)$
$= 0$

$\vec{i} \cdot \vec{k} = [1, 0, 0] \cdot [0, 0, 1]$
$= 1(0) + 0(0) + 0(1)$
$= 0$

Since $\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{i} \cdot \vec{k} = 0$, then $\vec{i}$, $\vec{j}$, and $\vec{k}$ are mutually orthogonal.

Chapter 7 Section 4 **Question 31 Page 401**

a) i) L.S. $= \vec{u} + \vec{v}$
$= [u_1, u_2, u_3] + [v_1, v_2, v_3]$
$= [u_1 + v_1, u_2 + v_2, u_3 + v_3]$
$= [v_1 + u_1, v_2 + u_2, v_3 + u_3]$
$= [v_1, v_2, v_3] + [u_1, u_2, u_3]$

R.S. $= \vec{v} + \vec{u}$
$= [v_1, v_2, v_3] + [u_1, u_2, u_3]$

Therefore, L.S. = R.S.

It is proven that $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

ii) L.S. $= \vec{u} \cdot \vec{v}$
$= [u_1, u_2, u_3] \cdot [v_1, v_2, v_3]$
$= u_1v_1 + u_2v_2 + u_3v_3$
$= v_1u_1 + v_2u_2 + v_3u_3$

R.S. $= \vec{v} \cdot \vec{u}$
$= [v_1, v_2, v_3] \cdot [u_1, u_2, u_3]$
$= v_1u_1 + v_2u_2 + v_3u_3$

Therefore, L.S. = R.S.

It is proven that $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.

b) The proofs depend on the commutative properties of addition and multiplication for real numbers and also on the definitions of vector addition and dot product for Cartesian vectors.

Chapter 7 Section 4 Question 32 Page 401

Let $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$, and $\vec{c} = [c_1, c_2, c_3]$.

$$\begin{aligned}
\text{L.S.} &= (\vec{a} + \vec{b}) + \vec{c} \\
&= ([a_1, a_2, a_3] + [b_1, b_2, b_3]) + [c_1, c_2, c_3] \\
&= [a_1 + b_1, a_2 + b_2, a_3 + b_3] + [c_1, c_2, c_3] \\
&= [(a_1 + b_1) + c_1, (a_2 + b_2) + c_2, (a_3 + b_3) + c_3] \\
&= [a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), a_3 + (b_3 + c_3)] \\
\text{R.S.} &= \vec{a} + (\vec{b} + \vec{c}) \\
&= [a_1, a_2, a_3] + ([b_1, b_2, b_3] + [c_1, c_2, c_3]) \\
&= [a_1, a_2, a_3] + [b_1 + c_1, b_2 + c_2, b_3 + c_3] \\
&= [a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), a_3 + (b_3 + c_3)]
\end{aligned}$$

Therefore, L.S. = R.S.

It is proven that $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

This proof depends on the associative property for addition of real numbers.

Chapter 7 Section 4 Question 33 Page 401

Let $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$, and $\vec{c} = [c_1, c_2, c_3]$.

a)

$$\begin{aligned}
\text{L.S.} &= k(\vec{a} + \vec{b}) \\
&= k([a_1, a_2, a_3] + [b_1, b_2, b_3]) \\
&= k[a_1 + b_1, a_2 + b_2, a_3 + b_3] \\
&= [k(a_1 + b), k(a_2 + b_2), k(a_3 + b_3)] \\
&= [ka_1 + kb_1, ka_2 + kb_2, ka_3 + kb_3] \\
\text{R.S.} &= k\vec{a} + k\vec{b} \\
&= k[a_1, a_2, a_3] + k[b_1, b_2, b_3] \\
&= [ka_1, ka_2, ka_3] + [kb_1, kb_2, kb_3] \\
&= [ka_1 + kb_1, ka_2 + kb_2, ka_3 + kb_3]
\end{aligned}$$

Therefore, L.S. = R.S.

It has been proven that $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$.

b) L.S. $= \vec{a} \cdot \left(\vec{b} + \vec{c}\right)$

$= \left[a_1, a_2, a_3\right] \cdot \left(\left[b_1, b_2, b_3\right] + \left[c_1, c_2, c_3\right]\right)$

$= \left[a_1, a_2, a_3\right] \cdot \left[b_1 + c_1, b_2 + c_2, b_3 + c_3\right]$

$= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3)$

$= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3$

R.S. $= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

$= \left[a_1, a_2, a_3\right] \cdot \left[b_1, b_2, b_3\right] + \left[a_1, a_2, a_3\right] \cdot \left[c_1, c_2, c_3\right]$

$= a_1b_1 + a_2b_2 + a_3b_3 + a_1c_1 + a_2c_2 + a_3c_3$

$= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3$

Therefore, L.S. = R.S.

It has been proven that $\vec{a} \cdot \left(\vec{b} + \vec{c}\right) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$.

c) L.S. $= k\left(\vec{a} \cdot \vec{b}\right)$

$= k\left(\left[a_1, a_2, a_3\right] \cdot \left[b_1, b_2, b_3\right]\right)$

$= k\left(a_1b_1 + a_2b_2 + a_3b_3\right)$

$= k(a_1b_1) + k(a_2b_2) + k(a_3b_3)$

$= ka_1b_1 + ka_2b_2 + ka_3b_3$

M.S. $= k\vec{a} \cdot \left(\vec{b}\right)$

$= \left(k\left[a_1, a_2, a_3\right]\right) \cdot \left[b_1, b_2, b_3\right]$

$= \left[ka_1, ka_2, ka_3\right] \cdot \left[b_1, b_2, b_3\right]$

$= (ka_1)b_1 + (ka_2)b_2 + (ka_3)b_3$

$= ka_1b_1 + ka_2b_2 + ka_3b_3$

R.S. $= \vec{a} \cdot \left(k\vec{b}\right)$

$= \left[a_1, a_2, a_3\right] \cdot \left(k\left[b_1, b_2, b_3\right]\right)$

$= \left[a_1, a_2, a_3\right] \cdot \left[kb_1, kb_2, kb_3\right]$

$= a_1(kb_1) + a_2(kb_2) + a_3(kb_3)$

$= a_1kb_1 + a_2kb_2 + a_3kb_3$

$= ka_1b_1 + ka_2b_2 + ka_3b_3$

Therefore, L.S. = M.S = R.S.

It has been proven that $k\left(\vec{a} \cdot \vec{b}\right) = \left(k\vec{a}\right) \cdot \vec{b} = \vec{a} \cdot \left(k\vec{b}\right)$.

Chapter 7 Section 4 **Question 34 Page 401**

$$\begin{aligned}\text{proj}_{\vec{v}}\vec{u} &= \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}\\ &= \left(\frac{[3, 4, 7]\cdot[1, 2, 3]}{[1, 2, 3]\cdot[1, 2, 3]}\right)[1, 2, 3]\\ &= \left(\frac{32}{14}\right)[1, 2, 3]\\ &= \left[\frac{16}{7}, \frac{32}{7}, \frac{48}{7}\right]\end{aligned}$$

The component vector for the $\vec{v}$ direction is $\left[\frac{16}{7}, \frac{32}{7}, \frac{48}{7}\right]$.

The component vector in the perpendicular direction can be found by vector subtraction.

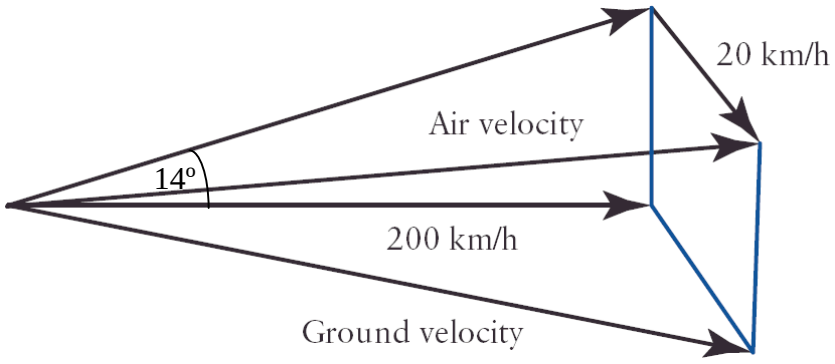
$$[3, 4, 7] - \left[\frac{16}{7}, \frac{32}{7}, \frac{48}{7}\right] = \left[\frac{5}{7}, \frac{-4}{7}, \frac{1}{7}\right]$$

Check perpendicularity using the dot product.

$$\begin{aligned}[3, 4, 7]\cdot\left[\frac{5}{7}, \frac{-4}{7}, \frac{1}{7}\right] &= 3\left(\frac{5}{7}\right) + 4\left(-\frac{4}{7}\right) + 7\left(\frac{1}{7}\right)\\ &= \frac{0}{7}\\ &= 0\end{aligned}$$

Therefore, the two perpendicular components are $\left[\frac{16}{7}, \frac{32}{7}, \frac{48}{7}\right]$ and $\left[\frac{5}{7}, \frac{-4}{7}, \frac{1}{7}\right]$.

Chapter 7 Section 4 **Question 35 Page 401**



Let east be the *x*-direction, north be the *y*-direction, and up be the *z*-direction.
The original vector for the airplane is [200, 0, 200tan 14°].
The wind vector is [0, 20, 0].

The air velocity is the sum of these 2 vectors, [200, 20, 200tan 14º].

The magnitude of the air velocity is $\sqrt{200^2 + 20^2 + (200\tan 14^\circ)^2}$ or approximately 207.1 km/h.

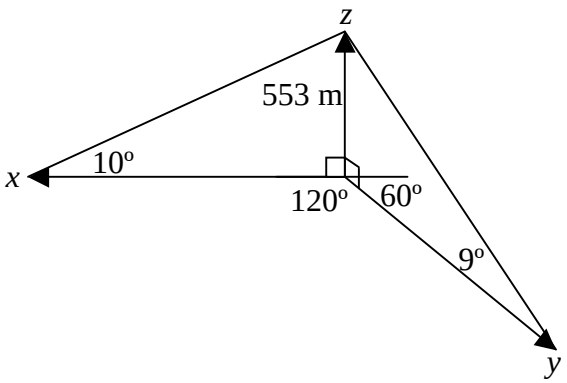
The final ground velocity is the sum of the original ground velocity and the wind velocity,
[200, 0, 0] + [0, 20, 0] = [200, 20, 0].

The magnitude of the final ground velocity is $\sqrt{200^2 + 20^2 + 0^2}$ or approximately 201.0 km/h.

Chapter 7 Section 4 Question 36 Page 401

This problem can be done with either geometric vectors or Cartesian vectors.

Use Cartesian vectors.
Set up a coordinate system so that south is the *x*-direction, east is the *y*-direction, and up is the *z*-direction.



The first person is located at $\left(\frac{553}{\tan 10^\circ}, 0, 0\right)$.

The second person is located at $\left(\frac{553}{\tan 9^\circ}\cos 120^\circ, \frac{553}{\tan 9^\circ}\sin 120^\circ, 0\right)$.

The displacement vector is $\left(\frac{553}{\tan 9^\circ}\cos 120^\circ, \frac{553}{\tan 9^\circ}\sin 120^\circ, 0\right) - \left(\frac{553}{\tan 10^\circ}, 0, 0\right)$

$$= \left[\frac{553}{\tan 9^\circ}\cos 120^\circ - \frac{553}{\tan 10^\circ}, \frac{553}{\tan 9^\circ}\sin 120^\circ, 0\right]$$

$$\doteq \left[-4882.0, 3023.7. 0\right]$$

The distance between these points is $\sqrt{\left(\frac{553}{\tan 9^\circ}\cos 120^\circ - \frac{553}{\tan 10^\circ}\right)^2 + \left(\frac{553}{\tan\ 9^\circ}\sin 120^\circ\right)^2 + 0^2} \doteq 5742.5$.

The distance between the two people is 5742.5 m.

Chapter 7 Section 4 **Question 37 Page 401**

a) Substitute in to the general formula for each case.

i) $\vec{s}(t) = \left[2t,\ 30t\cos 20^\circ,\ 30t\sin 20^\circ - 9.8t^2\right]$

ii) $\vec{s}(t) = \left[6t,\ 18t\cos 35^\circ,\ 18t\sin 35^\circ - 9.8t^2\right]$

b) Let $t = 1$ in each case.

i) $\vec{s}(1) \doteq \left[2,\ 28.2,\ 0.5\right]$

ii) $\vec{s}(1) = \left[6,\ 14.7,\ 0.5\right]$

c) Let $t = 3$ in each case.

i) $\vec{s}(3) = \left[6,\ 84.6,\ -57.4\right]$

ii) $\vec{s}(3) = \left[18,\ 44.2,\ -57.2\right]$

Note that the projectile is below the origin after 3 s.

d) The height is determined only by the z-coordinate.

i) $h(t) = 30t\sin 20^\circ - 9.8t^2$

$h'(t) = 30\sin 20^\circ - 19.6t$

The maximum height occurs when $h'(t) = 0$.

$30\sin 20^\circ - 19.6t = 0$

$$t = \frac{30\sin 20^\circ}{19.6}$$

$t \doteq 0.52$

$\vec{s}(0.52) \doteq \left[1.04,\ 14.66,\ 2.69\right]$

ii) $h(t) = 18t\sin 35^\circ - 9.8t^2$

$h'(t) = 18\sin 35^\circ - 19.6t$

The maximum height occurs when $h'(t) = 0$.

$18\sin 35^\circ - 19.6t = 0$

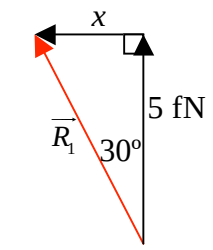
$$t = \frac{18\sin 35^\circ}{19.6}$$

$t \doteq 0.53$

$\vec{s}(0.53) \doteq \left[3.18,\ 7.81,\ 2.72\right]$

Chapter 7 Section 4 **Question 38 Page 402**

a)

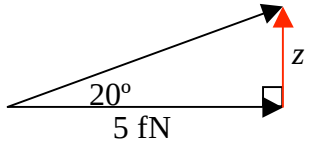


$x = 5\tan 30^\circ$
$\doteq 2.9$ fN

$\left|\vec{R}_1\right| = \sqrt{5^2 + 2.9^2}$
$\doteq 5.8$

The component of the force to the left is about 2.9 fN.

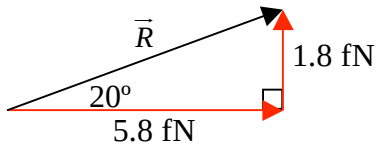
b)



$z = 5\tan 20^\circ$
$\doteq 1.8$ fN

The component of the force up is about 1.8 fN.

c) The magnitude of the resultant is found by combining the horizontal resultant from part a) and the vertical component from part b).

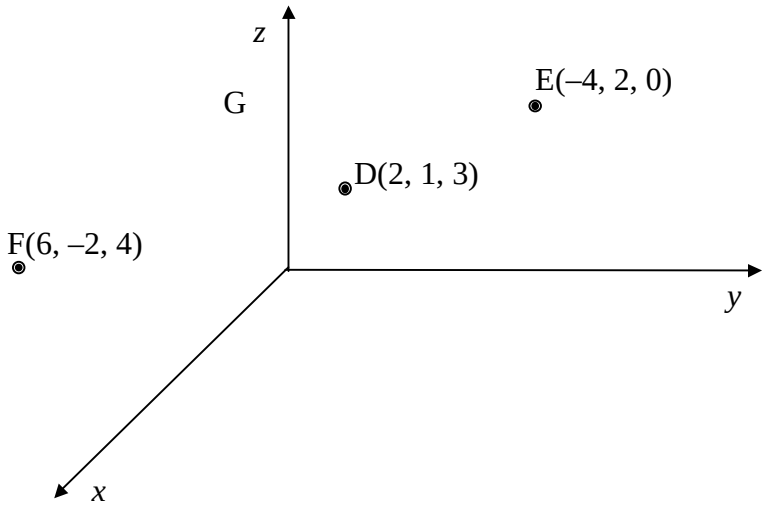


Use the Pythagorean theorem.

$\left|\vec{R}\right| = \sqrt{5.8^2 + 1.8^2}$
$\doteq 6.1$

The magnitude of the resultant force acting on the electron is about 6.1 fN.

Chapter 7 Section 4 **Question 39 Page 402**



Since GFDE is a parallelogram, then FG = DE and EG = FD.

$$\text{FG} = \text{DE}$$
$$(x - 6, y + 2, z - 4) = (-6, 1, -3)$$

$x - 6 = -6$ $\quad$ $y + 2 = 1$ $\quad$ $z - 4 = -3$
$x = 0$ $\quad$ $y = -1$ $\quad$ $z = 1$

$$\text{EG} = \text{FD}$$
$$(x + 4, y - 2, z - 0) = (-4, 3, -1)$$

$x + 4 = -4$ $\quad$ $y - 2 = -3$ $\quad$ $z - 0 = 1$
$x = -8$ $\quad$ $y = -1$ $\quad$ $z = 1$

$$\overrightarrow{OG} = \overrightarrow{OD} + \overrightarrow{OF} - \overrightarrow{OE}$$
$$= [6 + 2 - (-4), -2 + 1 - 2, 4 + 3 - 0]$$
$$= [12, -3, 7]$$

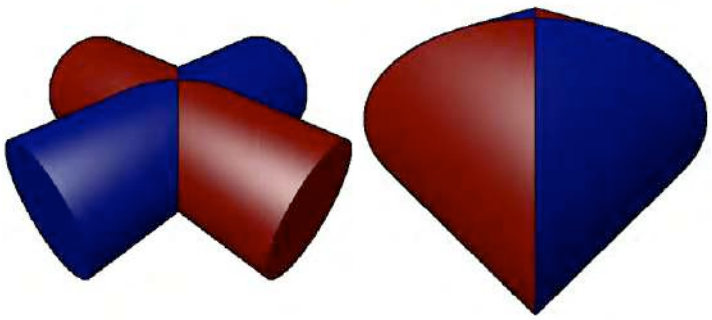
The fourth point has coordinates (0, –1, 1), (–8, –1, 1), or (12, –3, 7).

Chapter 7 Section 4 Question 40 Page 402

The set of points that are 10 units from the *x*-axis form a cylinder around the *x*-axis line. The cylinder has a diameter of 20 units and infinite height. Similarly, the set of points 10 units from the *y*-axis also form a 20-unit-diameter cylinder but with the *y*-axis as its central core and having an infinite height.
Points that are 10 units from both axes lie on the intersection of these two cylinders. This is not a simple curve.
Some points that satisfy the conditions are (10, 10, 0), (10, –10, 0), (–10, –10, 0), –10, 10, 0), (6, 8, 6), $\left(\pm 5, \pm 5\sqrt{3}, \pm 5\right)$, etc.

For a point (x, y, z) to satisfy the conditions, you need $x^2 + y^2 = 100$ and $y^2 + z^2 = 100$.

The resulting surface is known as the Steinmetz solid. Apparently the surface area and volume of this solid were calculated by Chinese mathematicians in about 500 B.C.



Chapter 7 Section 4 Question 41 Page 402

Answers may vary. For example: The vector $\vec{v} = [w, x, y, z]$ can take into consideration the position of the vector in 4-D space in terms of time as well as in terms of position.

The vector $\vec{v} = [w, x, y, z]$ can also represent a position vector in 4-D space, that is, a space that has four mutually perpendicular axes. Although it is not possible to visualize this, vectors in 4-D have the same operations and properties as those in 3-D such as addition, subtraction, scalar multiplication, dot product, commutative, associative, and distributive properties, etc.

Chapter 7 Section 4 Question 42 Page 402

a) Let the required vector be $\vec{v} = [x, y, z]$.

Since $\vec{v} \perp \vec{a}$, $\vec{v} \cdot \vec{a} = 0$.

$$[x, y, z] \cdot [2, 3, 1] = 0$$
$$2x + 3y + z = 0$$

Since $\vec{v} \perp \vec{b}$, $\vec{v} \cdot \vec{b} = 0$.

$$[x, y, z] \cdot [4, 5, -2] = 0$$
$$4x + 5y - 2z = 0$$

There are three variables but only two equations. Therefore, choose arbitrary values for *z*, and solve for *x* and *y*.

Let $z = 0$.

$2x + 3y = 0$ ①

$\underline{4x + 5y = 0}$ ②

$y = 0$ $2① - ②$

$x = 0$

$\vec{v} = [0, 0, 0]$

Let $z = 1$.

$2x + 3y = -1$ ①

$\underline{4x + 5y = 2}$ ②

$y = -4$ $2① - ②$

$x = \frac{11}{2}$

$\vec{v} = \left[\frac{11}{2}, -4, 1\right]$

Let $z = 2$.

$2x + 3y = -2$ ①

$\underline{4x + 5y = 4}$ ②

$y = -8$ $2① - ②$

$x = 11$

$\vec{v} = [11, -8, 2]$

b) Let $z = k$.

$2x + 3y = -k$ ①

$\underline{4x + 5y = 2k}$ ②

$y = -4k$ $2① - ②$

$x = \frac{11}{2}k$

$\vec{v} = \left[\frac{11k}{2}, -4k, k\right]$

This last example shows that there is a vector $\vec{v}$ for every choice of parameter k. Therefore, there are infinitely many solutions.

Chapter 7 Section 4 **Question 43 Page 402**

a) $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|}$

$= \dfrac{\left[2, 1, 3\right]\cdot\left[-1, 2, 4\right]}{\left|\left[2, 1, 3\right]\right|\left|\left[-1, 2, 4\right]\right|}$

$= \dfrac{2(-1)+1(2)+3(4)}{\sqrt{2^2+1^2+3^2}\sqrt{(-1)^2+2^2+4^2}}$

$= \dfrac{12}{17.1464}$

$\theta = \cos^{-1}\left(\dfrac{12}{17.1464}\right)$

$\doteq 45.6^\circ$

$\cos\theta = \dfrac{\vec{b}\cdot\vec{c}}{\left|\vec{b}\right|\left|\vec{c}\right|}$

$= \dfrac{\left[-1, 2, 4\right]\cdot\left[3, 4, 10\right]}{\left|\left[-1, 2, 4\right]\right|\left|\left[3, 4, 10\right]\right|}$

$= \dfrac{-1(3)+2(4)+4(10)}{\sqrt{(-1)^2+2^2+4^2}\sqrt{3^2+4^2+10^2}}$

$= \dfrac{45}{51.2348}$

$\theta = \cos^{-1}\left(\dfrac{45}{51.2348}\right)$

$\doteq 28.6^\circ$

$\cos\theta = \dfrac{\vec{a}\cdot\vec{c}}{\left|\vec{a}\right|\left|\vec{c}\right|}$

$= \dfrac{\left[2, 1, 3\right]\cdot\left[3, 4, 10\right]}{\left|\left[2, 1, 3\right]\right|\left|\left[3, 4, 10\right]\right|}$

$= \dfrac{2(3)+1(4)+3(10)}{\sqrt{2^2+1^2+3^2}\sqrt{3^2+4^2+10^2}}$

$= \dfrac{40}{41.8330}$

$\theta = \cos^{-1}\left(\dfrac{40}{41.8330}\right)$

$\doteq 17.0^\circ$

The three angles are 45.6º, 28.6º, and 17.0º.

b) $\cos\theta = \dfrac{\vec{d}\cdot\vec{e}}{\left|\vec{d}\right|\left|\vec{e}\right|}$

$= \dfrac{[5, 1, 2]\cdot[-3, 1, 4]}{\left|[5, 1, 2]\right|\left|[-3, 1, 4]\right|}$

$= \dfrac{5(-3)+1(1)+2(4)}{\sqrt{5^2+1^2+2^2}\sqrt{(-3)^2+1^2+4^2}}$

$= \dfrac{-6}{27.9285}$

$\theta = \cos^{-1}\left(\dfrac{-6}{27.9285}\right)$

$\doteq 102.4°$

$\cos\theta = \dfrac{\vec{e}\cdot\vec{f}}{\left|\vec{e}\right|\left|\vec{f}\right|}$

$= \dfrac{[-3, 1, 4]\cdot[6, -2, 3]}{\left|[-3, 1, 4]\right|\left|[6, -2, 3]\right|}$

$= \dfrac{-3(6)+1(-2)+4(3)}{\sqrt{(-3)^2+1^2+4^2}\sqrt{6^2+(-2)^2+3^2}}$

$= \dfrac{-8}{35.6931}$

$\theta = \cos^{-1}\left(\dfrac{-8}{35.6931}\right)$

$\doteq 103.0°$

$\cos\theta = \dfrac{\vec{d}\cdot\vec{f}}{\left|\vec{d}\right|\left|\vec{f}\right|}$

$= \dfrac{[5, 1, 2]\cdot[6, -2, 3]}{\left|[5, 1, 2]\right|\left|[6, -2, 3]\right|}$

$= \dfrac{5(6)+1(-2)+2(3)}{\sqrt{5^2+1^2+2^2}\sqrt{6^2+(-2)^2+3^2}}$

$= \dfrac{34}{38.3406}$

$\theta = \cos^{-1}\left(\dfrac{34}{38.3406}\right)$

$\doteq 27.5°$

The three angles are 102.4°, 103.0°, and 27.5°.

Chapter 7 Section 4 **Question 44 Page 402**

Proof 1 ($\Rightarrow$)

Suppose $\vec{u} \perp \vec{v}$.

Then $\vec{u} \cdot \vec{v} = 0$ and $\vec{v} \cdot \vec{u} = 0$.

$$\begin{aligned}
\text{L.S.} &= \left|\vec{u} + \vec{v}\right| \\
&= \left(\vec{u} + \vec{v}\right) \cdot \left(\vec{u} + \vec{v}\right) \\
&= \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} \\
&= \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v}
\end{aligned}$$

$$\begin{aligned}
\text{R.S.} &= \left|\vec{u} - \vec{v}\right| \\
&= \left(\vec{u} - \vec{v}\right) \cdot \left(\vec{u} - \vec{v}\right) \\
&= \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} \\
&= \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v}
\end{aligned}$$

Therefore, L.S. = R.S.

Proof 2 ($\Leftarrow$)

Suppose $\left|\vec{u} + \vec{v}\right| = \left|\vec{u} - \vec{v}\right|$.

Show that $\vec{u} \cdot \vec{v} = 0$.

$$\left|\vec{u} + \vec{v}\right| = \left|\vec{u} - \vec{v}\right|$$

$$\left(\vec{u} + \vec{v}\right) \cdot \left(\vec{u} + \vec{v}\right) = \left(\vec{u} - \vec{v}\right) \cdot \left(\vec{u} - \vec{v}\right)$$

$$\vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} = \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v}$$

$$\vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{v} = -\vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{v}$$

$$4\vec{u} \cdot \vec{v} = 0$$

$$\vec{u} \cdot \vec{v} = 0$$

Therefore, $\vec{u} \cdot \vec{v} = 0$ and $\vec{u} \perp \vec{v}$.

Chapter 7 Section 4 **Question 45 Page 402**

a) Let $\vec{a} = \overrightarrow{\text{OA}}$ and $\vec{b} = \overrightarrow{\text{OB}}$ be position vectors for two points in space and let $\vec{r} = \overrightarrow{\text{OP}}$ be the position vector for any point satisfying the relationship.

$\vec{r} - \vec{a} = \overrightarrow{\text{AP}}$ and $\vec{r} - \vec{b} = \overrightarrow{\text{BR}}$

Since $\left(\vec{r} - \vec{a}\right) \cdot \left(\vec{r} - \vec{b}\right) = 0$, $\overrightarrow{\text{AP}} \perp \overrightarrow{\text{BP}}$.

Points P are all those points such that $\angle\text{APB}$ is a right angle.

This is a property of the sphere that has points A and B as endpoints for one of its diameters.

b) A and B are the two endpoints of a diameter of the sphere. ($\vec{a}$ and $\vec{b}$ are the position vectors for A and B.)

Chapter 7 Section 4 Question 46 Page 402

The last digit can only be an 8 or a 9 and the third digit can only be a 5 or a 6. The time must end in 58 or 59.
The first two digits can only be 1 and 0. The time is either 10:58 or 10:59.
In 12 h, there are $12 \times 60 = 720$ possible display times.

The probability is $\frac{2}{720}$ or $\frac{1}{360}$ or about 0.28%.

Chapter 7 Section 4 Question 47 Page 402

The equation of the line through Q and R is the line $y = 1$.
Find the points P and S using algebra.
Since Q and R are intersection points, $(x - 1)$ and $(x - 4)$ are factors.

Using long division or synthetic division, $x^4 - 10x^3 + 24x^2 + 5x - 20 = (x - 1)(x - 4)(x^2 - 5x - 5)$.
The other intersection points, P and S, have

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(-5)}}{2}$$
$$= \frac{5 \pm \sqrt{45}}{2}$$
$$= \frac{5 \pm 3\sqrt{5}}{2}$$

Then, $\frac{QR}{RS} = \frac{3}{\left(\frac{5+3\sqrt{5}}{2}\right) - 4}$

$$= \frac{6}{5 + 3\sqrt{5} - 8}$$
$$= \frac{6}{-3 + 3\sqrt{5}} \times \frac{3 + 3\sqrt{5}}{3 + 3\sqrt{5}}$$
$$= \frac{6(3)\left(1 + \sqrt{5}\right)}{-9 + 45}$$
$$= \frac{1 + \sqrt{5}}{2}$$

$\frac{RQ}{QP} = \frac{3}{1 - \left(\frac{5-3\sqrt{5}}{2}\right)}$

$$= \frac{6}{2 - 5 + 3\sqrt{5}}$$
$$= \frac{6}{-3 + 3\sqrt{5}} \times \frac{3 + 3\sqrt{5}}{3 + 3\sqrt{5}}$$
$$= \frac{6(3)\left(1 + \sqrt{5}\right)}{-9 + 45}$$
$$= \frac{1 + \sqrt{5}}{2}$$

Chapter 7 Section 5 **The Cross Product and Its Properties**

Chapter 7 Section 5 **Question 1 Page 410**

a) $\vec{u} \times \vec{v} = \left|\vec{u}\right|\left|\vec{v}\right|(\sin\theta)\hat{n}$

$= 60(80)\sin 55^\circ \hat{n}$

$\doteq 3931.9\hat{n}$

b) $\vec{u} \times \vec{v} = \left|\vec{u}\right|\left|\vec{v}\right|(\sin\theta)\hat{n}$

$= 112(128)\sin 164^\circ \hat{n}$

$\doteq 3951.5\hat{n}$

Chapter 7 Section 5 **Question 2 Page 410**

a) $\vec{a} \times \vec{b} = [3, -2, 9] \times [1, 1, 6]$

$= [-2(6) - 1(9), 9(1) - 6(3), 3(1) - 1(-2)]$

$= [-21, -9, 5]$

b) $\vec{a} \times \vec{b} = [6, 3, 2] \times [-5, 5, 9]$

$= [3(9) - 5(2), 2(-5) - 9(6), 6(5) - (-5)(3)]$

$= [17, -64, 45]$

c) $\vec{a} \times \vec{b} = [-8, 10, 3] \times [2, 0, 5]$

$= [10(5) - 0(3), 3(2) - 5(-8), (-8)(0) - 2(10)]$

$= [50, 46, -20]$

d) $\vec{a} \times \vec{b} = [4.3, 5.7, -0.2] \times [12.3, -4.9, 8.8]$

$= [5.7(8.8) - (-4.9)(-0.2), (-0.2)(12.3) - 8.8(4.3), 4.3(-4.9) - 12.3(5.7)]$

$= [49.18, -40.3, -91.18]$

Chapter 7 Section 5 **Question 3 Page 411**

a) $\vec{u} \times \vec{v} = [5, -3, 7] \times [-1, 6, 2]$

$= [-3(2) - 6(7), 7(-1) - 2(5), 5(6) - (-1)(-3)]$

$= [-48, -17, 27]$

Use the dot product to check if $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$.

$\vec{u} \times \vec{v} \cdot \vec{u} = [-48, -17, 27] \cdot [5, -3, 7]$

$= -48(5) + (-17)(-3) + 27(7)$

$= 0$

$\vec{u} \times \vec{v} \cdot \vec{v} = [-48, -17, 27] \cdot [-1, 6, 2]$

$= -48(-1) + (-17)(6) + 27(2)$

$= 0$

Therefore, $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$.

b) $\vec{u} \times \vec{v} = [-2, 1, 5] \times [3, 2, 0]$

$= [1(0) - 2(5), 5(3) - 0(-2), -2(2) - 3(1)]$

$= [-10, 15, -7]$

Use the dot product to check if $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$.

$\vec{u} \times \vec{v} \cdot \vec{u} = [-10, 15, -7] \cdot [-2, 1, 5]$

$= -10(-2) + 15(1) + (-7)5$

$= 0$

$\vec{u} \times \vec{v} \cdot \vec{v} = [-10, 15, -7] \cdot [3, 2, 0]$

$= -10(3) + 15(2) + (-7)(0)$

$= 0$

Therefore, $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$.

c) $\vec{u} \times \vec{v} = [4, -6, 2] \times [6, 8, -3]$

$= [-6(-3) - 8(2), 2(6) - (-3)(4), 4(8) - 6(-6)]$

$= [2, 24, 68]$

Use the dot product to check if $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$.

$$\begin{aligned}\vec{u}\times\vec{v}\cdot\vec{u} &= [2, 24, 68]\cdot[4, -6, 2]\\ &= 2(4) + 24(-6) + 68(2)\\ &= 0\end{aligned}$$

$$\begin{aligned}\vec{u}\times\vec{v}\cdot\vec{v} &= [2, 24, 68]\cdot[6, 8, -3]\\ &= 2(6) + 24(8) + 68(-3)\\ &= 0\end{aligned}$$

Therefore, $\vec{u}\times\vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$.

Chapter 7 Section 5 Question 4 Page 411

The area of the parallelogram defined by $\vec{p}$ and $\vec{q}$ is $\left|\vec{p}\times\vec{q}\right| = \left|\vec{p}\right|\left|\vec{q}\right|\left|\sin\theta\right|$.

a) $\vec{p}\times\vec{q} = [6, 3, 8]\times[3, 3, 5]$

$$\begin{aligned}&= [3(5) - 3(8), 8(3) - 5(6), 6(3) - 3(3)]\\ &= [-9, -6, 9]\end{aligned}$$

$$\begin{aligned}\left|\vec{p}\times\vec{q}\right| &= \sqrt{(-9)^2 + (-6)^2 + 9^2}\\ &\doteq 14.1\end{aligned}$$

The area of the parallelogram is about 14.1 square units.

b) $\left|\vec{u}\times\vec{v}\right| = \left|\vec{u}\right|\left|\vec{v}\right|\left|\sin\theta\right|$

$$\begin{aligned}&= 43(27)\sin 32^\circ\\ &\doteq 615.2\end{aligned}$$

The area of the parallelogram is about 615.2 square units.

Chapter 7 Section 5 Question 5 Page 411

Let the cross product $\vec{a}\times\vec{b}$ represent the situation where the wrench undergoes a clockwise rotation and the direction of the cross product is up. Then situation where the wrench undergoes a counterclockwise rotation and the direction of the cross product would be down is $\vec{b}\times\vec{a}$. Since the two cross products are vector quantities that are opposite in direction, therefore $\vec{a}\times\vec{b} = -\left(\vec{b}\times\vec{a}\right)$.

Chapter 7 Section 5 **Question 6 Page 411**

a) Let $\vec{u} = [-3, 4, 7]$ and $\vec{v} = [2, 8, 3]$.

$$\begin{aligned}
\vec{u} \times \vec{v} &= [-3, 4, 7] \times [2, 8, 3] \\
&= [4(3) - 8(7), 7(2) - 3(-3), (-3)(8) - 2(4)] \\
&= [-44, 23, -32] \\
\left|\vec{u} \times \vec{v}\right| &= \sqrt{(-44)^2 + 23^2 + (-32)^2} \\
&= \sqrt{3489} \\
\left|\vec{u}\right| &= \sqrt{(-3)^2 + 4^2 + 7^2} \\
&= \sqrt{74} \\
\left|\vec{v}\right| &= \sqrt{2^2 + 8^2 + 3^2} \\
&= \sqrt{77} \\
\vec{u} \cdot \vec{v} &= -3(2) + 4(8) + 7(3) \\
&= 47 \\
\sqrt{\left|\vec{u}\right|^2 \left|\vec{v}\right|^2 - \left(\vec{u} \cdot \vec{v}\right)^2} &= \sqrt{74(77) - 47^2} \\
&= \sqrt{3489}
\end{aligned}$$

Therefore, it has been verified that $\left|\vec{u} \times \vec{v}\right| = \sqrt{\left|\vec{u}\right|^2 \left|\vec{v}\right|^2 - \left(\vec{u} \cdot \vec{v}\right)^2}$.

b) Let $\vec{u}$ and $\vec{v}$ be any two vectors.

$$\begin{aligned}
\text{L.S.} &= \left|\vec{u} \times \vec{v}\right| \\
&= \left|\vec{u}\right|\left|\vec{v}\right|\left|\sin\theta\right| \\
\text{R.S.} &= \sqrt{\left|\vec{u}\right|^2 \left|\vec{v}\right|^2 - \left(\vec{u} \cdot \vec{v}\right)^2} \\
&= \sqrt{\left|\vec{u}\right|^2 \left|\vec{v}\right|^2 - \left(\left|\vec{u}\right|\left|\vec{v}\right|\cos\theta\right)^2} \\
&= \sqrt{\left|\vec{u}\right|^2 \left|\vec{v}\right|^2 - \left|\vec{u}\right|^2 \left|\vec{v}\right|^2 \cos^2\theta} \\
&= \sqrt{\left|\vec{u}\right|^2 \left|\vec{v}\right|^2 (1 - \cos^2\theta)} \\
&= \sqrt{\left|\vec{u}\right|^2 \left|\vec{v}\right|^2 \sin^2\theta} \qquad \sin^2\theta + \cos^2\theta = 1 \\
&= \left|\vec{u}\right|\left|\vec{v}\right|\left|\sin\theta\right|
\end{aligned}$$

Therefore, L.S. = R.S.

Chapter 7 Section 5 **Question 7 Page 411**

a) $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\left[2,-6,3\right]\times\left(\left[-1,5,8\right]+\left[-4,5,6\right]\right)$

$=\left[2,-6,3\right]\times\left[-5,10,14\right]$

$=\left[-6(14)-10(3),\ 3(-5)-14(2),\ 2(10)-(-5)(-6)\right]$

$=\left[-114,-43,-10\right]$

b) $\left(\vec{b}+\vec{c}\right)\times\vec{a}=\left(\left[-1,5,8\right]+\left[-4,5,6\right]\right)\times\left[2,-6,3\right]$

$=\left[-5,10,14\right]\times\left[2,-6,3\right]$

$=\left[10(3)-(-6)(14),\ 14(2)-3(-5),\ -5(-6)-2(10)\right]$

$=\left[114,43,10\right]$

c) $\vec{a}\times\vec{b}-\left(\vec{a}\times\vec{c}\right)=\left[2,-6,3\right]\times\left[-1,5,8\right]-\left[2,-6,3\right]\times\left[-4,5,6\right]$

$=\left[-6(8)-5(3),\ 3(-1)-8(2),\ 2(5)-(-1)(-6)\right]-\left[-6(6)-5(3),\ 3(-4)-6(2),\ 2(5)-(-4)(-6)\right]$

$=\left[-63,-19,4\right]-\left[-51,-24,-14\right]$

$=\left[-12,5,18\right]$

d) $\vec{a}\times5\vec{a}=\left[2,-6,3\right]\times\left[10,-30,15\right]$

$=\left[-6(15)-(-30)(3),\ 3(10)-15(2),\ 2(-30)-10(-6)\right]$

$=\left[0,0,0\right]$

e) $\left|\vec{a}\times\vec{c}\right|=\left|\left[2,-6,3\right]\times\left[-4,5,6\right]\right|$

$=\left|\left[-6(6)-5(3),\ 3(-4)-6(2),\ 2(5)-(-4)(-6)\right]\right|$

$=\left|\left[-51,-24,-14\right]\right|$

$=\sqrt{(-51)^2+(-24)^2+(-14)^2}$

$=\sqrt{3373}$

f) $\left|\vec{b}\times\left(\vec{c}-\vec{a}\right)\right|=\left|\left[-1,5,8\right]\times\left(\left[-4,5,6\right]-\left[2,-6,3\right]\right)\right|$

$=\left|\left[-1,5,8\right]\times\left[-6,11,3\right]\right|$

$=\left|\left[5(3)-11(8),\ 8(-6)-3(-1),\ -1(11)-(-6)(5)\right]\right|$

$=\left|\left[-73,-45,19\right]\right|$

$=\sqrt{(-73)^2+(-45)^2+19^2}$

$=\sqrt{7715}$

Chapter 7 Section 5 **Question 8 Page 411**

Use $\vec{c}\times\vec{d}$ and $\vec{d}\times\vec{c}$.

$$\begin{aligned}\vec{c}\times\vec{d} &= [4, 6, -1]\times[-2, 10, 11]\\ &= [6(11)-10(-1), -1(-2)-11(4), 4(10)-(-2)(6)]\\ &= [76, -42, 52]\end{aligned}$$

Two possible orthogonal vectors are $[76, -42, 52]$ and $[-76, 42, -52]$.

Chapter 7 Section 5 **Question 9 Page 411**

$$\begin{aligned}\vec{u}\times\vec{v} &= [3, -4, 1]\times[2, 3, -4]\\ &= [-4(-4)-3(1), 1(2)-(-4)(3), 3(3)-2(-4)]\\ &= [13, 14, 17]\\ |\vec{u}\times\vec{v}| &= \sqrt{13^2+14^2+17^2}\\ &= \sqrt{654}\\ \vec{p} &= \frac{1}{\sqrt{654}}\left(\vec{u}\times\vec{v}\right)\\ &= \left[\frac{13}{\sqrt{654}}, \frac{14}{\sqrt{654}}, \frac{17}{\sqrt{654}}\right]\end{aligned}$$

A vector orthogonal to both $\vec{u}$ and $\vec{v}$ is $\left[\frac{13}{\sqrt{654}}, \frac{14}{\sqrt{654}}, \frac{17}{\sqrt{654}}\right]$.

Chapter 7 Section 5 **Question 10 Page 411**

a) $\overrightarrow{PQ} = \overrightarrow{OQ}-\overrightarrow{OP}$
$= [1, 6, 2]-[0, 2, 5]$
$= [1, 4, -3]$

and $\overrightarrow{SR} = \overrightarrow{OR}-\overrightarrow{OS}$
$= [7, 4, 2]-[6, 0, 5]$
$= [1, 4, -3]$

Therefore, $\overrightarrow{PQ} = \overrightarrow{SR}$.

This gives one pair of side equal and parallel.
This is a sufficient condition to prove that quadrilateral PQRS is a parallelogram.

b) The area is found by calculating $\left|\overrightarrow{PQ} \times \overrightarrow{PS}\right|$.

$$\begin{aligned}\overrightarrow{PQ} \times \overrightarrow{PS} &= [1, 4, -3] \times [6, -2, 0] \\ &= [4(0) - (-2)(-3), -3(6) - 0(1), 1(-2) - 6(4)] \\ &= [6, -18, -26]\end{aligned}$$

$$\begin{aligned}\left|\overrightarrow{PQ} \times \overrightarrow{PS}\right| &= \sqrt{(-6)^2 + (-18)^2 + (-26)^2} \\ &= 2\sqrt{259} \\ &\doteq 32.2\end{aligned}$$

The area of the parallelogram is about 32.2 square units.

c) The easiest test is to check if two adjacent sides are perpendicular. Use the dot product.

$$\begin{aligned}\overrightarrow{PQ} \cdot \overrightarrow{PS} &= [1, 4, -3] \cdot [6, -2, 0] \\ &= 1(6) + 4(-2) + (-3)(0) \\ &= -2 \\ &\neq 0\end{aligned}$$

No. The parallelogram is not a rectangle since $\overrightarrow{PQ}$ is not perpendicular to $\overrightarrow{PS}$.

Chapter 7 Section 5 **Question 11 Page 411**

$$\begin{aligned}\vec{g} \times \vec{h} &= [4, 5, 2] \times [-2, 6, 1] \\ &= [5(1) - 6(2), 2(-2) - 1(4), 4(6) - (-2)(5)] \\ &= [-7, -8, 34]\end{aligned}$$

$$\begin{aligned}\left|\vec{g} \times \vec{h}\right| &= \sqrt{(-7)^2 + (-8)^2 + 34^2} \\ &= \sqrt{1269}\end{aligned}$$

$$\begin{aligned}\left|\vec{g}\right| &= \sqrt{4^2 + 5^2 + 2^2} \\ &= \sqrt{45}\end{aligned}$$

$$\begin{aligned}\left|\vec{h}\right| &= \sqrt{(-2)^2 + 6^2 + 1^2} \\ &= \sqrt{41}\end{aligned}$$

$$\begin{aligned}\vec{g} \cdot \vec{h} &= 4(-2) + 5(6) + 2(1) \\ &= 24\end{aligned}$$

Using the cross product:

$$\sin\theta = \frac{\left|\vec{g}\times\vec{h}\right|}{\left|\vec{g}\right|\left|\vec{h}\right|}$$
$$= \frac{\sqrt{1269}}{\sqrt{45\sqrt{41}}}$$

$$\theta = 56.0^{o}$$

Using the dot product:

$$\cos\theta = \frac{\vec{g}\cdot\vec{h}}{\left|\vec{g}\right|\left|\vec{h}\right|}$$
$$= \frac{24}{\sqrt{45}\sqrt{41}}$$

$$\theta \doteq 56.0°$$

Chapter 7 Section 5 **Question 12 Page 411**

The area of the parallelogram is $\left|\vec{a}\times\vec{b}\right| = \left|\vec{a}\right|\left|\vec{b}\right|\left|\sin\theta\right|$ where $\vec{a}$ and $\vec{b}$ are two adjacent sides and θ is the angle between the sides.

$$85 = 10(9)\left|\sin\theta\right|$$
$$\left|\sin\theta\right| = \frac{85}{90}$$
$$\theta = \sin^{-1}\left(\pm\frac{85}{90}\right)$$

$$\theta \doteq 70.8^{o} \text{ and } 109.2^{o}$$

Since adjacent interior angles of a parallelogram add to 180º, the four interior angles have measures 70.8º, 109.2º, 70.8º, and 109.2º.

Chapter 7 Section 5 **Question 13 Page 411**

$$\text{L.S.} = \left(\vec{u}\times\vec{v}\right)\times\vec{w}$$
$$= \left(\left[3, 2, 9\right]\times\left[8, 0, 3\right]\right)\times\left[6, 2, 6\right]$$
$$= \left[2(3)-0(9), 9(8)-3(3), 3(0)-8(2)\right]\times\left[6, 2, 6\right]$$
$$= \left[6, 63, -16\right]\times\left[6, 2, 6\right]$$
$$= \left[63(6)-2(-16), (-16)(6)-6(6), 6(2)-6(63)\right]$$
$$= \left[410, -132, -366\right]$$

$$\begin{aligned}\text{R.S.} &= \vec{u}\times(\vec{v}\times\vec{w})\\ &= [3, 2, 9]\times([8, 0, 3]\times[6, 2, 6])\\ &= [3, 2, 9]\times[0(6)-2(3), 3(6)-6(8), 8(2)-6(0)]\\ &= [3, 2, 9]\times[-6, -30, 16]\\ &= [2(16)-(-30)(9), 9(-6)-16(3), 3(-30)-(-6)(2)]\\ &= [302, -102, -78]\end{aligned}$$

Therefore, L.S. ≠ R.S.

Therefore, $(\vec{u}\times\vec{v})\times\vec{w}\neq\vec{u}\times(\vec{v}\times\vec{w})$.

Chapter 7 Section 5 Question 14 Page 411

Solutions for Achievement Checks are shown in the Teacher Resource.

Chapter 7 Section 5 Question 15 Page 411

The area of the parallelogram is $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}||\sin\theta|$ where $\vec{a}$ and $\vec{b}$ are two adjacent sides and θ is the angle between the sides.
The area will be zero when either of the two sides have length zero or when $\sin\theta = 0$ which means that $\theta = 0^\circ$ or 180° and the two sides are parallel.
The latter case occurs when the vertices of the parallelogram are collinear. For example, if three of the points of a parallelogram are A = (3, 0, 0), B = (4, 0, 0), and C = (5, 0, 0), then $\overrightarrow{AB} = [1, 0, 0]$, $\overrightarrow{AC} = [1, 0, 0]$, and $\overrightarrow{AB}\times\overrightarrow{AC} = \vec{0}$, and the area is zero.

Chapter 7 Section 5 Question 16 Page 411

a) Verify a specify case of the left distributive law for vector cross product over vector addition.

$$\begin{aligned}\text{L.S.} &= \vec{a}\times(\vec{b}+\vec{c})\\ &= [-2, 4, 3]\times([6, 1, 2]+[5, -3, -2])\\ &= [-2, 4, 3]\times[11, -2, 0]\\ &= [4(0)-(-2)(3), 3(11)-0(-2), -2(-2)-11(4)]\\ &= [6, 33, -40]\end{aligned}$$

R.S. $= \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

$= [-2, 4, 3] \times [6, 1, 2] + [-2, 4, 3] \times [5, -3, -2]$

$= [4(2) - 1(3), 3(6) - 2(-2), -2(1) - 6(4)] + [4(-2) - (-3)(3), 3(5) - (-2)(-2), -2(-3) - 5(4)]$

$= [5, 22, -26] + [1, 11, -14]$

$= [6, 33, -40]$

Therefore, L.S. = R.S.

Therefore, $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

b) Let $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$, and $\vec{c} = [c_1, c_2, c_3]$.

L.S. $= \vec{a} \times (\vec{b} + \vec{c})$

$= [a_1, a_2, a_3] \times ([b_1, b_2, b_3] + [c_1, c_2, c_3])$

$= [a_1, a_2, a_3] \times [b_1 + c_1, b_2 + c_2, b_3 + c_3]$

$= [a_2(b_3 + c_3) - (b_2 + c_2)a_3, a_3(b_1 + c_1) - (b_3 + c_3)a_1, a_1(b_2 + c_2) - (b_1 + c_1)a_2]$

$= [a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2, a_3b_1 + a_3c_1 - a_1b_3 - a_1c_3, a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1]$

R.S. $= \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

$= [a_1, a_2, a_3] \times [b_1, b_2, b_3] + [a_1, a_2, a_3] \times [c_1, c_2, c_3]$

$= [a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1] + [a_2c_3 - a_3c_2, a_3c_1 - a_1c_3, a_1c_2 - a_2c_1]$

$= [a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2, a_3b_1 + a_3c_1 - a_1b_3 - a_1c_3, a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1]$

Therefore, L.S. = R.S.

Therefore, $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

c) Verify a specify case of the right distributive law for vector cross product over vector addition.

L.S. $= (\vec{a} + \vec{b}) \times \vec{c}$

$= ([-2, 4, 3] + [6, 1, 2]) \times [5, -3, -2]$

$= [4, 5, 5] \times [5, -3, -2]$

$= [5(-2) - (-3)(5), 5(5) - (-2)(4), 4(-3) - 5(5)]$

$= [5, 33, -37]$

$$\begin{aligned}\text{R.S.} &= \vec{a}\times\vec{c}+\vec{b}\times\vec{c}\\ &= [-2, 4, 3]\times[5, -3, -2]+[6, 1, 2]\times[5, -3, -2]\\ &= [4(-2)-(-3)(3), 3(5)-(-2)(-2), -2(-3)-5(4)]+[1(-2)-(-3)(2), 2(5)-(-2)(6), 6(-3)-5(1)]\\ &= [1, 11, -14]+[4, 22, -23]\\ &= [5, 33, -37]\end{aligned}$$

Therefore, L.S. = R.S.

Therefore, $(\vec{a}+\vec{b})\times\vec{c}=\vec{a}\times\vec{c}+\vec{b}\times\vec{c}$.

d) Let $\vec{a}=[a_1, a_2, a_3]$, $\vec{b}=[b_1, b_2, b_3]$, and $\vec{c}=[c_1, c_2, c_3]$.

$$\begin{aligned}\text{L.S.} &= (\vec{a}+\vec{b})\times\vec{c}\\ &= ([a_1, a_2, a_3]+[b_1, b_2, b_3])\times[c_1, c_2, c_3]\\ &= [a_1+b_1, a_2+b_2, a_3+b_3]\times[c_1, c_2, c_3]\\ &= [(a_2+b_2)c_3-c_2(a_3+b_3), (a_3+b_3)c_1-c_3(a_1+b_1), (a_1+b_1)c_2-c_1(a_2+b)]\\ &= [a_2c_3+b_2c_3-a_3c_2-b_3c_2, a_3c_1+b_3c_1-a_1c_3-b_1c_3, a_1c_2+b_1c_2-a_2c_1-b_2c_1]\end{aligned}$$

$$\begin{aligned}\text{R.S.} &= \vec{a}\times\vec{c}+\vec{b}\times\vec{c}\\ &= [a_1, a_2, a_3]\times[c_1, c_2, c_3]+[b_1, b_2, b_3]\times[c_1, c_2, c_3]\\ &= [a_2c_3-a_3c_2, a_3c_1-a_1c_3, a_1c_2-a_2c_1]+[b_2c_3-b_3c_2, b_3c_1-b_1c_3, b_1c_2-b_2c_1]\\ &= [a_2c_3+b_2c_3-a_3c_2-b_3c_2, a_3c_1+b_3c_1-a_1c_3-b_1c_3, a_1c_2+b_1c_2-a_2c_1-b_2c_1]\end{aligned}$$

Therefore, the L.S. = R.S.

Therefore, $(\vec{a}+\vec{b})\times\vec{c}=\vec{a}\times\vec{c}+\vec{b}\times\vec{c}$.

Chapter 7 Section 5 Question 17 Page 411

a) Verify a specify case.

$$\begin{aligned}\text{L.S.} &= k(\vec{u}\times\vec{v})\\ &= 2([-1, 4, 1]\times[3, -2, 4])\\ &= 2[4(4)-(-2)(1), 1(3)-4(-1), -1(-2)-3(4)]\\ &= [2(18), 2(7), 2(-10)]\\ &= [36, 14, -20]\end{aligned}$$

$$\begin{aligned}
\text{M.S.} &= (k\vec{u}) \times \vec{v} \\
&= \left(2\left[-1, 4, 1\right]\right) \times \left[3, -2, 4\right] \\
&= \left[-2, 8, 2\right] \times \left[3, -2, 4\right] \\
&= \left[8(4) - (-2)(2), 2(3) - 4(-2), -2(-2) - 3(8)\right] \\
&= \left[36, 14, -20\right] \\
\text{R.S.} &= \vec{u} \times (k\vec{v}) \\
&= \left[-1, 4, 1\right] \times \left(2\left[3, -2, 4\right]\right) \\
&= \left[-1, 4, 1\right] \times \left[6, -4, 8\right] \\
&= \left[4(8) - (-4)(1), 1(6) - 8(-1), -1(-4) - 6(4)\right] \\
&= \left[36, 14, -20\right]
\end{aligned}$$

Therefore, L.S. = M.S. = R.S.

Therefore, $k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v})$.

b) Let $\vec{u} = [u_1, u_2, u_3]$ and $\vec{v} = [v_1, v_2, v_3]$.

$$\begin{aligned}
\text{L.S.} &= k(\vec{u} \times \vec{v}) \\
&= k\left(\left[u_1, u_2, u_3\right] \times \left[v_1, v_2, v_3\right]\right) \\
&= k\left[u_2v_3 - v_2u_3, u_3v_1 - v_3u_1, u_1v_2 - v_1u_2\right] \\
&= \left[k(u_2v_3 - v_2u_3), k(u_3v_1 - v_3u_1), k(u_1v_2 - v_1u_2)\right] \\
&= \left[ku_2v_3 - kv_2u_3, ku_3v_1 - kv_3u_1, ku_1v_2 - kv_1u_2\right] \\
\text{M.S.} &= (k\vec{u}) \times \vec{v} \\
&= \left(k\left[u_1, u_2, u_3\right]\right) \times \left[v_1, v_2, v_3\right] \\
&= \left[ku_1, ku_2, ku_3\right] \times \left[v_1, v_2, v_3\right] \\
&= \left[ku_2v_3 - kv_2u_3, ku_3v_1 - kv_3u_1, ku_1v_2 - kv_1u_2\right] \\
\text{R.S.} &= \vec{u} \times (k\vec{v}) \\
&= \left[u_1, u_2, u_3\right] \times \left(k\left[v_1, v_2, v_3\right]\right) \\
&= \left[u_1, u_2, u_3\right] \times \left[kv_1, kv_2, kv_3\right] \\
&= \left[ku_2v_3 - kv_2u_3, ku_3v_1 - kv_3u_1, ku_1v_2 - kv_1u_2\right]
\end{aligned}$$

Therefore, L.S. = M.S. = R.S.

Therefore, $k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v})$.

Chapter 7 Section 5 Question 18 Page 412

No. Choose three collinear vectors.

Let $\vec{a} = [1, 2, 3]$, $\vec{b} = [2, 4, 6]$, and $\vec{c} = [3, 6, 9]$.

$$\begin{aligned}\vec{a} \times \vec{b} &= [1, 2, 3] \times [2, 4, 6] \\ &= [2(6) - 4(3), 3(2) - 6(1), 1(4) - 2(2)] \\ &= [0, 0, 0]\end{aligned}$$

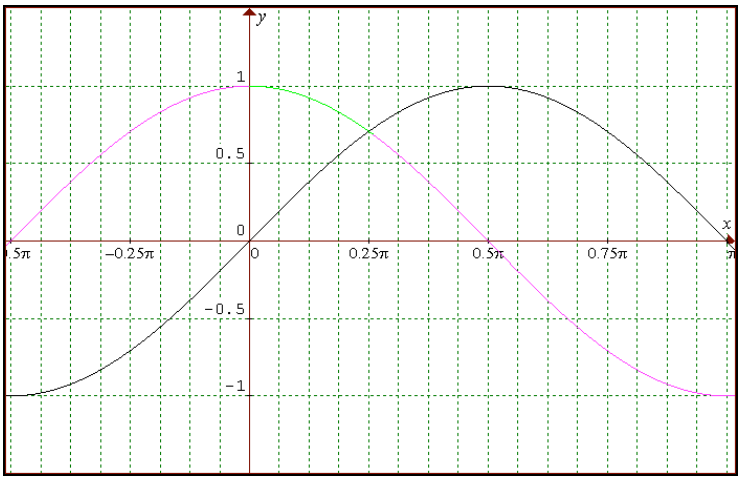
$$\begin{aligned}\vec{a} \times \vec{c} &= [1, 2, 3] \times [3, 6, 9] \\ &= [2(9) - 6(3), 3(3) - 9(1), 1(6) - 3(2)] \\ &= [0, 0, 0]\end{aligned}$$

Clearly $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ but $\vec{b} \neq \vec{c}$. Any three collinear vectors will show this.

Chapter 7 Section 5 Question 19 Page 412

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}||\sin\theta|$ and $|\vec{a} \cdot \vec{b}| = |\vec{a}||\vec{b}|\cos\theta$.

Therefore, $|\vec{a} \times \vec{b}| < |\vec{a} \cdot \vec{b}|$ if $|\sin\theta| < \cos\theta$.



Examining the related functions $y = \sin\theta$ and $y = \cos\theta$ in the interval [0°, 180°], it can be seen that $0° < \theta < 45°$ when $|\sin\theta| < \cos\theta$.

$|\vec{a} \times \vec{b}| < |\vec{a} \cdot \vec{b}|$ if $\theta < 45°$.

$|\vec{a} \times \vec{b}| = |\vec{a} \cdot \vec{b}|$ if $\theta = 45°$.

$|\vec{a} \times \vec{b}| > |\vec{a} \cdot \vec{b}|$ if $\theta > 45°$.

The easy test is to determine the measure of θ is $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$.

b) $[2, 1, -1] \cdot [-1, -2, 1] = -5$

$\left\|[2, 1, -1]\right\| = \sqrt{6}$

$\left\|[-1, -2, 1]\right\| = \sqrt{6}$

$$\theta = \cos^{-1}\left(\frac{-5}{\sqrt{6}\sqrt{6}}\right)$$

$\doteq 146.4^{\circ}$

In this case, $\left|\vec{a} \times \vec{b}\right| > \left|\vec{a} \cdot \vec{b}\right|$.

c) $[2, 1, 1] \cdot [3, 1, 2] = 9$

$\left\|[2, 1, 1]\right\| = \sqrt{6}$

$\left\|[3, 1, 2]\right\| = \sqrt{14}$

$$\theta = cos^{-1}\left(\frac{9}{\sqrt{6}\sqrt{14}}\right)$$

$\doteq 10.9^{\circ}$

In this case, $\left|\vec{a} \times \vec{b}\right| < \left|\vec{a} \cdot \vec{b}\right|$.

d) The most likely random case is $\left|\vec{a} \times \vec{b}\right| > \left|\vec{a} \cdot \vec{b}\right|$ since θ is more likely to be in the interval (45°, 180°] than in the interval [0°, 45°).

Chapter 7 Section 5 Question 20 Page 412

a) From question 6, it is known that $|\vec{u} \times \vec{v}| = \sqrt{|\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2}$.

The expression under the square root symbol must be greater than or equal to zero (since $\vec{u} \times \vec{v}$ is always defined).

$|\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2 \geq 0$

$|\vec{u}|^2 |\vec{v}|^2 \geq (\vec{u} \cdot \vec{v})^2$

$|\vec{u}||\vec{v}| \geq |\vec{u} \cdot \vec{v}|$

[Note: Actually, the stronger result, $|\vec{u}||\vec{v}| \geq \vec{u} \cdot \vec{v}$, is true.]

b) *Proof 1* $(\Rightarrow)$

Let $\vec{u}$ and $\vec{v}$ be collinear. Then $\vec{u} = k\vec{v}$.

$$|\vec{u} \times \vec{v}| = \sqrt{|\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2}$$

$$|k\vec{v} \times \vec{v}| = \sqrt{|\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2}$$

$$0 = \sqrt{|\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2} \qquad \vec{u} \times \vec{v} = \vec{0} \text{ for collinear vectors.}$$

$$0 = |\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2$$

$$(\vec{u} \cdot \vec{v})^2 = |\vec{u}|^2 |\vec{v}|^2$$

$$|\vec{u} \cdot \vec{v}| = |\vec{u}||\vec{v}| \qquad \text{Take the positive square root of both sides.}$$

Proof 2 $(\Leftarrow)$

Let $|\vec{u} \cdot \vec{v}| = |\vec{u}||\vec{v}|$

$$|\vec{u} \cdot \vec{v}| = |\vec{u}||\vec{v}|$$

$$(\vec{u} \cdot \vec{v})^2 = |\vec{u}|^2 |\vec{v}|^2 \qquad \text{Square both sides}$$

$$0 = |\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2$$

$$0 = \sqrt{|\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2}$$

$$0 = |\vec{u} \times \vec{v}| \qquad \text{(From question 6.)}$$

$$0 = |\vec{u}||\vec{v}| \sin\theta$$

If $\sin\theta = 0$, then $\theta = 0^\circ$ and $\vec{u}$ and $\vec{v}$ are collinear.

If $|\vec{u}| = 0$ or $|\vec{v}| = 0$, then $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$ and the vectors are (trivially) collinear.

Therefore, $\vec{u}$ and $\vec{v}$ be collinear if and only if $(\Leftrightarrow)$ $|\vec{u} \cdot \vec{v}| = |\vec{u}||\vec{v}|$.

Chapter 7 Section 5 **Question 21 Page 412**

Vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are position vectors for the three vertices of the triangle.

Two of the sides of the triangle can be represented by $\vec{b} - \vec{a}$ and $\vec{c} - \vec{a}$.

The area of the parallelogram defined by $\vec{b} - \vec{a}$ and $\vec{c} - \vec{a}$ is $\left|\left(\vec{b} - \vec{a}\right) \times \left(\vec{c} - \vec{a}\right)\right|$.

The area of the triangle is one-half of this expression.

$$\Delta\text{Area} = \frac{1}{2}\left|\left(\vec{b}-\vec{a}\right)\times\left(\vec{c}-\vec{a}\right)\right|$$

$$= \frac{1}{2}\left|\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}+\vec{a}\times\vec{a}\right| \quad \text{Distributive property}$$

$$= \frac{1}{2}\left|\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}\right| \quad \vec{a}\times\vec{a}=\vec{0}$$

$$= \frac{1}{2}\left|\vec{b}\times\vec{c}+\vec{a}\times\vec{b}+\vec{c}\times\vec{a}\right| \quad \vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$$

$$= \frac{1}{2}\left|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right| \quad \text{Commutative property}$$

Chapter 7 Section 5 **Question 22 Page 412**

a) Two examples of $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)\times\vec{c}$:

Since $\vec{i}\times\left(\vec{j}\times\vec{k}\right)=\vec{i}\times\vec{i}=\vec{0}$
and $\left(\vec{i}\times\vec{j}\right)\times\vec{k}=\vec{k}\times\vec{k}=\vec{0}$,
$\vec{i}\times\left(\vec{j}\times\vec{k}\right)=\left(\vec{i}\times\vec{j}\right)\times\vec{k}$

and

Since $\vec{j}\times\left(\vec{i}\times\vec{k}\right)=\vec{j}\times\left(-\vec{j}\right)=\vec{0}$
and $\left(\vec{j}\times\vec{i}\right)\times\vec{k}=\left(-\vec{k}\right)\times\vec{k}=\vec{0}$,
$\vec{j}\times\left(\vec{i}\times\vec{k}\right)=\left(\vec{j}\times\vec{i}\right)\times\vec{k}$

b) Let $\vec{a}=[a_1, a_2, a_3]$, $\vec{b}=[b_1, b_2, b_3]$, and $\vec{c}=[c_1, c_2, c_3]$.

$$\text{L.S.} = \vec{a}\times\left(\vec{b}\times\vec{c}\right)$$

$$= [a_1, a_2, a_3]\times\left([b_1, b_2, b_3]\times[c_1, c_2, c_3]\right)$$

$$= [a_1, a_2, a_3]\times[b_2c_3-b_3c_2,\ b_3c_1-b_1c_3,\ b_1c_2-b_2c_1]$$

$$= [a_2(b_1c_2-b_2c_1)-a_3(b_3c_1-b_1c_3)\,a_3(b_2c_3-b_3c_2)-a_1(b_1c_2-b_2c_1),\ a_1(b_3c_1-b_1c_3)-a_2(b_2c_3-b_3c_2)]$$

$$= [a_2b_1c_2-a_2b_2c_1-a_3b_3c_1+a_3b_1c_3,\ a_3b_2c_3-a_3b_3c_2-a_1b_1c_2+a_1b_2c_1,\ a_1b_3c_1-a_1b_1c_3-a_2b_2c_3+a_2b_3c_2]$$

$$\text{R.S.} = \left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$$

$$= \left([a_1, a_2, a_3]\cdot[c_1, c_2, c_3]\right)[b_1, b_2, b_3]-\left([a_1, a_2, a_3]\cdot[b_1, b_2, b_3]\right)[c_1, c_2, c_3]$$

$$= (a_1c_1+a_2c_2+a_3c_3)[b_1, b_2, b_3]-(a_1b_1+a_2b_2+a_3b_3)[c_1, c_2, c_3]$$

$$= [a_1b_1c_1+a_2b_1c_2+a_3b_1c_3,\ a_1b_2c_1+a_2b_2c_2+a_3b_2c_3,\ a_1b_3c_1+a_2b_3c_2+a_3b_3c_3]$$
$$-[a_1b_1c_1+a_2b_2c_1+a_3b_3c_1,\ a_1b_1c_2+a_2b_2c_2+a_3b_3c_2,\ a_1b_1c_3+a_2b_2c_3+a_3b_3c_3]$$

$$= [a_2b_1c_2-a_2b_2c_1-a_3b_3c_1+a_3b_1c_3,\ a_3b_2c_3-a_3b_3c_2-a_1b_1c_2+a_1b_2c_1,\ a_1b_3c_1-a_1b_1c_3-a_2b_2c_3+a_2b_3c_2]$$

Therefore, L.S. = R.S.

Therefore, $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$.

Chapter 7 Section 5 Question 23 Page 412

See Question 12 on page 359.

For a pentagon with vertices (x_1,y_1), (x_2,y_2), (x_3,y_3), (x_4,y_4), and (x_5,y_5), the area is

$$A = \frac{1}{2}\left|x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_4 - x_4y_3 + x_4y_5 - x_5y_4 + x_5y_1 - x_1y_5\right|$$

Therefore, $A = \frac{1}{2}\left|0(7) - 2(5) + 2(6) - 5(7) + 5(4) - 6(6) + 6(2) - 1(4) + 1(5) - 0(2)\right|$

$$= \frac{1}{2}\left|-36\right|$$

$$= 18$$

The area is 18 square units.

Chapter 7 Section 5 Question 24 Page 412

Assume the interior of each square is 20 mm by 20 mm.
The total area of the grid is (120 + 7)(100 + 6).
The centre of the marble can pass through a square that is 10 mm by 10 mm in each grid hole.
Therefore, the area for "success" is 30(100).

The probability it will pass through (success) is $\frac{30(100)}{(127)(106)} \doteq 0.223$ or 22.3%.

Chapter 7 Section 6 **Applications of the Dot Product and Cross Product**

Chapter 7 Section 6 **Question 1 Page 418**

a) $\left|\vec{\tau}\right| = \left|\vec{r}\right|\left|\vec{F}\right|\sin\theta$

$= 0.15(90)\sin 70^\circ$

$\doteq 12.7$

The torque is approximately 12.7 N•m .

b) The torque vector is upward from the material. Therefore, the head of the bolt is being loosened and it will move upward out of the material. [Note: In a real situation, this is true only if the bolt has "regular" right-handed thread.]

Chapter 7 Section 6 **Question 2 Page 418**

a) $\text{proj}_{\vec{v}}\vec{u} = \left(\dfrac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$

$= \left(\dfrac{[3, 1, 4]\cdot[6, 2, 7]}{[6, 2, 7]\cdot[6, 2, 7]}\right)[6, 2, 7]$

$= \left(\dfrac{3(6)+1(2)+4(7)}{6(6)+2(2)+7(7)}\right)[6, 2, 7]$

$= \dfrac{48}{89}[6, 2, 7]$

$\left|\text{proj}_{\vec{v}}\vec{u}\right| = \left|\dfrac{48}{89}[6, 2, 7]\right|$

$= \dfrac{48}{89}\sqrt{6^2+2^2+7^2}$

$= \dfrac{48\sqrt{89}}{89}$

$\doteq 5.1$

b) $\text{proj}_{\vec{v}}\vec{u} = \left(\dfrac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$

$= \left(\dfrac{[5, -4, 8]\cdot[3, 7, 6]}{[3, 7, 6]\cdot[3, 7, 6]}\right)[3, 7, 6]$

$= \left(\dfrac{5(3)+(-4)(7)+8(6)}{3(3)+7(7)+6(6)}\right)[3, 7, 6]$

$= \dfrac{35}{94}[3, 7, 6]$

$$\left|\text{proj}_{\vec{v}}\vec{u}\right| = \left|\frac{35}{94}[3, 7, 6]\right|$$
$$= \frac{35}{94}\sqrt{3^2 + 7^2 + 6^2}$$
$$= \frac{35\sqrt{94}}{94}$$
$$\doteq 3.6$$

c) $$\text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$
$$= \left(\frac{[-2, -7, 3]\cdot[6, 1, -8]}{[6, 1, -8]\cdot[6, 1, -8]}\right)[6, 1, -8]$$
$$= \left(\frac{-2(6) + (-7)(1) + 3(-8)}{6(6) + 1(1) + (-8)(-8)}\right)[6, 1, -8]$$
$$= \frac{-43}{101}[6, 1, -8]$$

$$\left|\text{proj}_{\vec{v}}\vec{u}\right| = \left|\frac{-43}{101}[6, 1, -8]\right|$$
$$= \frac{43}{101}\sqrt{6^2 + 1^2 + (-8)^2}$$
$$= \frac{43\sqrt{101}}{101}$$
$$\doteq 4.3$$

d) $$\text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$
$$= \left(\frac{[1, 0, -1]\cdot[9, 1, 0]}{[9, 1, 0]\cdot[9, 1, 0]}\right)[9, 1, 0]$$
$$= \left(\frac{1(9) + 0(1) + (-1)(0)}{9(9) + 1(1) + 0(0)}\right)[9, 1, 0]$$
$$= \frac{9}{82}[9, 1, 0]$$

$$\left|\text{proj}_{\vec{v}}\vec{u}\right| = \left|\frac{9}{82}[9, 1, 0]\right|$$
$$= \frac{9}{82}\sqrt{9^2 + 1^2 + 0^2}$$
$$= \frac{9\sqrt{82}}{82}$$
$$\doteq 1.0$$

Chapter 7 Section 6 **Question 3 Page 418**

a) To calculate the work done against gravity, use only the vertical components of $\vec{F}$ and $\vec{s}$.

$$W_{\text{gravity}} = \vec{F} \cdot \vec{s}$$
$$= [0, 0, 12] \cdot [0, 0, 8]$$
$$= 96$$

The work done against gravity is 96 J.

$$W = \vec{F} \cdot \vec{s}$$
$$= [3, 5, 12] \cdot [0, 0, 8]$$
$$= 96$$

The work done in the direction of travel is 96 J.

b) $$W_{\text{gravity}} = \vec{F} \cdot \vec{s}$$
$$= [0, 0, 12] \cdot [0, 0, 10]$$
$$= 120$$

The work done against gravity is 120 J.

$$W = \vec{F} \cdot \vec{s}$$
$$= [3, 5, 12] \cdot [2, 0, 10]$$
$$= 126$$

The work done in the direction of travel is 126 J.

c) $$W_{\text{gravity}} = \vec{F} \cdot \vec{s}$$
$$= [0, 0, 12] \cdot [0, 0, 6]$$
$$= 72$$

The work done against gravity is 72 J.

$$W = \vec{F} \cdot \vec{s}$$
$$= [3, 5, 12] \cdot [2, 1, 6]$$
$$= 83$$

The work done in the direction of travel is 83 J.

Chapter 7 Section 6 **Question 4 Page 418**

a) $\vec{a} \times \vec{b} \cdot \vec{c} = (\vec{a} \times \vec{b}) \cdot \vec{c}$
$$= [-2, 3, 5] \times [4, 0, -1] \cdot [2, -2, 3]$$
$$= [3(-1) - 0(5), 5(4) - (-1)(-2), -2(0) - 4(3)] \cdot [2, -2, 3]$$
$$= [-3, 18, -12] \cdot [2, -2, 3]$$
$$= -3(2) + 18(-2) + (-12)(3)$$
$$= -78$$

b) $\vec{a} \cdot \vec{b} \times \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$
$$= [-2, 3, 5] \cdot [4, 0, -1] \times [2, -2, 3]$$
$$= [-2, 3, 5] \cdot [0(3) - (-2)(-1), (-1)(2) - 3(4), 4(-2) - 2(0)]$$
$$= [-2, 3, 5] \cdot [-2, -14, -8]$$
$$= -2(-2) + 3(-14) + 5(-8)$$
$$= -78$$

c) $\vec{a} \times \vec{c} \cdot \vec{b} = (\vec{a} \times \vec{c}) \cdot \vec{b}$
$$= [-2, 3, 5] \times [2, -2, 3] \cdot [4, 0, -1]$$
$$= [3(3) - (-2)(5), 5(2) - 3(-2), -2(-2) - 2(3)] \cdot [4, 0, -1]$$
$$= [19, 16, -2] \cdot [4, 0, -1]$$
$$= 19(4) + 16(0) + (-2)(-1)$$
$$= 78$$

d) $\vec{b} \cdot \vec{a} \times \vec{c} = \vec{b} \cdot (\vec{a} \times \vec{c})$
$$= [4, 0, -1] \cdot [-2, 3, 5] \times [2, -2, 3]$$
$$= [4, 0, -1] \cdot [3(3) - (-2)(5), 5(2) - 3(-2), -2(-2) - 2(3)]$$
$$= [4, 0, -1] \cdot [19, 16, -2]$$
$$= 4(19) + 0(16) + (-1)(-2)$$
$$= 78$$

Chapter 7 Section 6 **Question 5 Page 418**

a) $V = \left|\vec{w} \cdot \vec{u} \times \vec{v}\right|$

$= \left|[1, 2, 7] \cdot [1, 4, 3] \times [2, 5, 6]\right|$

$= \left|[1, 2, 7] \cdot [4(6) - 5(3), 3(2) - 6(1), 1(5) - 2(4)]\right|$

$= \left|[1, 2, 7] \cdot [9, 0, -3]\right|$

$= |-12|$

$= 12$

The volume is 12 cubic units.

b) $V = \left|\vec{w} \cdot \vec{u} \times \vec{v}\right|$

$= \left|[1, 3, 5] \cdot [-2, 5, 1] \times [3, -4, 2]\right|$

$= \left|[1, 3, 5] \cdot [5(2) - (-4)(1), 1(3) - 2(-2), -2(-4) - 3(5)]\right|$

$= \left|[1, 3, 5] \cdot [14, 7, -7]\right|$

$= |0|$

$= 0$

The volume is 0 cubic units.

c) $V = \left|\vec{w} \cdot \vec{u} \times \vec{v}\right|$

$= \left|[-2, 0, 5] \cdot [1, 1, 9] \times [0, 0, 4]\right|$

$= \left|[-2, 0, 5] \cdot [1(4) - 0(9), 9(0) - 4(1), 1(0) - 0(1)]\right|$

$= \left|[-2, 0, 5] \cdot [4, -4, 0]\right|$

$= |-8|$

$= 8$

The volume is 8 cubic units.

Chapter 7 Section 6 Question 6 Page 418

The area of the parallelogram defined by $\vec{a}$ and $\vec{b}$ is $\left|\vec{a}\times\vec{b}\right|$ where $\vec{a}=\overrightarrow{\text{AB}}$ and $\vec{b}=\overrightarrow{\text{AC}}$.

For triangle ΔABC, calculate $\frac{1}{2}\left|\vec{a}\times\vec{b}\right|$.

$$\overrightarrow{\text{AB}}=[9, 7, -7]$$
$$\overrightarrow{\text{AC}}=[7, -1, -1]$$
$$\vec{a}\times\vec{b}=[9, 7, -7]\times[7, -1, -1]$$
$$=[7(-1)-(-1)(-7), -7(7)-(-1)(9), 9(-1)-7(7)]$$
$$=[-14, -40, -58]$$
$$\left|\vec{a}\times\vec{b}\right|=\sqrt{(-14)^2+(-40)^2+(-58)^2}$$
$$\doteq 71.83$$

$$A=\frac{1}{2}(71.83)$$
$$\doteq 35.9$$

The area of the triangle is about 35.9 square units.

Chapter 7 Section 6 Question 7 Page 418

$$\left|\vec{\tau}\right|=\left|\vec{r}\right|\left|\vec{F}\right|\sin\theta$$
$$=0.15(75)\sin 80^{\circ}$$
$$\doteq 11.1$$

The magnitude of the torque is about 11.1 N·m.

Chapter 7 Section 6 Question 8 Page 418

The torques created by each chills should be equal and opposite.
Let x be the distance of the girl from the balance point.
The angles made by the children will be θ and $-\theta$.

$$65(0.6)\sin\theta=-40(x)\sin(-\theta)$$
$$x=\frac{65(0.6)}{40}$$
$$x=0.975$$

The girl should sit 0.975 m form the balance point.

Chapter 7 Section 6 **Question 9 Page 418**

$$\begin{aligned}\vec{u}\times\vec{v} &= [2, 2, 3]\times[1, 3, 4]\\ &= [2(4)-3(3), 3(1)-4(2), 2(3)-1(2)]\\ &= [-1, -5, 4]\end{aligned}$$

$$\begin{aligned}\vec{v}\times\vec{w} &= [1, 3, 4]\times[6, 2, 1]\\ &= [3(1)-2(4), 4(6)-1(1), 1(2)-6(3)]\\ &= [-5, 23, -16]\end{aligned}$$

$$\begin{aligned}\vec{u}\times\vec{w} &= [2, 2, 3]\times[6, 2, 1]\\ &= [2(1)-2(3), 3(6)-1(2), 2(2)-6(2)]\\ &= [-4, 16, -8]\end{aligned}$$

a) $\left|\vec{u}\times\vec{v}\right|^2-\left(\vec{w}\cdot\vec{w}\right)^2 = \left|[-1, -5, 4]\right|^2-\left([6, 2, 1]\cdot[6, 2, 1]\right)^2$

$$\begin{aligned}&= \left(\sqrt{(-1)^2+(-5)^2+4^2}\right)^2-(6(6)+2(2)+1(1))^2\\ &= 1+25+16-1681\\ &= -1639\end{aligned}$$

b) $\left|\vec{u}\times\vec{u}\right|+\vec{u}\cdot\vec{u} = \left|\vec{0}\right|+[2, 2, 3]\cdot[2, 2, 3]$

$$\begin{aligned}&= 0+2^2+2^2+3^2\\ &= 17\end{aligned}$$

c) $\vec{u}\times\vec{v}\cdot\vec{v}\times\vec{w} = [-1, -5, 4]\cdot[-5, 23, -16]$

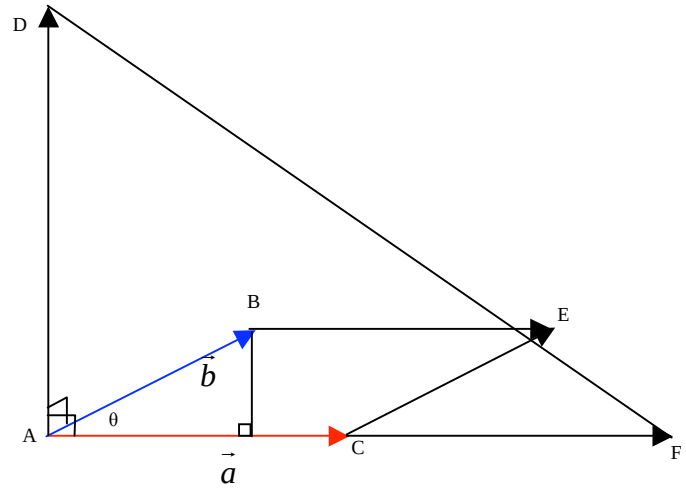
$$\begin{aligned}&= -1(-5)+(-5)(23)+4(-16)\\ &= -174\end{aligned}$$

d) $\vec{u}\times\vec{v}\cdot\vec{u}\times\vec{w} = [-1, -5, 4]\cdot[-4, 16, -8]$

$$\begin{aligned}&= -1(-4)+(-5)(16)+4(-8)\\ &= -108\end{aligned}$$

Chapter 7 Section 6 **Question 10 Page 418**

a)



Given vectors $\vec{a}$ and $\vec{b}$.

Area ABEC = $\left|\vec{a}\times\vec{b}\right|$.

Let $\overrightarrow{AD} = \vec{a}\times\vec{b}$.

Let $\overrightarrow{AF}$ be the product of the projection of $\vec{b}$ on $\vec{a}$ and the magnitude of $\vec{a}$.

$$\left|\overrightarrow{AF}\right| = \left(\left|\vec{b}\right|\cos\theta\right)\left|\vec{a}\right| = \vec{a}\cdot\vec{b}$$

b) By the Pythagorean theorem, $AD^2 + AF^2 = DF^2$.

Therefore, $\left|DF\right|^2 = \left|\vec{a}\times\vec{b}\right|^2 + \left|\vec{a}\cdot\vec{b}\right|^2$.

c) The proof or verification of $\left|\vec{a}\times\vec{b}\right|^2 + \left|\vec{a}\cdot\vec{b}\right|^2 = \left|\vec{a}\right|^2\left|\vec{b}\right|^2$ can be done geometrically or using coordinates.

First show a proof using coordinates.

Let $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$.

$$\text{L.S.} = \left|\vec{a}\times\vec{b}\right|^2 + \left|\vec{a}\cdot\vec{b}\right|^2$$

$$= \left|[a_1, a_2, a_3]\times[b_1, b_2, b_3]\right|^2 + \left|[a_1, a_2, a_3]\cdot[b_1, b_2, b_3]\right|^2$$

$$= \left|[a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1]\right|^2 + \left|a_1b_1 + a_2b_2 + a_3b_3\right|^2$$

$$= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 + (a_1b_1 + a_2b_2 + a_3b_3)^2$$

$$= {a_2}^2{b_3}^2 - 2a_2b_3a_3b_2 + {a_3}^2{b_2}^2 + {a_3}^2{b_1}^2 - 2a_1b_3a_3b_1 + {a_1}^2{b_3}^2 + {a_1}^2{b_2}^2 - 2a_2b_1a_1b_2 + {a_2}^2{b_1}^2$$
$$+ {a_1}^2{b_1}^2 + {a_2}^2{b_2}^2 + {a_3}^2{b_3}^2 + 2a_1b_1a_2b_2 + 2a_2b_2a_3b_3 + 2a_1b_1a_3b_3$$

$$= {a_1}^2{b_1}^2 + {a_1}^2{b_2}^2 + {a_1}^2{b_3}^2 + {a_2}^2{b_1}^2 + {a_2}^2{b_2}^2 + {a_2}^2{b_3}^2 + {a_3}^2{b_1}^2 + {a_3}^2{b_2}^2 + {a_3}^2{b_3}^2$$

$$\text{R.S.} = |\vec{a}|^2 |\vec{b}|^2$$
$$= \left\| a_1, a_2, a_3 \right\|^2 \left\| b_1, b_2, b_3 \right\|^2$$
$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$
$$= a_1^2 b_1^2 + a_1^2 b_2^2 + a_1^2 b_3^2 + a_2^2 b_1^2 + a_2^2 b_2^2 + a_2^2 b_3^2 + a_3^2 b_1^2 + a_3^2 b_2^2 + a_3^2 b_3^2$$

Therefore, L.S. = R.S.

Now show a proof using geometric vectors.

$$\text{L.S.} = |\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2$$
$$= \left| |\vec{a}||\vec{b}| \sin\theta \right|^2 + \left| |\vec{a}||\vec{b}| \cos\theta \right|^2$$
$$= |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta + |\vec{a}|^2 |\vec{b}|^2 \cos^2\theta$$
$$= |\vec{a}|^2 |\vec{b}|^2 (\sin^2\theta + \cos^2\theta)$$
$$= |\vec{a}|^2 |\vec{b}|^2 (1)$$
$$= |\vec{a}|^2 |\vec{b}|^2$$
$$\text{R.S.} = |\vec{a}|^2 |\vec{b}|^2$$

Therefore, L.S. = R.S.

Chapter 7 Section 6 Question 11 Page 418

Assume each force makes a 90º angle with the radius. However, one angle is clockwise (positive) and one is counterclockwise (negative).

$$|\vec{\tau}| = |\vec{r}||\vec{F}| \sin\theta$$

$$|\vec{\tau}_{\text{total}}| = (0.75)10 \sin 90^\circ + 0.35(5)\sin(-90^\circ)$$
$$\doteq 5.75$$

The magnitude of the total torque is about 5.75 N·m. The direction will be that of the 10-N torque (along the axle, away from you).

Chapter 7 Section 6 **Question 12 Page 419**

$$\left|\vec{\tau}\right| = \left|\vec{r}\right|\left|\vec{F}\right|\sin\theta$$
$$10 = 0.20(80)\sin\theta$$
$$\sin\theta = \frac{10}{(0.20)80}$$
$$\theta \doteq 38.7$$

The force is applied to the wrench at about a 38.7º angle.

Chapter 7 Section 6 **Question 13 Page 419**

Answers will vary.

• Torque is the force responsible for getting a vehicle to start moving from a stop position and it pulls the vehicle up steep hills. This measurement is important to know when

Horsepower is responsible for moving a vehicle along and allows it to cruise on the highway and accelerate in normal conditions.

A higher torque rating is more important than a higher horsepower rating when pulling a trailer, hauling a heavy load, or driving on a long, steep road.

A higher horsepower is more important than a higher torque rating when determining how much time it takes to go from stoplight to stoplight or if performing a quick acceleration.

• Torque is measured in pound-feet.

The abbreviated measurement unit for horsepower is hp.
1 hp = 550 foot-pounds per second

$$\text{Horsepower} = \frac{\text{torque} \times \text{engine speed}}{5252}$$

• When plotting graphs involving torque, the independent variable or torque is measured in Newton•metres. The dependent variable may be measured in revolutions per minute—a measure of engine speed or amperes—a measure of current.

Chapter 7 Section 6 **Question 14 Page 419**

The statement is false.

$\left|\vec{e}\times\vec{f}\right| = 0$ if $\vec{e}$ and $\vec{f}$ are collinear and the angle between them is 0° or 180°.
$\left|\vec{e}\cdot\vec{f}\right| = 0$ if $\vec{e}$ and $\vec{f}$ are perpendicular and the angle between them is 90°.

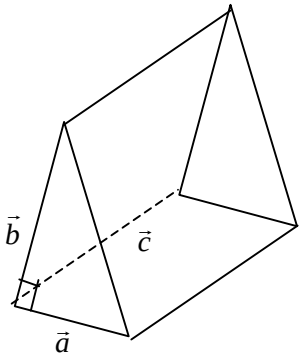
Chapter 7 Section 6 **Question 15 Page 419**

$$\left|\vec{\tau}\right| = \left|\vec{r}\right|\left|\vec{F}\right|\sin\theta$$
$$100 = 0.30\left|\vec{F}\right|\sin 40^\circ$$
$$\left|\vec{F}\right| = \frac{100}{(0.30)\sin 40^\circ}$$
$$\doteq 518.6$$

The magnitude of the force is about 518.6 N.

Chapter 7 Section 6 **Question 16 Page 419**

a)



b) $V = \frac{1}{2}\left|\vec{a}\right|\left|\vec{b}\right|\left|\vec{c}\right|$
$$= \frac{1}{2}\sqrt{14}\sqrt{20}\sqrt{180}$$
$$= 7.5\sqrt{14}$$
$$\doteq 28.1$$

The volume of the prism is approximately 28.1 cubic units.

c) If the prism was not a right triangular prism, the volume of the prism can be determined by the same formula but the perpendicular distance between the faces must be found first before the volume formula can be applied.

The volumes of the right and non-right prism would not be the equivalent given equal bases and height of the triangle.

Chapter 7 Section 4 Question 17 Page 419

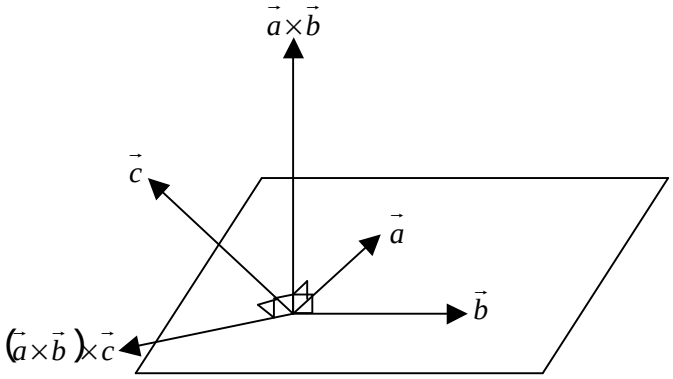
Let $\vec{u} = [u_1, u_2, u_3]$ and $\vec{v} = [v_1, v_2, v_3]$

$$\begin{aligned}
\text{L.S.} &= \vec{u} \times \vec{v} \cdot \vec{w} \\
&= \vec{u} \times \vec{v} \cdot \left(k\vec{u} + m\vec{v}\right) \\
&= [u_1, u_2, u_3] \times [v_1, v_2, v_3] \cdot \left[k[u_1, u_2, u_3] + m[v_1, v_2, v_3]\right] \\
&= [u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1] \cdot [ku_1 + mv_1,\ ku_2 + mv_2,\ ku_3 + mv_3] \\
&= (u_2v_3 - u_3v_2)(ku_1 + mv_1) + (u_3v_1 - u_1v_3)(ku_2 + mv_2) + (u_1v_2 - u_2v_1)(ku_3 + mv_3) \\
&= ku_1u_2v_3 + mu_2v_1v_3 - ku_1u_3v_2 - mu_3v_1v_2 + ku_2u_3v_1 + mu_3v_1v_2 - ku_1u_2v_3 - mu_1v_2v_3 \\
&\quad + ku_1u_3v_2 + mu_1v_2v_3 - ku_2u_3v_1 - mu_2v_1v_3 \\
&= 0 \\
\text{R.S.} &= 0
\end{aligned}$$

Therefore, L.S. = R.S.

Chapter 7 Section 6 Question 18 Page 419

For an informal proof, draw a diagram.



Suppose $\vec{a}$ and $\vec{b}$ lie in a plane and $\vec{c}$ is a vector mot in this plane.
$\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.
$(\vec{a} \times \vec{b}) \times \vec{c}$ is perpendicular $\vec{a} \times \vec{b}$ and $\vec{c}$. But the vectors perpendicular to $\vec{a} \times \vec{b}$ can only be in the plane containing both $\vec{a}$ and $\vec{b}$. Therefore $(\vec{a} \times \vec{b}) \times \vec{c}$ lies in the plane containing $\vec{a}$ and $\vec{b}$.

For a more formal proof, show that the volume of the parallelepiped formed by $(\vec{a} \times \vec{b}) \times \vec{c}$, $\vec{a}$, and $\vec{b}$ has volume zero; i.e., $(\vec{a} \times \vec{b}) \times \vec{c}$, $\vec{a}$, and $\vec{b}$ are coplanar vectors.

Let $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$, and $\vec{c} = [c_1, c_2, c_3]$.

$$
\begin{aligned}
V &= \left|\left(\left(\vec{a}\times\vec{b}\right)\times\vec{c}\right)\cdot\vec{a}\times\vec{b}\right| \\
&= \left|\left(\left(\left[a_1, a_2, a_3\right]\times\left[b_1, b_2, b_3\right]\right)\times\left[c_1, c_2, c_3\right]\right)\cdot\left[a_1, a_2, a_3\right]\times\left[b_1, b_2, b_3\right]\right| \\
&= \left|\left[a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\right]\times\left[c_1, c_2, c_3\right]\cdot\left[a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\right]\right| \\
&= \left|\begin{array}{l}\left[(a_3b_1 - a_1b)c_3 - c_2(a_1b_2 - a_2b_1),\ (a_1b_2 - a_2b)c_1 - c_3(a_2b_3 - a_3b_2),\ (a_2b_3 - a_3b_2)c_2 - c_1(a_3b_1 - a_1b_3)\right] \\ \cdot\left[a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\right]\end{array}\right| \\
&= \left|\begin{array}{l}\left[a_3b_1c_3 - a_1b_3c_3 - a_1b_2c_2 + a_2b_1c_2,\ a_1b_2c_1 - a_2b_1c_1 - a_2b_3c_3 + a_3b_2c_3,\ a_2b_3c_2 - a_3b_2c_2 - a_3b_1c_1 + a_1b_3c_1\right] \\ \cdot\left[a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\right]\end{array}\right| \\
&= \left|\begin{array}{l}(a_3b_1c_3 - a_1b_3c_3 - a_1b_2c_2 + a_2b_1c_2)(a_2b_3 - a_3b_2) + (a_1b_2c_1 - a_2b_1c_1 - a_2b_3c_3 + a_3b_2c_3)(a_3b_1 - a_1b_3) \\ +(a_2b_3c_2 - a_3b_2c_2 - a_3b_1c_1 + a_1b_3c_1)(a_1b_2 - a_2b_1)\end{array}\right| \\
&= \left|\begin{array}{l}a_2a_3b_1b_3c_3 - a_1a_2b_3b_3c_3 - a_1a_2b_2b_3c_2 + a_2a_2b_1b_3c_2 - a_3a_3b_1b_2c_3 + a_1a_3b_2b_3c_3 + a_1a_3b_2b_2c_2 - a_2a_3b_1b_2c_2 \\ +a_1a_3b_1b_2c_1 - a_2a_3b_1b_1c_1 - a_2a_3b_1b_3c_3 + a_3a_3b_1b_2c_3 - a_1a_1b_2b_3c_1 + a_1a_2b_1b_3c_1 + a_1a_2b_3b_3c_3 - a_1a_3b_2b_3c_3 \\ +a_1a_2b_2b_3c_2 - a_1a_3b_2b_2c_2 - a_1a_3b_1b_2c_1 + a_1a_1b_2b_3c_1 - a_2a_2b_1b_3c_2 + a_2a_3b_1b_2c_2 + a_2a_3b_1b_1c_1 - a_1a_2b_1b_3c_1\end{array}\right| \\
&= |0| \\
&= 0
\end{aligned}
$$

Chapter 7 Section 6 Question 19 Page 419

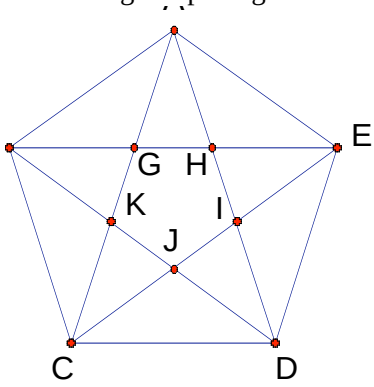
If $\vec{u}, \vec{v},$ and $\vec{w}$ are mutually orthogonal, they could be considered to form axes for 3-space in the same way as do $\vec{i}, \vec{j},$ and $\vec{k}$. They create three orthogonal planes: *uv*- plane, *uw*-plane, and *vw*-plane.
Then you can say that $\vec{u}+\vec{v}$ lies in the *uv*- plane, $\vec{u}+\vec{w}$ lies in the *uw*-plane, and $\vec{v}+\vec{w}$ lies in the *vw*-plane.
In general, these three new vectors are not themselves orthogonal. For example, if $\vec{i}, \vec{j},$ and $\vec{k}$ are the original vectors, then the new vectors are $\vec{i}+\vec{j} = [1, 1, 0]$, $\vec{j}+\vec{k} = [0, 1, 1]$, and $\vec{i}+\vec{k} = [1, 0, 1]$.

$$
\begin{aligned}
\left(\vec{i}+\vec{j}\right)\cdot\left(\vec{j}+\vec{k}\right) &= [1, 1, 0]\cdot[0, 1, 1] \\
&= 1 \\
&\neq 0
\end{aligned}
$$

Therefore, $\vec{i}+\vec{j}$ is not perpendicular to $\vec{j}+\vec{k}$.

Chapter 7 Section 6 **Question 20 Page 419**

Draw a regular pentagon ABCDE. The interior angles are 540º ÷ 5 = 108º.



There are many isosceles triangles in the diagram. This leads to labelling all of the angles as either 36º, 72º, or 108º.

ΔBKG is an isosceles triangle having the same shape as the triangle in the contest question.
ΔACD is similar to ΔBKG. (equal angles)
Let CD = 1 and AD = x. By similar triangles, x is the measure needed for the question.

Consider the triangles ΔHAE ~ ΔEAD.

$$\frac{HA}{AE} = \frac{DE}{AD}$$

$$\frac{HA}{1} = \frac{1}{x}$$

$$HA = \frac{1}{x}$$

ΔDHE is isosceles, so DE = DH = 1.

Therefore,

$$1 + \frac{1}{x} = x$$

$$x^2 - x - 1 = 0$$

$$x = \frac{1 \pm \sqrt{5}}{2} \qquad \text{(Quadratic formula)}$$

Since x must be positive, $x = \frac{1 + \sqrt{5}}{2}$ cm.

Chapter 7 Section 6 Question 21 Page 419

The first term is 333; the second term counts the symbols involved in the first term, namely, 3 threes.
The second term is 33; the third term counts these symbols: 2 threes.
The third term is 23; the fourth term counts these symbols: 1 two and 1 three.
Additional terms are 112113, 11121113, 1111211113, 111112111113, …

Chapter 7 Review

Chapter 7 Review **Question 1 Page 420**

a) $\vec{v} = [-6, 3]$

$= -6\vec{i} + 3\vec{j}$

b) $|\vec{v}| = \sqrt{(-6)^2 + 3^3}$

$= \sqrt{45}$

$= 3\sqrt{5}$

Unit vectors collinear with $\vec{v}$ have the form $\pm\frac{1}{|\vec{v}|}\vec{v}$

The required unit vectors are $\left[-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right]$ and $\left[\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right]$.

c) Let B(x, y) be the point.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= [x, y] - [2, 9]$

$= [x - 2, y - 9]$

$= [-6, 3]$

$x - 2 = -6$ $\quad$ $y - 9 = 3$

$x = -4$ $\quad$ $y = 12$

Therefore, B(–4, 12) is the required point.

Chapter 7 Review **Question 2 Page 420**

a) $-5[5, -2] = [-25, 10]$

b) $\vec{u} + \vec{v} = [5, -2] + [8, 5]$

$= [13, 3]$

c) $4\vec{u} + 2\vec{v} = 4[5, -2] + 2[8, 5]$

$= [20, -8] + [16, 10]$

$= [36, 2]$

d) $3\vec{u} - 7\vec{v} = 3[5, -2] - 7[8, 5]$

$= [15, -6] - [56, 35]$

$= [-41, -41]$

Chapter 7 Review **Question 3 Page 420**

The airplane's vector is $[345\cos 50^\circ, 345\sin 50^\circ] \doteq [221.8, 264.3]$.

The wind's vector is $[18\cos 183^\circ, 18\sin 183^\circ] \doteq [-17.98, -0.94]$.

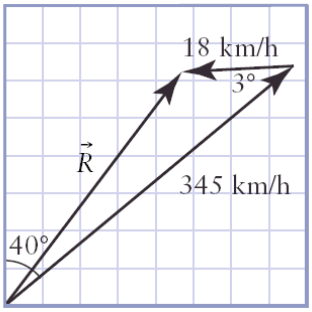
The resultant vector is [203.82, 263.36].

The magnitude of the resultant is $\sqrt{(203.82)^2 + (263.36)^2} \doteq 333.02$.

The angle of the resultant with the *east* direction is $\tan^{-1}\left(\frac{263.36}{203.82}\right) \doteq 52.3^\circ$.

The bearing is $90^\circ - 52.3^\circ = 37.7^\circ$.

The ground velocity of the airplane is about 333.0 km/h on a bearing of 37.7°.



Chapter 7 Review **Question 4 Page 420**

a) $\vec{u} \cdot \vec{v} = 20(15)\cos 70^\circ$

$\doteq 102.6$

b) $\vec{p} \cdot \vec{q} = 425(300)\cos 110^\circ$

$\doteq -43\,607.6$

Chapter 7 Review **Question 5 Page 420**

a) $\vec{u} \cdot \vec{v} = 5(-6) + 2(7)$

$= -16$

b) $\vec{u} \cdot \vec{v} = (-3)(3) + 2(7)$

$= 5$

c) $\vec{u} \cdot \vec{v} = 3(4) + 2(-6)$

$= 0$

Chapter 7 Review **Question 6 Page 420**

The vectors in part c) are orthogonal since $\vec{u} \cdot \vec{v} = 0$.

Chapter 7 Review **Question 7 Page 420**

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|}$$

$$= \frac{20}{5.2(7.3)}$$

$$\theta = \cos^{-1}\left(\frac{20}{5.2(7.3)}\right)$$

$$\doteq 58.2^{\circ}$$

Chapter 7 Review **Question 8 Page 420**

a) $\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|}$

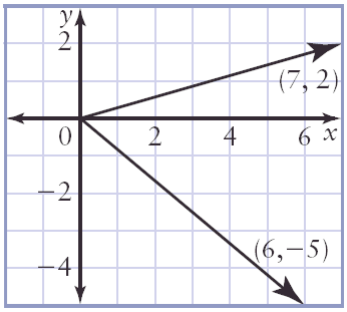
$$= \frac{[6, -5] \cdot [7, 2]}{\left|[6, -5]\right|\left|[7, 2]\right|}$$

$$= \frac{6(7) + (-5)(2)}{\sqrt{6^2 + (-5)^2}\sqrt{7^2 + 2^2}}$$

$$= \frac{32}{56.8595}$$

$$\theta = \cos^{-1}\left(\frac{32}{56.8595}\right)$$

$$\doteq 55.8^{\circ}$$



b) $\cos\theta = \dfrac{\vec{p}\cdot\vec{q}}{\left|\vec{p}\right|\left|\vec{q}\right|}$

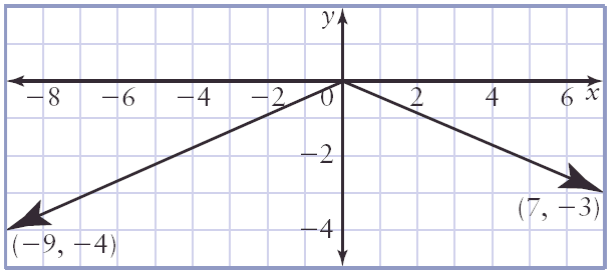
$= \dfrac{[-9,-4]\cdot[7,-3]}{\left|[-9,-4]\right|\left|[7,-3]\right|}$

$= \dfrac{-9(7)+(-4)(-3)}{\sqrt{(-9)^2+(-4)^2}\sqrt{7^2+(-3)^2}}$

$= \dfrac{-51}{75.0067}$

$\theta = \cos^{-1}\left(\dfrac{-51}{75.0067}\right)$

$\doteq 132.8^\circ$



Chapter 7 Review **Question 9 Page 420**

a) $\left|\text{proj}_{\vec{v}}\vec{u}\right| = \left|\vec{u}\right|\cos\theta$

$= 56\cos 125^\circ$

$\doteq -32.1$

The projection has magnitude 32.1 and has direction opposite to $\vec{v}$.

b) $\text{proj}_{\vec{v}}\vec{u} = \left(\dfrac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$

$= \left(\dfrac{[7,1]\cdot[9,-3]}{[9,-3]\cdot[9,-3]}\right)[9,-3]$

$= \left(\dfrac{60}{90}\right)[9,-3]$

$\doteq [6,-2]$

Chapter 7 Review **Question 10 Page 420**

a) $\vec{F} \cdot \vec{d} = [16, 12] \cdot [3, 9]$
$= 16(3) + 12(9)$
$= 156$

The work done is 156 $\text{N} \cdot \text{m}$ or 156 J.

b) $\vec{F} \cdot \vec{d} = [200, 2000] \cdot [3, 45]$
$= 200(3) + 2000(45)$
$= 90\ 600$

The work done is 90 600 $\text{N} \cdot \text{m}$ or 90 6000 J.

Chapter 7 Review **Question 11 Page 420**

a) The total revenue can be represented by [125, 70]·[229, 329].

b) $R = 125(229) + 70(329)$
$= 51\ 655$

The total revenue from sales is $51 655.00

Chapter 7 Review **Question 12 Page 420**

a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
$= [-5, 9, -1] - [2, 7, 8]$
$= [-7, 2, -9]$
$\left|\overrightarrow{AB}\right| = \sqrt{(-7)^2 + 2^2 + (-9)^2}$
$= \sqrt{134}$

b) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$
$= [4, -9, 7] - [0, 3, 6]$
$= [4, -12, 1]$
$\left|\overrightarrow{PQ}\right| = \sqrt{4^2 + (-12)^2 + 1^2}$
$= \sqrt{161}$

Chapter 7 Review **Question 13 Page 420**

a) $5\vec{a} - 4\vec{b} + 3\vec{c} = 5\left[3, -7, 8\right] - 4\left[-6, 3, 4\right] + 3\left[2, 5, 7\right]$

$= \left[15, -35, 40\right] - \left[-24, 12, 16\right] + \left[6, 15, 21\right]$

$= \left[45, -32, 45\right]$

b) $-5\vec{a} \cdot \vec{c} = -5\left[3, -7, 8\right] \cdot \left[2, 5, 7\right]$

$= \left[-15, 35, -40\right] \cdot \left[2, 5, 7\right]$

$= -15(2) + 35(5) + (-40)(7)$

$= -135$

c) $\vec{b} \cdot \left(\vec{c} - \vec{a}\right) = \left[-6, 3, 4\right] \cdot \left(\left[2, 5, 7\right] - \left[3, -7, 8\right]\right)$

$= \left[-6, 3, 4\right] \cdot \left[-1, 12, -1\right]$

$= (-6)(-1) + 3(12) + 4(-1)$

$= 38$

Chapter 7 Review **Question 14 Page 420**

Orthogonal vectors have a dot product equal to zero.

$\left[6, 1, 8\right] \cdot \left[k, -4, 5\right] = 0$

$6k - 4 + 40 = 0$

$6k = -36$

$k = -6$

Chapter 7 Review **Question 15 Page 420**

a) $\vec{u} \times \vec{v} = \left|\vec{u}\right|\left|\vec{v}\right|(\sin\theta)\hat{n}$

$= 200(350)\sin 110^\circ \hat{n}$

$\doteq 65\,778.5\hat{n}$

b) $\vec{u} \times \vec{v} = \left[4, 1, -3\right] \times \left[3, 7, 8\right]$

$= \left[1(8) - 7(-3), -3(3) - 8(4), 4(7) - 3(1)\right]$

$= \left[29, -41, 25\right]$

Chapter 7 Review **Question 16 Page 420**

$$\begin{aligned}
A &= \left|\vec{u}\times\vec{v}\right| \\
&= \left|\left[6, 8, 9\right]\times\left[3, -1, 2\right]\right| \\
&= \left|\left[8(2)-(-1)(9), 9(3)-2(6), 6(-1)-3(8)\right]\right| \\
&= \left|\left[25, 15, -30\right]\right| \\
&= \sqrt{25^2+15^2+(-30)^2} \\
&\doteq 41.8
\end{aligned}$$

Chapter 7 Review **Question 17 Page 420**

Let $\vec{a}=\left[1, 0, 0\right]$, $\vec{b}=\left[1, 1, 1\right]$, and $\vec{c}=\left[0, 0, 1\right]$.

$$\begin{aligned}
\text{L.S.} &= \vec{a}\times\left(\vec{b}+\vec{c}\right) \\
&= \left[1, 0, 0\right]\times\left(\left[1, 1, 1\right]+\left[0, 0, 1\right]\right) \\
&= \left[1, 0, 0\right]\times\left[1, 1, 2\right] \\
&= \left[0(2)-1(0), 0(1)-2(1), 1(1)-1(0)\right] \\
&= \left[0, -2, 1\right]
\end{aligned}$$

$$\begin{aligned}
\text{R.S.} &= \vec{a}\times\vec{b}+\vec{a}\times\vec{c} \\
&= \left[1, 0, 0\right]\times\left[1, 1, 1\right]+\left[1, 0, 0\right]\times\left[0, 0, 1\right] \\
&= \left[0(1)-1(0), 0(1)-1(1), 1(1)-1(0)\right]+\left[0(1)-0(0), 0(0)-1(1), 1(0)-0(0)\right] \\
&= \left[0, -1, 1\right]+\left[0, -1, 0\right] \\
&= \left[0, -2, 1\right]
\end{aligned}$$

Therefore, L.S. = R.S. and $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$.

Chapter 7 Review **Question 18 Page 420**

a) $\left|\vec{\tau}\right| = \left|\vec{r}\right|\left|\vec{F}\right|\sin\theta$

$$\begin{aligned}
&= 0.10(200)\sin 80^\circ \\
&\doteq 19.7
\end{aligned}$$

The torque is approximately 19.7 $\text{N}\cdot\text{m}$.

b) The torque vector points downward into the material, along the direction of the bolt. The bolt is being tightened.
(Note this practical example assumes that the bolt has normal right-handed thread.)

Chapter 7 Review **Question 19 Page 420**

$$\begin{aligned}
\text{proj}_{\vec{v}}\vec{u} &= \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} \\
&= \left(\frac{[-2, 5, 3]\cdot[4, -8, 9]}{[4, -8, 9]\cdot[4, -8, 9]}\right)[4, -8, 9] \\
&= \left(\frac{-2(4)+5(-8)+3(9)}{4(4)+(-8)(-8)+9(9)}\right)[4, -8, 9] \\
&= \frac{-21}{161}[4, -8, 9]
\end{aligned}$$

$$\begin{aligned}
\left|\text{proj}_{\vec{v}}\vec{u}\right| &= \left|\frac{-21}{161}[4, -8, 9]\right| \\
&= \frac{21}{161}\sqrt{4^2+(-8)^2+9^2} \\
&= \frac{21\sqrt{161}}{161} \\
&\doteq 1.7
\end{aligned}$$

Chapter 7 Practice Test

Chapter 7 Practice Test **Question 1 Page 422**

B is the best answer.

$$\begin{aligned}[3, -4]\cdot[6, k] &= 0\\ 18 - 4k &= 0\\ 4k &= 18\\ k &= 4.5\end{aligned}$$

Chapter 7 Practice Test **Question 2 Page 422**

D is incorrect.

Chapter 7 Practice Test **Question 3 Page 422**

D is the best answer.

A = 1(1)sin 90º
= 1

B = 1(1)sin 60º
= 0.9

C = 4(1)sin 180º
= 0

D = 4(0.5)sin 90º
= 2

Chapter 7 Practice Test **Question 4 Page 422**

D is the best answer.
Position vectors for points in the *yz*-plane are of the form [0, *y*, *z*].
Some examples are [0, 1, 1], [0, 4, 3], and [0, –2, –5].

Chapter 7 Practice Test **Question 5 Page 422**

C is the best answer.

A [–8, –6]·[–6, 8] = 0

B [8, 6]·[–6, 8] = 0

C [4, –3]·[–6, 8] = –48

D [–4, –3]·[–6, 8] = 0

Chapter 7 Practice Test **Question 6 Page 422**

C is the best answer.

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA}\\ &= [0, 2, -4] - [1, 3, -7]\\ &= [-1, -1, 3]\\ \left|\overrightarrow{AB}\right| &= \sqrt{(-1)^2 + (-1)^2 + 3^2}\\ &= \sqrt{11}\end{aligned}$$

Chapter 7 Practice Test **Question 7 Page 422**

a) $$\begin{aligned}\vec{u} \cdot \vec{v} &= [8, 3] \cdot [2, 7]\\ &= 8(2) + 3(7)\\ &= 37\end{aligned}$$

b) $$\begin{aligned}\cos\theta &= \frac{\vec{u} \cdot \vec{v}}{\left|\vec{u}\right|\left|\vec{v}\right|}\\ &= \frac{37}{\left|[8, 3]\right|\left|[2, 7]\right|}\\ &= \frac{37}{\sqrt{8^2 + 3^2}\sqrt{2^2 + 7^2}}\\ &= \frac{37}{62.2013}\\ \theta &= \cos^{-1}\left(\frac{37}{62.2013}\right)\\ &\doteq 53.5^\circ\end{aligned}$$

c) $$\begin{aligned}\text{proj}_{\vec{u}}\vec{v} &= \left(\frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\right)\vec{u}\\ &= \left(\frac{37}{[8, 3] \cdot [8, 3]}\right)[8, 3]\\ &= \left(\frac{37}{8(8) + 3(3)}\right)[8, 3]\\ &= \frac{37}{73}[8, 3]\\ &\doteq [4.1, 1.5]\end{aligned}$$

d) No. The projections are not equal. $\text{Proj}_{\vec{v}}\vec{u}$ is in the direction of $\vec{v}$ while the direction of $\text{proj}_{\vec{u}}\vec{v}$ is in the direction of $\vec{u}$, and these are clearly different directions.

Chapter 7 Practice Test **Question 8 Page 422**

Let the price vector be $\vec{p} = [399, 129]$ and the sales vector be $\vec{s} = [100, 240]$.

$$\begin{aligned}\vec{p}\cdot\vec{s} &= [399, 129]\cdot[100, 240]\\ &= 399(100) + 129(240)\\ &= 70\ 860\end{aligned}$$

The total revenue is \$70 860.

Chapter 7 Practice Test **Question 9 Page 422**

a)
$$\begin{aligned}\vec{u}\times\vec{v} &= [5, 8, 2]\times[-7, 3, 6]\\ &= [8(6) - 3(2), 2(-7) - 6(5), 5(3) - (-7)(8)]\\ &= [42, -44, 71]\end{aligned}$$
$$\begin{aligned}\vec{u}\cdot\vec{v} &= [5, 8, 2]\cdot[-7, 3, 6]\\ &= 5(-7) + 8(3) + 2(6)\\ &= 1\end{aligned}$$

b)
$$\begin{aligned}\vec{u}\times\vec{v} &= [1, 2, -5]\times[3, -4, 0]\\ &= [2(0) - (-4)(-5), -5(3) - 0(1), 1(-4) - 3(2)]\\ &= [-20, -15, -10]\end{aligned}$$
$$\begin{aligned}\vec{u}\cdot\vec{v} &= [1, 2, -5]\cdot[3, -4, 0]\\ &= 1(3) + 2(-4) + (-5)(0)\\ &= -5\end{aligned}$$

c)
$$\begin{aligned}\vec{u}\times\vec{v} &= [-3, 0, 0]\times[7, 0, 0]\\ &= [0(0) - 0(0), 0(7) - 0(-3), -3(0) - 7(0)]\\ &= [0, 0, 0]\end{aligned}$$
$$\begin{aligned}\vec{u}\cdot\vec{v} &= [-3, 0, 0]\cdot[7, 0, 0]\\ &= -3(7) + 0(0) + 0(0)\\ &= -21\end{aligned}$$

d) $\vec{u} \times \vec{v} = [1, -9, 7] \times [9, 1, 0]$

$= [-9(0) - 1(7), 7(9) - 0(1), 1(1) - 9(-9)]$

$= [-7, 63, 82]$

$\vec{u} \cdot \vec{v} = [1, -9, 7] \cdot [9, 1, 0]$

$= 1(9) + (-9)(1) + 7(0)$

$= 0$

Chapter 7 Practice Test **Question 10 Page 422**

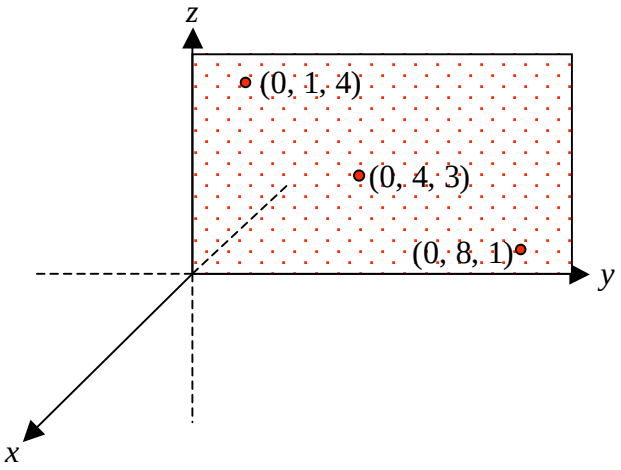
a) The vectors in part c are collinear since $\vec{u} \times \vec{v} = \vec{0}$.

b) The vectors in part d are orthogonal since $\vec{u} \cdot \vec{v} = 0$.

Chapter 7 Practice Test **Question 11 Page 423**

a) Position vectors for points in the *yz*-plane are of the form [0, *y*, *z*].
Some examples of points in the plane are (0, 1, 4), (0, 4, 3), and (0, –2, –5).

The diagram shows some of the points with positive *y*- and *z*-coordinates.



b) These points form a cylinder in 3-space with the *x*-axis as the central axis of symmetry of the cylinder. The cylinder is defined by the points $(x, 10\cos\theta, 10\sin\theta)$. Each point on the cylinder is 10 units from the *x*-axis.

The equation of the cylinder is $y^2 + z^2 = 100$.

Chapter 7 Practice Test **Question 12 Page 423**

a) $\left|\vec{u}\right| = \sqrt{2^2 + 4^2 + (-9)^2}$

$= \sqrt{101}$

Unit vectors collinear with $\vec{u}$ have the form $\pm\frac{1}{\left|\vec{u}\right|}\vec{u}$.

The two unit vectors collinear with $\vec{u}$ are $\left[\frac{2}{\sqrt{101}}, \frac{4}{\sqrt{101}}, -\frac{9}{\sqrt{101}}\right]$ and $\left[-\frac{2}{\sqrt{101}}, -\frac{4}{\sqrt{101}}, \frac{9}{\sqrt{101}}\right]$.

b) Choose any vector $\vec{v}$ that is perpendicular to $\vec{u}$. You want a vector that makes the dot product $\vec{u} \cdot \vec{v} = 0$.

Choose a convenient vector such as $\vec{v} = [2, -1, 0]$.

$\left|\vec{v}\right| = \sqrt{2^2 + (-1)^2 + 0^2}$

$= \sqrt{5}$

Therefore two possible unit vectors collinear with $\vec{u}$ are $\left[\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, 0\right]$ and $\left[-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right]$.

c) The choice of vector $\vec{v}$ was somewhat arbitrary. You just need the dot product $\vec{u} \cdot \vec{v} = 0$.

In general, if $\vec{v} = [v_1, v_2, v_3]$, then $2v_1 + 4v_2 - 9v_3 = 0$. For every choice of v_1 and v_2, there will be a (unique) value for v_3. Every $\vec{v}$ created in this way, will lead to two orthogonal (to $\vec{u}$) unit vectors. Hence, there are an infinite number of solutions.

Chapter 7 Practice Test **Question 13 Page 423**

The area of the triangle is one-half of the area of the parallelogram defined by two of the sides of the triangle.

$A_\Delta = \frac{1}{2}\left|\vec{a} \times \vec{b}\right|$

$= \frac{1}{2}\left|\vec{a}\right|\left|\vec{b}\right|\left|\sin\theta\right|$

$= \frac{1}{2}(14)(17)\sin 24^\circ$

$\doteq 48.4$

The area of the triangle is approximately 48.4 square units.

Chapter 7 Practice Test **Question 14 Page 423**

One vector perpendicular to $\vec{c}$ and $\vec{d}$ is $\vec{c}\times\vec{d}$.

$$\begin{aligned}\vec{c}\times\vec{d} &= [-5, 6, 2]\times[3, 3, -8]\\ &= [6(-8)-3(2),\ 2(3)-(-8)(-5),\ -5(3)-3(6)]\\ &= [-54, -34, -33]\end{aligned}$$

$$\begin{aligned}|\vec{c}\times\vec{d}| &= \sqrt{(-54)^2+(-34)^2+(-33)^2}\\ &= \sqrt{5161}\end{aligned}$$

The required unit vectors are $\left[-\frac{54}{\sqrt{5161}}, -\frac{34}{\sqrt{5161}}, -\frac{33}{\sqrt{5161}}\right]$ and $\left[\frac{54}{\sqrt{5161}}, \frac{34}{\sqrt{5161}}, \frac{33}{\sqrt{5161}}\right]$.

Chapter 7 Practice Test **Question 15 Page 423**

a) The standard formula for computing the area of a parallelogram requires 3-D vectors. You can achieve this by embedding the given vectors in 3-space by including a *z*-coordinate of zero for each vector.

$$\begin{aligned}A &= |\vec{a}\times\vec{b}|\\ &= |[1, -4, 0]\times[3, 5, 0]|\\ &= |[-4(0)-5(0),\ 0(3)-0(1),\ 1(5)-3(-4)]|\\ &= |[0, 0, 17]|\\ &= 17\end{aligned}$$

The area is 17 square units.

b)

$$\begin{aligned}A &= |\vec{u}\times\vec{v}|\\ &= |[-3, 2, 0]\times[6, -4, 2]|\\ &= |[2(2)-(-4)(0),\ 0(6)-2(-3),\ -3(-4)-6(2)]|\\ &= |[4, 6, 0]|\\ &= \sqrt{4^2+6^2+0^2}\\ &\doteq 7.2\end{aligned}$$

The area is approximately 7.2 square units.

Chapter 7 Practice Test **Question 16 Page 423**

$$\begin{aligned}
V &= \left|\vec{u}\cdot\vec{v}\times\vec{w}\right| \\
&= \left|\left[1, 0, -4\right]\cdot\left[0, -3, 2\right]\times\left[2, -2, 0\right]\right| \\
&= \left|\left[1, 0, -4\right]\cdot\left[-3(0)-(-2)(2),\ 2(2)-0(0),\ 0(-2)-2(-3)\right]\right| \\
&= \left|\left[1, 0, -4\right]\cdot\left[4, 4, 6\right]\right| \\
&= \left|1(4)+0(4)+(-4)(6)\right| \\
&= \left|-20\right| \\
&= 20
\end{aligned}$$

The volume is 20 cubic units.

Chapter 7 Practice Test **Question 17 Page 423**

Many solutions are possible. Check for right angles using the dot product.

$$\overrightarrow{AB} = \left[-8, 4\right]$$
$$\overrightarrow{BC} = \left[4, -9\right]$$
$$\overrightarrow{AC} = \left[-4, -5\right]$$

$$\begin{aligned}
\overrightarrow{AB}\cdot\overrightarrow{BC} &= [-8, 4]\cdot\left[4, -9\right] \\
&= -68 \\
&\neq 0 \\
\overrightarrow{AB}\cdot\overrightarrow{AC} &= [-8, 4]\cdot\left[-4, -5\right] \\
&= 12 \\
&\neq 0 \\
\overrightarrow{BC}\cdot\overrightarrow{AC} &= \left[4, -9\right]\cdot\left[-4, -5\right] \\
&= 29 \\
&\neq 0
\end{aligned}$$

Therefore, ΔABC is not a right triangle.

Chapter 7 Practice Test **Question 18 Page 423**

If you assume the force is parallel to the ramp, then the angle between $\vec{F}$ and $\vec{s}$ is 0°.

$$W = \vec{F} \cdot \vec{s}$$
$$= \left|\vec{F}\right|\left|\vec{s}\right|\cos\theta$$
$$= 40(5)\cos 0^\circ$$
$$= 200$$

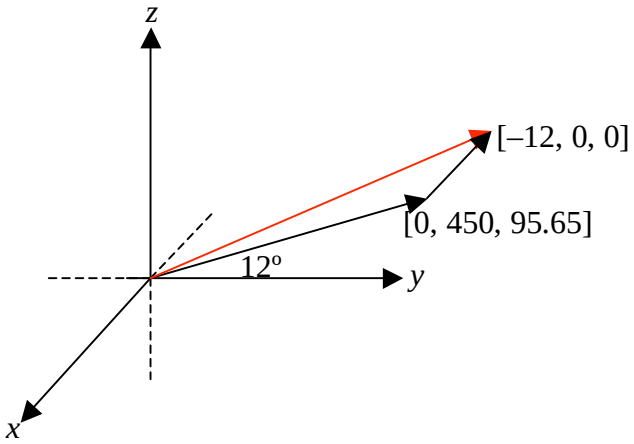
The work done in pulling the cart up the ramp is 200 J.

Chapter 7 Practice Test **Question 19 Page 423**

Represent the velocities as vectors in 3-space. Let the *y*-axis point north, the *x*-axis point east, and the *z*-axis point vertical.
The jet's vector is [0, 450, 450tan 12º].
The wind vector is [–12, 0, 0].
The resultant air velocity is the vector sum [–12, 450, 95.65].
The magnitude of this vector (the air speed) is $\sqrt{(-12)^2 + 450^2 + 95.65^2}$ or approximately 460.2 km/h.

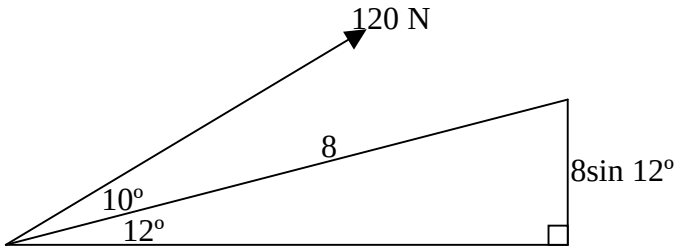
The ground velocity vector can be calculated by ignoring the vertical (*z*) components of the vectors.

The ground velocity vector is [–12, 450, 0]. Its magnitude is $\sqrt{(-12)^2 + 450^2 + 0^2}$ or approximately 450.2 kmh



Chapter 7 Practice Test **Question 20 Page 423**

Draw a diagram of the situation.



The force can be resolved into rectangular components, 120cos 22° (vertical) and 20sin 22° (horizontal). To calculate the mechanical work done against gravity, consider only the vertical component of the force and the vertical distance moved.

$W_g = (120\sin 22^\circ)(8\sin 12^\circ)$

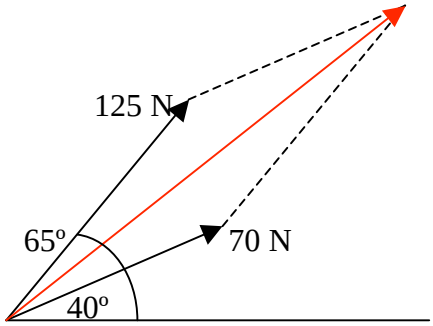
$\doteq 74.8$

The work against gravity is approximately 74.8 J.

The mechanical work done pulling the cart up the ramp is $120(8)\cos 10^\circ$ or approximately 945.4 J.

Chapter 7 Practice Test **Question 21 Page 423**

Draw a diagram.



In component form, the vectors are $\left[70\cos 40^\circ, 70\sin 40^\circ\right]$ and $\left[125\cos 65^\circ, 125\sin 65^\circ\right]$.

The resultant vector is the vector sum.

$$\vec{R} = \left[70\cos 40^\circ + 125\cos 65^\circ,\ 70\sin\ \ 40^\circ + 125\sin 65^\circ\right]$$
$$\doteq \left[106.45,\ 158.28\right]$$
$$\left|\vec{R}\right| = \sqrt{106.5^2 + 158.3^2}$$
$$\doteq 190.7$$
$$\theta = \tan^{-1}\left(\frac{158.28}{106.45}\right)$$
$$\doteq 56.1^\circ$$

The resultant has a magnitude of 190.7 N and its direction is 56.1° (above the horizontal).
The equilibrant vector is the opposite vector, having a magnitude of 190.7 N and a direction of −123.9° (below the horizontal).

Chapter 7 Practice Test **Question 22 Page 423**

One component can be found using the projection formula.

$$\text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$
$$= \left(\frac{\left[7, 9\right]\cdot\left[6, 2\right]}{\left[6, 2\right]\cdot\left[6, 2\right]}\right)\left[6, 2\right]$$
$$= \frac{60}{40}\left[6, 2\right]$$
$$= \left[9, 3\right]$$

The other component can be found by vector subtraction.

$$\left[7, 9\right] - \left[9, 3\right] = \left[-2, 6\right]$$

The two rectangular components are $\left[9, 3\right]$ and $\left[-2, 6\right]$.

Chapter 7 Practice Test **Question 23 Page 423**

Let the θ be the angle between the two sides $\vec{a}$ and $\vec{b}$ of the parallelogram.

$$A = \left|\vec{a}\times\vec{b}\right|$$
$$A = \left|\vec{a}\right|\left|\vec{b}\right|\left|\sin\theta\right|$$
$$200 = 90(30)\sin\theta$$
$$\sin\theta = \frac{200}{2700}$$
$$\theta = \sin^{-1}\left(\frac{2}{27}\right)$$
$$\theta \doteq 4.2^\circ$$

The interior angles of the parallelogram are 4.2° and 180° – 4.2° = 175.8°.

Chapter 7 Practice Test **Question 24 Page 423**

Use the formula for torque.

$$\left|\vec{\tau}\right| = \left|\vec{r}\right|\left|\vec{F}\right|\sin\theta$$

$$175 = 0.25\left|\vec{F}\right|\sin 78^\circ$$

$$\left|\vec{F}\right| = \frac{175}{0.25\sin 78^\circ}$$

$$\left|\vec{F}\right| \doteq 715.6$$

The magnitude of the force is about 715.6 N.

Chapter 7 Practice Test **Question 25 Page 423**

The statement is false.

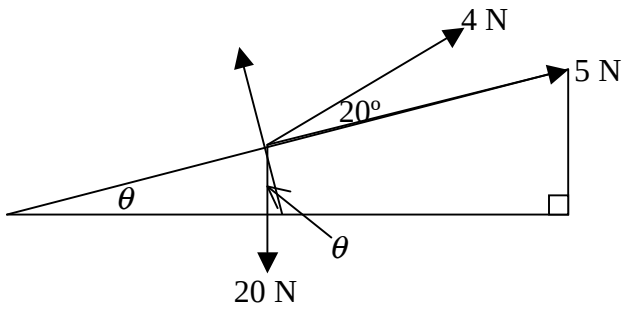
$\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$

If $\vec{u}\times\vec{v} = \vec{v}\times\vec{u}$, then $\vec{u}\times\vec{v} = \vec{0}$.

The cross product of two vectors is the zero vector if and only if the vectors are collinear; that is, the vectors have either an angle of 0º or 180º between them.

Chapter 7 Practice Test **Question 26 Page 423**

Draw a diagram of the situation.



Resolve the forces along the surface of the ramp.

$$20\sin\theta = 5 + 4\cos 20^\circ$$

$$\sin\theta = \frac{5 + 4\cos 20^\circ}{20}$$

$$\theta = \sin^{-1}\left(\frac{5 + 4\cos 20^\circ}{20}\right)$$

$$\theta \doteq 26.0^\circ$$

Chapter 7 Practice Test **Question 27 Page 423**

a) i) $\vec{u} \times \vec{v} \cdot \vec{w} = [4, -8, 12] \times [-11, 17, 12] \cdot [15, 10, 8]$

$= [-8(12) - 17(12), 12(-11) - 12(4), 4(17) - (-11)(-8)] \cdot [15, 10, 8]$

$= [-300, -180, -20] \cdot [15, 10, 8]$

$= -300(15) + (-180)(10) + (-20)(8)$

$= -6460$

ii) $\vec{u} \cdot \vec{v} \times \vec{w} = [4, -8, 12] \cdot [-11, 17, 12] \times [15, 10, 8]$

$= [4, -8, 12] \cdot [17(8) - 10(12), 12(15) - 8(-11), -11(10) - 15(17)]$

$= [4, -8, 12] \cdot [16, 268, -365]$

$= 4(16) + (-8)(269) + 12(-365)$

$= -6460$

b) Yes, the two answers are the same. The absolute value of each expression represents the volume of the parallelepiped defined by three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$. These expressions should be either equal or opposite to each other.

Chapter 7 Practice Test **Question 28 Page 423**

Let $\vec{a} = [a_1, a_2]$, $\vec{b} = [b_1, b_2]$, and $\vec{c} = [c_1, c_2]$.

a) L.S. $= (\vec{a} + \vec{b}) + \vec{c}$

$= ([a_1, a_2] + [b_1, b_2]) + [c_1, c_2]$

$= [a_1 + b_1, a_2 + b_2] + [c_1, c_2]$

$= [(a_1 + b_1) + c_1, (a_2 + b_2) + c_2]$

R.S. $= \vec{a} + (\vec{b} + \vec{c})$

$= [a_1, a_2] + ([b_1, b_2] + [c_1, c_2])$

$= [a_1, a_2] + [b_1 + c_1. b_2 + c_2]$

$= [a_1 + (b_1 + c_1), a_2 + (b_2 + c_2)]$

$= [(a_1 + b_1) + c_1, (a_2 + b_2) + c_2]$ Associative property for real numbers

Therefore, L.S. = R.S.

b) L.S. $= k\left(\vec{a} + \vec{b}\right)$

$= k\left(\left[a_1, a_2\right] + \left[b_1, b_2\right]\right)$

$= k\left[a_1 + b_1, a_2 + b_2\right]$

$= \left[k(a_1 + b_1), k(a_2 + b_2)\right]$

$= \left[ka_1 + kb_1, ka_2 + kb_2\right]$ Distributive property for real numbers

R.S. $= k\vec{a} + k\vec{b}$

$= k\left[a_1, a_2\right] + k\left[b_1, b_2\right]$

$= \left[ka_1, ka_2\right] + \left[kb_1. kb_2\right]$

$= \left[ka_1 + kb_1, ka_2 + kb_2\right]$

Therefore, L.S. = R.S.

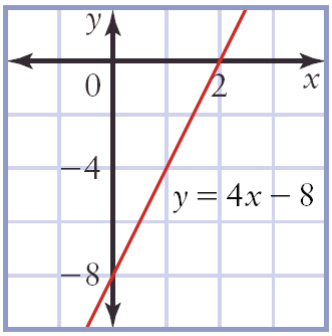
Chapter 8 **Lines and Planes**

Chapter 8 Prerequisite Skills

Chapter 8 Prerequisite Skills **Question 1 Page 426**

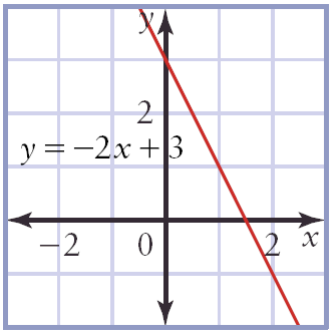
a) $y = 4x - 8$

x	y
1	–4
2	0
3	4
0	–8
–1	–12
–2	–16



b) $y = -2x + 3$

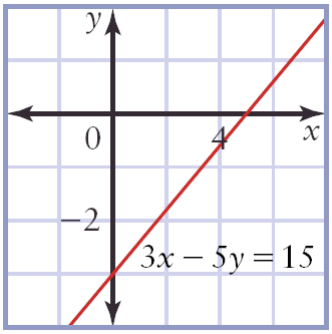
x	y
1	1
2	–1
3	–3
0	3
–1	5
–2	7



c) $3x - 5y = 15$

$$y = \frac{3}{5}x - 3$$

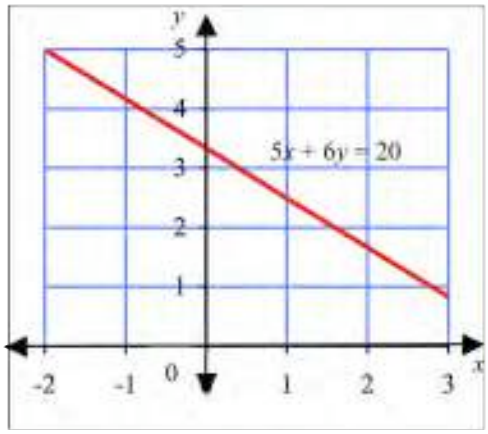
x	y
5	0
10	3
15	6
0	–3
–5	–6
-10	–9



d) $5x + 6y = 20$

$$y = -\frac{5}{6}x + \frac{20}{6}$$

x	y
1	$\frac{15}{6}$
2	$\frac{10}{6}$
3	$\frac{5}{6}$
0	$\frac{20}{6}$
–1	$\frac{25}{6}$
–2	5



Chapter 8 Prerequisite Skills **Question 2 Page 426**

To find the x-intercept, let $y = 0$ and solve for x.
To find the y-intercept, let $x = 0$ and solve for y.

a) $0 = 3x + 7$

$$x = -\frac{7}{3}$$

$y = 3(0) + 7$
$y = 7$

b) $0 = 5x - 10$
$x = 2$

$y = 5(0) - 10$
$y = -10$

c) $2x - 9(0) = 18$
$x = 9$

$2(0) - 9y = 18$
$y = -2$

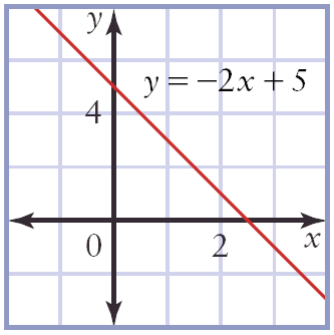
d) $4x + 8(0) = 9$

$$x = \frac{9}{4}$$
$$= 2.25$$

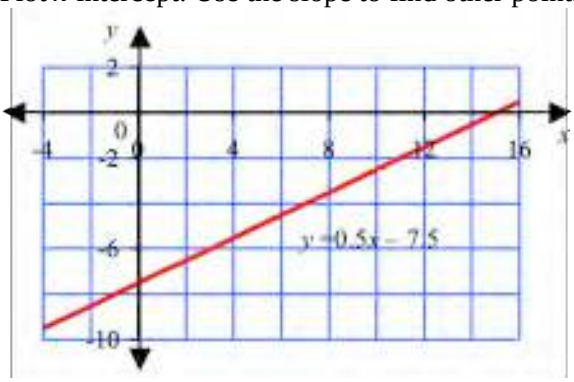
$$4(0) + 8y = 9$$
$$y = \frac{9}{8}$$
$$= 1.125$$

Chapter 8 Prerequisite Skills **Question 3 Page 426**

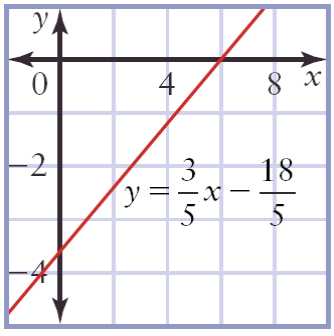
a) Plot y-intercept. Use the slope to find other points, such as (1, 3) and (2, 1).



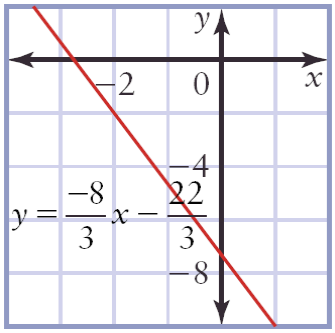
b) Plot x-intercept. Use the slope to find other points, such as (5, –5) and (7, –4)



c) Plot point (1, –3). Use the slope to plot other points. Move 5 right and 3 up to point (6, 0). Again move 5 right and 3 up to point (11, 3).



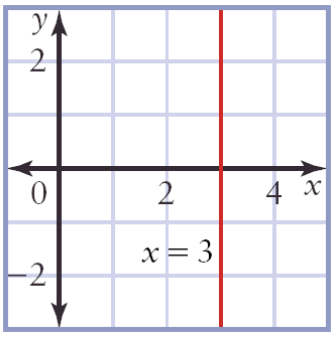
d) Plot the point (–5, 6). Use the slope to plot other points. Move 3 right and 8 down to point (–2, –2). Again, move 3 right and 8 down to point (1, –10).



e) $2x - 6 = 0$

$x = 3$

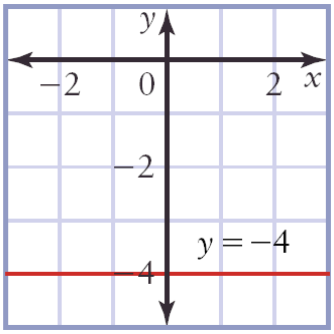
All points on graph have $x = 3$. It is a vertical line.



f) $y + 4 = 0$

$y = -4$

All points on the graph have $y = -4$. It is a horizontal line.



Chapter 8 Prerequisite Skills Question 4 Page 426

a) By observation, the point of intersection is (7, 2).

b) By observation, the point of intersection is (–2, 2).

c) By observation, the point of intersection is not obvious.
One line passes through (0, 2) and (2, 1).
Slope:

$$\frac{1-2}{2-0} = -\frac{1}{2}$$

y-intercept is 2.

The equation is $y = -\frac{1}{2}x + 2$ or $x + 2y = 4$.

The other line passes through (0, –1) and (1, 1).
Slope:

$$\frac{1-(-1)}{1-0} = 2$$

y-intercept is –1.
The equation is $y = 2x - 1$ or $2x - y = 1$.

Solve the system of equations using substitution.

$$x + 2(2x - 1) = 4$$
$$5x = 6$$
$$x = 1.2$$

$$y = 2(1.2) - 1$$
$$y = 1.4$$

The intersection point is (1.2, 1.4).

Chapter 8 Prerequisite Skills Question 5 Page 426

a) Use elimination.

$$y = 3x + 2 \quad ①$$
$$y = -x - 2 \quad ②$$
$$0 = 4x + 4 \quad ① - ②$$
$$x = -1$$

Substitute $x = -1$ into equation ①.

$$y = 3(-1) + 2$$
$$y = -1$$

The point of intersection is (–1, –1).

b) Use elimination.

$x + 2y = 11$ ①

$x + 3y = 16$ ②

$0x - y = -5$ ①–②

$y = 5$

Substitute $y = 5$ into equation ①.

$x + 2(5) = 11$

$x = 1$

The point of intersection is (1, 5)

c) Use elimination.

$4x + 3y = -20$ ①

$5x - 2y = 21$ ②

$8x + 6y = -40$ 2①

$15x - 6y = 63$ 3②

$23x = 23$ 2①+3②

$x = 1$

Substitute $x = 1$ into equation ①.

$4(1) + 3y = -20$

$3y = -24$

$y = -8$

The point of intersection is (1, –8).

d) Use elimination.

$$\begin{aligned} 2x+4y&=15 && ① \\ 4x-6y&=-15 && ② \\ 4x+8y&=30 && 2① \\ -4x+6y&=15 && -② \\ \hline 14y&=45 && 2①-② \\ y&=\frac{45}{14} \end{aligned}$$

Substitute $y=\frac{45}{14}$ into equation ①.

$$2x+4\left(\frac{45}{14}\right)=15$$

$$28x+180=210$$

$$28x=30$$

$$x=\frac{15}{14}$$

The point of intersection is $\left(\frac{15}{14}, \frac{45}{14}\right)$.

Chapter 8 Prerequisite Skills **Question 6 Page 426**

a) Parallel lines have equal slopes. The line $y=3x+5$ has slope 3.

The x-intercept 10 corresponds to the point (10, 0).

Use the point-slope form of the equation of a line.

$$\frac{y-0}{x-10}=3$$

$$y=3x-30$$

The equation of the line is $y=3x-30$.

b) Parallel lines have equal slopes. The line $4x+5y=7$ has slope $-\frac{4}{5}$.

The slope and a point on the line are known.

Use the point-slope form of the equation of a line.

$$\frac{y-6}{x-(-2)}=\frac{-4}{5}$$

$$5y-30=-4x-8$$

$$5y=-4x+22$$

$$y=-\frac{4}{5}x+\frac{22}{5}$$

The equation of the line is $y=-\frac{4}{5}x+\frac{22}{5}$.

c) Perpendicular lines have negative reciprocal slopes. The line $y = -\frac{3}{2}x + 6$ has slope $-\frac{3}{2}$.

The required line will have slope $\frac{2}{3}$.

The x-intercept of $5x - 2y = 20$ is 4 (let $y = 0$). Therefore the point (4, 0) is on the required line.

The slope and a point on the line are known.

Use the point-slope form of the equation of a line.

$$\frac{y - 0}{x - 4} = \frac{2}{3}$$

$$3y = 2x - 8$$

$$y = \frac{2}{3}x - \frac{8}{3}$$

The equation of the line is $y = \frac{2}{3}x - \frac{8}{3}$.

d) Perpendicular lines have negative reciprocal slopes. The line $7x + 5y = 20 \Rightarrow y = -\frac{7}{5}x + 4$ has slope $-\frac{7}{5}$.

The required line will have slope $\frac{5}{7}$.

The y-intercept of $6x - 5y = 15 \Rightarrow y = \frac{6}{5}x - 3$ is -3. Therefore the point $(0, -3)$ is on the required line.

The slope and a point on the line are known.

Use the point-slope form of the equation of a line.

$$\frac{y - (-3)}{x - 0} = \frac{5}{7}$$

$$7y + 21 = 5x$$

$$y = \frac{5}{7}x - 3$$

The equation of the line is $y = \frac{5}{7}x - 3$.

e) Lines parallel to the y-axis have the form $x = a$.

Since the required line passes through (–3, 0), the required equation is $x = -3$.

Chapter 8 Prerequisite Skills **Question 7 Page 427**

a) $\vec{a} \cdot \vec{b} = [3, 1] \cdot [5, 7]$

$= 3(5) + 1(7)$

$= 22$

Since $\vec{a} \cdot \vec{b} \neq 0$, $\vec{a}$ and $\vec{b}$ are not perpendicular.

b) $\vec{a} \cdot \vec{b} = [-4, 5] \cdot [-9, 1]$

$= (-4)(-9) + 5(1)$

$= 41$

Since $\vec{a} \cdot \vec{b} \neq 0$, $\vec{a}$ and $\vec{b}$ are not perpendicular.

c) $\vec{a} \cdot \vec{b} = [6, 1] \cdot [-2, 12]$

$= 6(-2) + 1(12)$

$= 0$

Since $\vec{a} \cdot \vec{b} = 0$, $\vec{a}$ and $\vec{b}$ are perpendicular.

d) $\vec{a} \cdot \vec{b} = [1, 9, -4] \cdot [3, -6, -2]$

$= 1(3) + 9(-6) + (-4)(-2)$

$= -43$

Since $\vec{a} \cdot \vec{b} \neq 0$, $\vec{a}$ and $\vec{b}$ are not perpendicular.

e) $\vec{a} \cdot \vec{b} = [3, 4, 1] \cdot [1, -1, 1]$

$= 3(1) + 4(-1) + 1(1)$

$= 0$

Since $\vec{a} \cdot \vec{b} = 0$, $\vec{a}$ and $\vec{b}$ are perpendicular.

f) $\vec{a} \cdot \vec{b} = [7, -3, 2] \cdot [1, 8, 10]$

$= 7(1) + (-3)(8) + 2(10)$

$= 3$

Since $\vec{a} \cdot \vec{b} \neq 0$, $\vec{a}$ and $\vec{b}$ are not perpendicular.

Chapter 8 Prerequisite Skills **Question 8 Page 427**

a) $\vec{a} \times \vec{b} = [2, -7, 3] \times [1, 9, 6]$

$= [(-7)(6) - 9(3), 3(1) - 6(2), 2(9) - 1(-7)]$

$= [-69, -9, 25]$

b) $\vec{a} \times \vec{b} = [8, 2, -4] \times [3, 7, -1]$

$= [2(-1) - 7(-4), -4(3) - (-1)(8), 8(7) - 3(2)]$

$= [26, -4, 50]$

c) $\vec{a} \times \vec{b} = [3, 3, 5] \times [5, 1, -1]$

$= [3(-1) - 1(5), 5(5) - (-1)(3), 3(1) - 5(3)]$

$= [-8, 28, -12]$

d) $\vec{a} \times \vec{b} = [2, 0, 0] \times [0, 7, 0]$

$= [0(0) - 7(0), 0(0) - 0(2), 2(7) - 0(0)]$

$= [0, 0, 14]$

Chapter 8 Prerequisite Skills **Question 9 Page 427**

The vectors are not unique, as any vector that is a scalar multiple of the given vector will be parallel.

a) [2, 10]

b) [–30, 40]

c) [4, 2, 14]

d) [1, 4, –5]

Chapter 8 Prerequisite Skills **Question 10 Page 427**

These vectors are not unique, as any vector that produces zero in a dot product with the given direction vector will be perpendicular.

a) [–5, 1] since [–5, 1]·[1, 5] = 0

b) [4, 3] since [4, 3]·[–3, 4] = 0

c) [3, 1, –1] since [3, 1, –1]·[2, 1, 7] = 0

d) [1, 1, 1] since [1, 1, 1]·[–1, –4, 5] = 0

Chapter 8 Prerequisite Skills **Question 11 Page 427**

Use the formula for dot product.

a) $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$\cos\theta = \dfrac{[1, 3]\cdot[2, 5]}{\sqrt{1^2+3^2}\sqrt{2^2+5^2}}$

$\cos\theta \doteq \dfrac{17}{17.0294}$

$\theta \doteq \cos^{-1}\left(\dfrac{17}{17.0294}\right)$

$\theta \doteq 3.4^\circ$

b) $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$\cos\theta = \dfrac{[-4, 1]\cdot[7, 2]}{\sqrt{(-4)^2+1^2}\sqrt{7^2+2^2}}$

$\cos\theta \doteq \dfrac{-26}{30.0167}$

$\theta \doteq \cos^{-1}\left(\dfrac{-26}{30.0167}\right)$

$\theta \doteq 150.0^\circ$

c) $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$\cos\theta = \dfrac{[1, 0, 2]\cdot[5, 3, 4]}{\sqrt{1^2+0^2+2^2}\sqrt{5^2+3^2+4^2}}$

$\cos\theta \doteq \dfrac{13}{15.8114}$

$\theta \doteq \cos^{-1}\left(\dfrac{13}{15.8114}\right)$

$\theta \doteq 34.7^\circ$

d) $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$$\cos\theta = \frac{[-3, 2, -8]\cdot[1, -2, 6]}{\sqrt{(-3)^2 + 2^2 + (-8)^2}\sqrt{1^2 + (-2)^2 + 6^2}}$$

$$\cos\theta \doteq \frac{-55}{56.1872}$$

$$\theta \doteq \cos^{-1}\left(\frac{-55}{56.1872}\right)$$

$$\theta \doteq 168.2^\circ$$

Chapter 8 Section 1 **Equations of Lines in Two-Space and Three-Space**

Chapter 8 Section 1 **Question 1 Page 437**

a) $[x, y] = [2, 7] + t[3, 1],\ t \in \mathbb{R}$

b) $[x, y] = [10, -4] + t[-2, 5],\ t \in \mathbb{R}$

c) $[x, y, z] = [9, -8, 1] + t[10, -3, 2],\ t \in \mathbb{R}$

d) $[x, y, z] = [-7, 1, 5] + t[0, 6, -1],\ t \in \mathbb{R}$

Chapter 8 Section 1 **Question 2 Page 437**

a) First determine the direction vector.

$$\begin{aligned}\vec{m} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= [4, 10] - [1, 7] \\ &= [3, 3]\end{aligned}$$

A possible vector equation of the line is $[x, y] = [4, 10] + t[3, 3],\ t \in \mathbb{R}$.

b) First determine the direction vector.

$$\begin{aligned}\vec{m} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= [-2, -8] - [-3, 5] \\ &= [1, -13]\end{aligned}$$

A possible vector equation of the line is $[x, y] = [-2, -8] + t[1, -13],\ t \in \mathbb{R}$.

c) First determine the direction vector.

$$\begin{aligned}\vec{m} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= [9, 2, 8] - [6, 2, 5] \\ &= [3, 0, 3]\end{aligned}$$

A possible vector equation of the line is $[x, y, z] = [9, 2, 8] + t[3, 0, 3],\ t \in \mathbb{R}$.

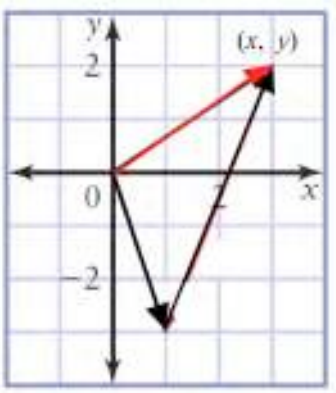
d) First determine the direction vector.

$$\begin{aligned}\vec{m} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= [1, -1, -5] - [1, 1, -3] \\ &= [0, -2, -2]\end{aligned}$$

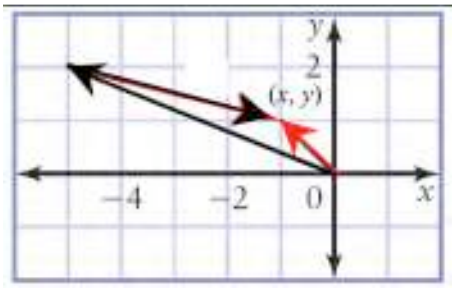
A possible vector equation of the line is $[x, y, z] = [1, -1, -5] + t[0, -2, -2],\ t \in \mathbb{R}$.

Chapter 8 Section 1 **Question 3 Page 437**

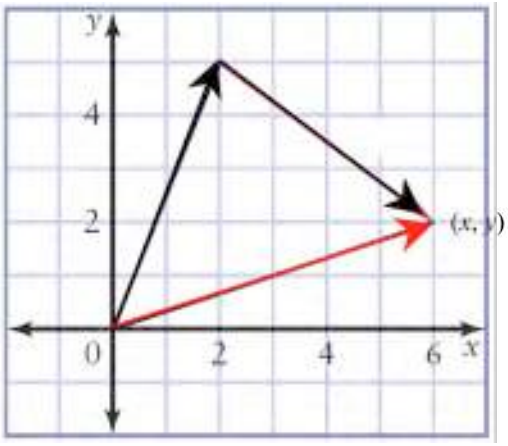
a)



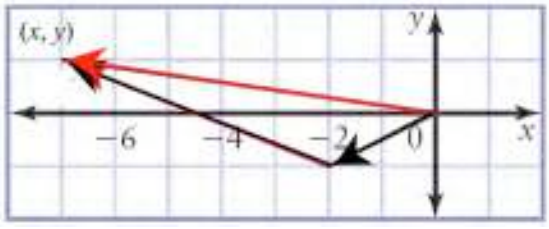
b)



c)



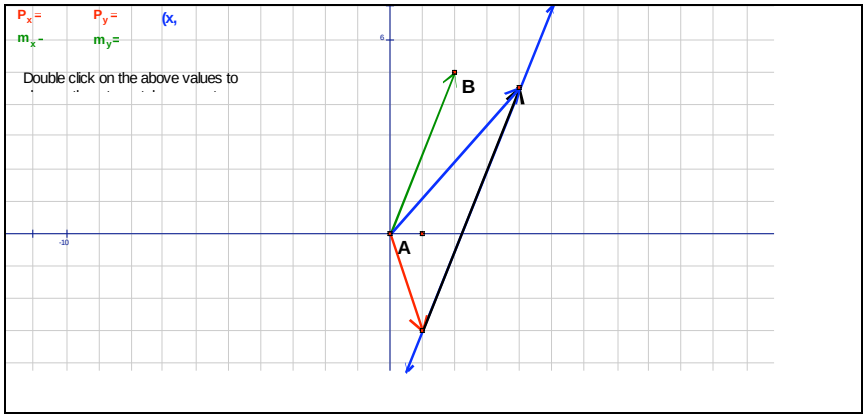
d)



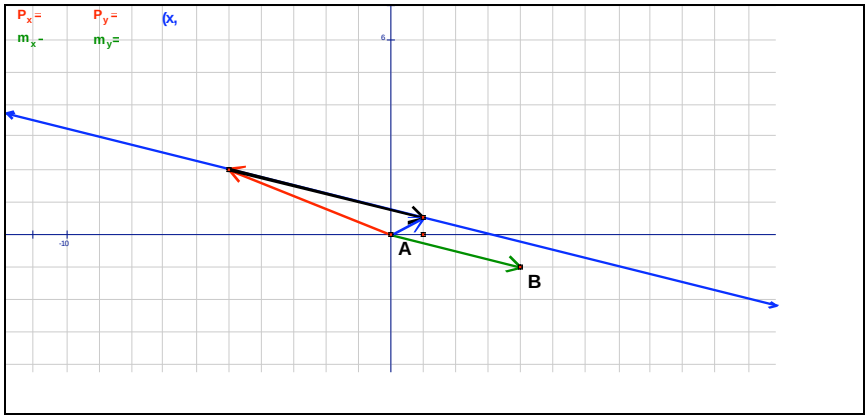
Chapter 8 Section 1 **Question 4 Page 437**

Use *The Geometer's Sketchpad*® file 8.1 VectorEquation.gsp. Some possible screens are shown below.

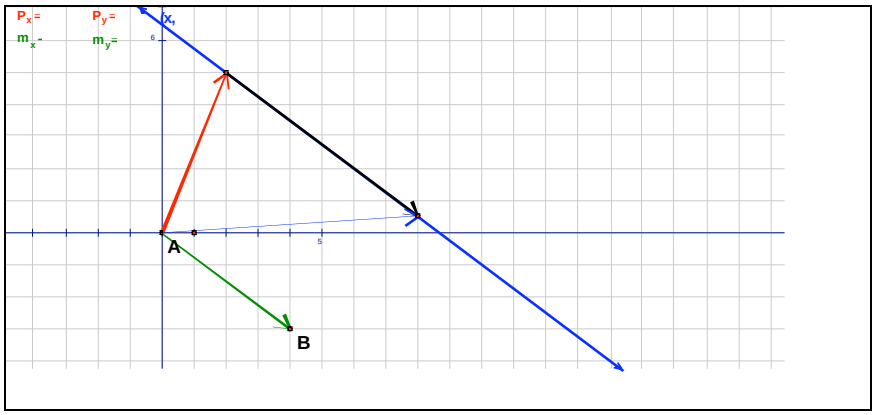
a)



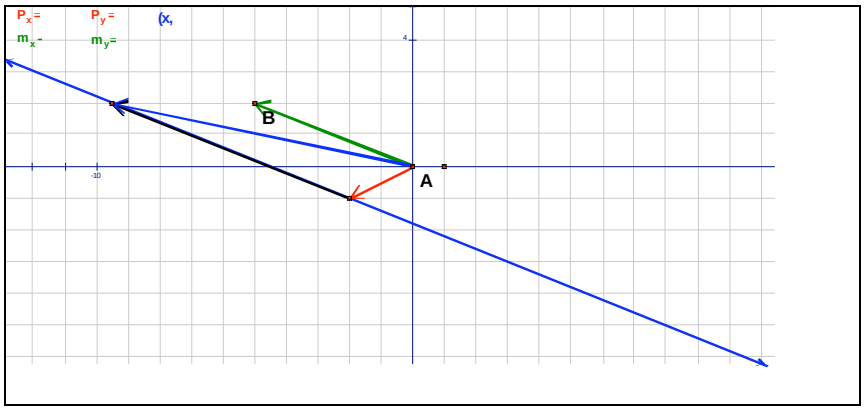
b)



c)



d)



Chapter 8 Section 1 **Question 5 Page 437**

a) The point P(–1, 11) corresponds to the position vector [–1, 11].
Substitute the coordinates into the vector equation.

$$[-1, 11] = [3, 1] + t[-2, 5]$$

Equate the *x*-coordinates.

$$-1 = 3 - 2t$$

$$t = 2$$

Equate the *y*-coordinates.

$$11 = 1 + 5t$$

$$t = 2$$

Since the *t* values are equal, the point P(–1, 11) does lie on the line.

b) The point P(9, –15) corresponds to the position vector [9, –15].
Substitute the coordinates into the vector equation.

$$[9, -15] = [3, 1] + t[-2, 5]$$

Equate the *x*-coordinates.

$$9 = 3 - 2t$$

$$t = -3$$

Equate the *y*-coordinates.

$$-15 = 1 + 5t$$

$$t = -\frac{16}{5}$$

Since the *t* values are not equal, the point P(9, –15) does not lie on the line.

c) The point P(–9, 21) corresponds to the position vector [–9, 21].
Substitute the coordinates into the vector equation.

$$[-9, 21] = [3, 1] + t[-2, 5]$$

Equate the x-coordinates.

$$-9 = 3 - 2t$$

$$t = 6$$

Equate the y-coordinates.

$$21 = 1 + 5t$$

$$t = 4$$

Since the t values are not equal, the point P(–9, 21) does not lie on the line.

d) The point P(−2, 13.5) corresponds to the position vector [−2, 13.5].
Substitute the coordinates into the vector equation.

$$[-2, 13.5] = [3, 1] + t[-2, 5]$$

Equate the x-coordinates.

$$-2 = 3 - 2t$$

$$t = \frac{5}{2}$$

Equate the y-coordinates.

$$13.5 = 1 + 5t$$

$$t = \frac{5}{2}$$

Since the t values are equal, the point P(−2, 13.5) does lie on the line.

Chapter 8 Section 1 **Question 6 Page 437**

$t \in \mathbb{R}$ for all equations.

a) $x = 10 + 13t$

$y = 6 + t$

b) $x = 12t$

$y = 5 - 7t$

c) $x = 3 + 6t$

$y = -9t$

$z = -1 + t$

d) $x = 11 + 3t$

$y = 2$

$z = 0$

Chapter 8 Section 1 **Question 7 Page 437**

a) $[x, y] = [3, 9] + t[5, 7],\ t \in \mathbb{R}$.

b) $[x, y] = [-5, 0] + t[-6, 11],\ t \in \mathbb{R}$.

c) $[x, y, z] = [1, -6, 2] + t[4, 1, -2],\ t \in \mathbb{R}$.

d) $[x, y, z] = [7, 0, 0] + t[0, -1, 0],\ t \in \mathbb{R}$.

Chapter 8 Section 1 **Question 8 Page 437**

a) Isolate t in each of the parametric equations.

$x = 1 + 2t$ $\qquad$ $y = 1 - 3t$

$t = \dfrac{x-1}{2}$ $\qquad$ $t = \dfrac{y-1}{-3}$

$$\frac{x-1}{2} = \frac{y-1}{-3}$$

$$-3x + 3 = 2y - 2$$

$$3x + 2y - 5 = 0$$

b) Isolate t in each of the parametric equations.

$x = -2 + t$ $\qquad$ $y = 4 + 5t$

$t = x + 2$ $\qquad$ $t = \dfrac{y-4}{5}$

$$x + 2 = \frac{y-4}{5}$$

$$5x + 10 = y - 4$$

$$5x - y + 14 = 0$$

c) Isolate t in each of the parametric equations.

$x = 5 + 7t$ $\qquad$ $y = -2 - 4t$

$t = \dfrac{x-5}{7}$ $\qquad$ $t = \dfrac{y+2}{-4}$

$$\frac{x-5}{7} = \frac{y+2}{-4}$$
$$-4x + 20 = 7y + 14$$
$$4x + 7y - 6 = 0$$

d) Isolate t in each of the parametric equations.

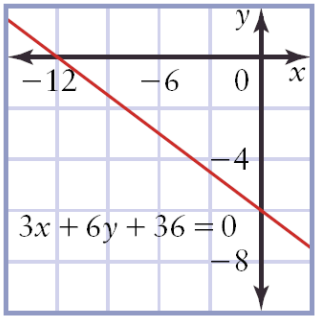
$x = 0.5 + 0.3t$ $\qquad$ $y = 1.5 - 0.2t$

$t = \dfrac{x-0.5}{0.3}$ $\qquad$ $t = \dfrac{y-1.5}{-0.2}$

$$\frac{x-0.5}{0.3} = \frac{y-1.5}{-0.2}$$
$$-0.2x + 0.1 = 0.3y - 0.45$$
$$0.2x + 0.3y - 0.55 = 0$$
$$2x + 3y - 5.5 = 0$$
$$4x + 6y - 11 = 0$$

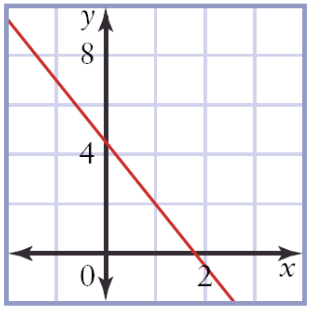
Chapter 8 Section 1 $\qquad$ **Question 9 Page 437**

a) Choose two points on the graph. The x-intercept (–12) and the y-intercept (–6) are easy to calculate.

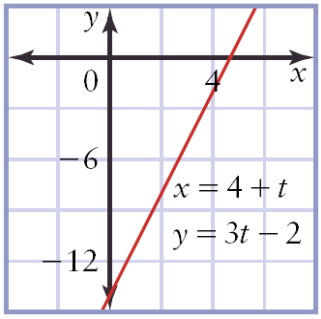


b) Find two points on the graph by letting $t = 0$ and 1, say. This gives the points (–1, 7) and (1, 2).

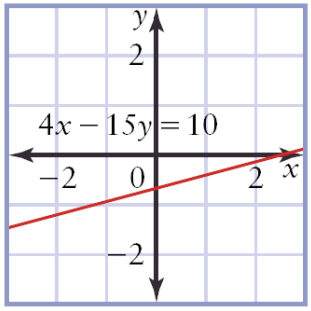
$[x, y] = [-1, 7] + t[2, -5]$



c) Find two points on the graph by letting $t = 0$ and 1, say. This gives the points (4, –2) and (5, 1).



d) Choose two points on the graph. The x-intercept (2.5) and the y-intercept $\left(-\frac{2}{3}\right)$ are easy to calculate.



Chapter 8 Section 1 **Question 10 Page 437**

The scalar equation of a line in two-space is of the form $Ax + By + C = 0$ where $\vec{n} = [A, B]$ is a normal vector for the line.

a) The scalar equation is of the form $3x + y + C = 0$. Substitute the point (2, 4) to determine the value of C.

$3(2) + 4 + C = 0$

$C = -10$

A scalar equation for the line is $3x + 2y - 10 = 0$.

b) The scalar equation is of the form $x - y + C = 0$. Substitute the point (–5, 1) to determine the value of C.

$-5 - 1 + C = 0$

$C = 6$

A scalar equation for the line is $x - y + 6 = 0$.

c) The scalar equation is of the form $y + C = 0$. Substitute the point (–3, –7) to determine the value of C.

$-7 + C = 0$

$C = 7$

A scalar equation for the line is $y + 7 = 0$.

d) The scalar equation is of the form $1.5x - 3.5y + C = 0$. Substitute the point (0.5, –2.5) to determine the value of C.

$1.5(0.5) - 3.5(-2.5) + C = 0$

$C = -9.5$

A scalar equation for the line is $1.5x - 3.5y - 9.5 = 0$ or $3x - 7y - 19 = 0$

Chapter 8 Section 1 **Question 11 Page 437**

$t \in \mathbb{R}$ for all equations.

a) Choose two points on the line, say (0, 3) and (6, 0). (Hint: consider the intercepts)
The vector joining these two points is a possible direction vector.

$\vec{m} = [6, 0] - [0, 3]$

$= [6, -3]$

A possible vector equation is $[x, y] = [0, 3] + t[6, -3]$.

Possible parametric equations are $x = 6t, y = 3 - 3t$.

b) Choose two points on the line, say (0, –12) and (3, 0). (Hint: consider the intercepts)
The vector joining these two points is a possible direction vector.

$\vec{m} = [3, 0] - [0, -12]$

$= [3, 12]$

A possible vector equation is $[x, y] = [3, 0] + t[3, 12]$.

Possible parametric equations are $x = 3 + 3t, y = 12t$.

c) Choose two points on the line, say (1, –4) and (3, 1). (Hint: consider convenient values for x and solve for y.)
The vector joining these two points is a possible direction vector.

$\vec{m} = [3, 1] - [1, -4]$

$= [2, 5]$

A possible vector equation is $[x, y] = [1, -4] + t[2, 5]$.

Possible parametric equations are $x = 1 + 2t, y = -4 + 5t$.

d) Choose two points on the line, say (–9, 3) and (0, –5). (Hint: consider convenient values for x and solve for y.)
The vector joining these two points is a possible direction vector.

$\vec{m} = [-9, 3] - [0, -5]$

$= [-9, 8]$

A possible vector equation is $[x, y] = [0, -5] + t[-9, 8]$.

Possible parametric equations are $x = -9t, y = -5 + 8t$.

Chapter 8 Section 1 **Question 12 Page 437**

The vector joining the two given two points is a possible direction vector.

$\vec{m} = [3, 4, -5] - [9, -2, 7]$

$= [-6, 6, -12]$

A possible vector equation is $[x, y, z] = [3, 4, -5] + t[-6, 6, -12]$

Possible parametric equations are $x = 3 - 6t$, $y = 4 + 6t$, and $z = -5 - 12t$.

$t \in \mathbb{R}$ for all equations.

Chapter 8 Section 1 **Question 13 Page 437**

a) The point $P_0(7, 0, 2)$ corresponds to the position vector [7, 0, 2].
Substitute the coordinates into the vector equation.

$[7, 0, 2] = [1, 3, -7] + t[2, -1, 3]$

Equate the *x*-coordinates.

$7 = 1 + 2t$

$t = 3$

Equate the *y*-coordinates.

$0 = 3 - t$

$t = 3$

Equate the *z*-coordinates.

$2 = -7 + 3t$

$t = 3$

Since the *t* values are equal, the point $P_0(7, 0, 2)$ does lie on the line.

b) The point $P_0(2, 1, -3)$ corresponds to the position vector [2, 1, –3].
Substitute the coordinates into the vector equation.

$[2, 1, -3] = [1, 3, -7] + t[2, -1, 3]$

Equate the *x*-coordinates.

$2 = 1 + 2t$

$t = \frac{1}{2}$

Equate the *y*-coordinates.

$1 = 3 - t$

$t = 2$

Equate the z-coordinates.

$-3 = -7 + 3t$

$t = \frac{4}{3}$

Since the t values are not equal, the point $P_0(2, 1, -3)$ does not lie on the line.

c) The point $P_0(13, -3, 11)$ corresponds to the position vector $[13, -3, 11]$.
Substitute the coordinates into the vector equation.

$[13, -3, 11] = [1, 3, -7] + t[2, -1, 3]$

Equate the x-coordinates.

$13 = 1 + 2t$

$t = 6$

Equate the y-coordinates.

$-3 = 3 - t$

$t = 6$

Equate the z-coordinates.

$11 = -7 + 3t$

$t = 6$

Since the t values are equal, the point $P_0(13, -3, 11)$ does lie on the line.

d) The point $P_0(-4, 0.5, -14.5)$ corresponds to the position vector $[-4, 0.5, -14.5]$.
Substitute the coordinates into the vector equation.

$[-4, 0.5, -14.5] = [1, 3, -7] + t[2, -1, 3]$

Equate the x-coordinates.

$-4 = 1 + 2t$

$t = -\frac{5}{2}$

Equate the y-coordinates.

$0.5 = 3 - t$

$t = 2.5$

$t = \frac{5}{2}$

Equate the z-coordinates.

$-14.5 = -7 + 3t$

$$t = -\frac{7.5}{3}$$

$$t = -\frac{5}{2}$$

Since the t values are not equal, the point $P_0(-4, 0.5, -14.5)$ does not lie on the line.

Chapter 8 Section 1 **Question 14 Page 438**

a) A direction vector to the first line can be [6, 4] and a direction vector for the second line can be [3, 2].
[6, 4] = 2[3, 2]
Since one direction vector is a scalar multiple of the other, the two lines are parallel.

b) A direction vector to the first line can be [–9, 1], and a direction vector for the second line can be [1, 9].
[–9, 1]·[1, 9] = –9(1) + 1(9)
= 0

Since the dot product of these two direction vectors is zero, the two lines are perpendicular.

Chapter 8 Section 1 **Question 15 Page 438**

a) This represents a horizontal line in two-space with a y-intercept of 3.

b) This represents a line that lies along the z-axis in three-space.

c) This represents a vertical line in two-space with x-intercept 1.

d) This represents a line parallel to the y-axis and passing through the point (–1, 3, 2).

Chapter 8 Section 1 **Question 16 Page 438**

a) The line is parallel to the x-axis. Choose $\vec{i} = [1, 0, 0]$ as a direction vector.
Point (3, –8) is on the line and has position vector $[3, -8]$.
A possible vector equation is $[x, y] = [3, -8] + t[1, 0]$, $t \in \mathbb{R}$.

b) A normal to the given line is $\vec{n} = [4, -3]$. This is a direction vector for the new line.
Point (–2, 4) is on the line and has position vector $[-2, 4]$
A possible vector equation is $[x, y] = [-2, 4] + t[4, -3]$, $t \in \mathbb{R}$.

c) The line is parallel to the z-axis. Choose $\vec{k} = [0, 0, 1]$ as a direction vector.
Point (1, 5, 10) is on the line and has position vector $[1, 5, 10]$
A possible vector equation is $[x, y, z] = [1, 5, 10] + t[0, 0, 1]$, $t \in \mathbb{R}$.

d) The given line has direction vector $\vec{m} = [3, -5, -9]$.

The position vector for the x-intercept of 10 is $[-10, 0, 0]$.

A possible vector equation is $[x, y, z] = [-10, 0, 0] + t[3, -5, -9]$, $t \in \mathbb{R}$.

e) The position vector for the x-intercept of the first line is $[3, 0, 0]$.

Let $x = 0$ and solve for t.

$0 = 6 + 3t$

$t = -2$

Let $y = 0$ and solve for t.

$0 = -2 - t$

$t = -2$

Substitute $t = -2$ and solve for z.

$z = -3 + (-2)(-2)$

$z = 1$

Thus, the position vector for the z-intercept is [0, 0, 1].

A direction vector for the line is $\vec{m} = [3, 0, 0] - [0, 0, 1] = [3, 0, -1]$, $t \in \mathbb{R}$.

A possible vector equation is $[x, y, z] = [0, 0, 1] + t[3, 0, -1]$.

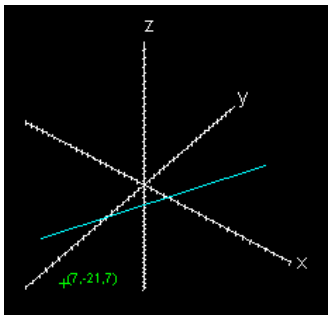
Chapter 8 Section 1 **Question 17 Page 438**

Answers may vary. For example:

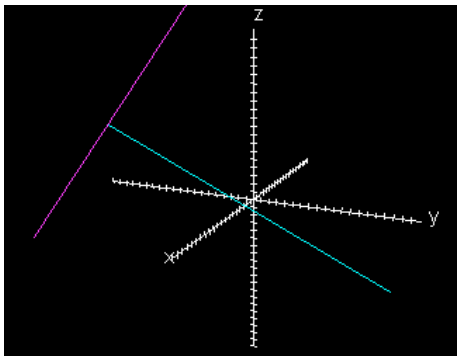
A scalar equation in three-space would be of the form $Ax + By + Cz + D = 0$. For such an equation, you could let y and z equal 0 and solve for x to find a unique x-intercept. Letting x and z equal 0 would lead to a unique y-intercept and letting x and y equal 0 would lead to a unique z-intercept. But it is easy to imagine lines in three-space that do not intersect even one axis to form an intercept. Therefore the original assumption must be wrong. A scalar equation in three-space must not represent a line.

Chapter 8 Section 1 **Question 18 Page 438**

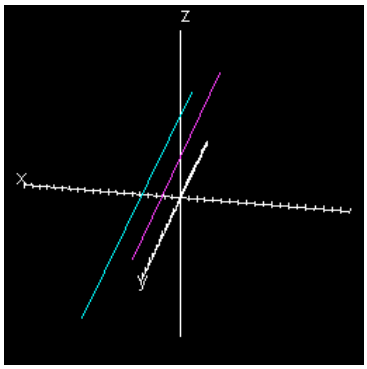
a) No.



b) Yes.



c) No.



Chapter 8 Section 1 **Question 19 Page 438**

a) The following are the answers to the questions in 8.1ChapterProblem.gsp.

2. The variable x is always twice the value of t and the variable y is always equal to t.
A possible equation is $x = 2y = 2t, \quad t \in \mathbb{R}$.

3. The rectangle seems to rotate 60º clockwise for every 1 increase in the value of t.
The centre of the rectangle seems to follow the line $y = 0.5x$.

4. Yes. It agrees with the observations from part 3. If the formula is changed for x or y, the rotation continues in the same way but the centre of rotation follows a different line. (If $y = 2t$, the centre follows the line $y = x$; this suggests that the centre follows the line $y = \frac{x \text{ parameter}}{y \text{ parameter}} x.$)

5. If you double the rotation angle (120º), the motion is twice as fast. If you halve the rotation angle (60º), the motion is half as fast.

6. If you change x to t^2, the rectangle (centre) will follow a parabolic path.

b) Answers may vary. A sample solution is shown.

Changing the value of a parameter can change both the location and the orientation of a figure (rectangle) defined by parametric equations.

Chapter 8 Section 1 **Question 20 Page 438**

a) The point P(7, –21, 7) corresponds to the position vector [7, –21, 7].
Substitute the coordinates into the vector equation.

$$[7, -21, 7] = [4, -3, 2] + t[1, 8, -3]$$

Equate the x-coordinates.

$$7 = 4 + t$$

$$t = 3$$

Equate the y-coordinates.

$$-21 = -3 + 8t$$

$$t = -\frac{18}{8}$$

$$t = -\frac{9}{4}$$

Equate the z-coordinates.

$$7 = 2 - 3t$$

$$t = -\frac{5}{3}$$

Since the t values are not equal, the point P(7, –21, 7) does not lie on the line.

b) If the lines intersect, there is one t-value and one s-value that will produce the same point.

Equate the x-coordinates.

$$4 + t = 2 + 4s$$

$$4s - t = 2 \quad ①$$

Equate the y-coordinates.

$$-3 + 8t = -19 - 5s$$

$$5s + 8t = -16 \quad ②$$

Equate the z-coordinates.

$$2 - 3t = 8 - 9s$$

$$9s - 3t = 6 \quad ③$$

Solve ① and ② for s and t.

$$4s - t = 2 \quad ①$$

$$\underline{5s + 8t = -16} \quad ②$$

$$37s = 0 \quad 8① + ②$$

$$s = 0$$

Substitute $s = 0$ into equation ①.

$$4(0) - t = 2$$

$$t = -2$$

When $t = -2$, the first line gives the point:
$(4 + (-2)(1), -3 + (-2)(8), 2 + (-2)(-3)) = (2, -19, 8)$

When $s = 0$, the second line gives the point:
$(2 + 0(4), -19 + (0)(-5), 8 + 0(-9)) = (2, -19, 8)$

Therefore, the two lines intersect at the point (2, –19, 8).

c) If the lines intersect, there is one t-value and one v-value that will produce the same point.

Equate the x-coordinates.
$4 + t = 1 + 4v$
$4v - t = 3$ ①

Equate the y-coordinates.
$-3 + 8t = 0 - 5v$
$5v + 8t = 3$ ②

Equate the z-coordinates.
$2 - 3t = 3 - 9v$
$9v - 3t = 1$ ③

Solve ① and ② for s and v.

$$\begin{aligned} 4v - t &= 3 && ① \\ \underline{5v + 8t = 3} && ② \\ 37v &= 27 && 8① + ② \end{aligned}$$

$$v = \frac{27}{37}$$

Substitute $v = \frac{27}{37}$ into equation ①.

$$4\left(\frac{27}{37}\right) - t = 3$$

$$t = \frac{108}{37} - 3$$

$$t = -\frac{3}{37}$$

When $t = -\frac{3}{37}$, the first line gives the point:

$$\left(4 + \left(-\frac{3}{37}\right)(1), -3 + \left(-\frac{3}{37}\right)(8), 2 + \left(-\frac{3}{37}\right)(-3)\right) = \left(\frac{145}{37}, \frac{134}{37}, \frac{83}{37}\right).$$

When $v = \frac{27}{37}$, the second line gives the point:

$$\left(2 + \left(\frac{27}{37}\right)(4), -19 + \left(\frac{27}{37}\right)(-5), 8 + \left(\frac{27}{37}\right)(-9)\right) = \left(\frac{182}{37}, \frac{-838}{37}, \frac{53}{37}\right).$$

Therefore, the two lines do not intersect.

Chapter 8 Section 1 **Question 21 Page 438**

a) The lines are parallel if their direction vectors are scalar multiples of each other.

$[4, -5] = p[7, k]$

Equate the x-coordinates.

$4 = 7p$

$p = \frac{4}{7}$

Equate the y-coordinates.

$-5 = pk$

$-5 = \frac{4}{7}k$

$k = -\frac{35}{4}$

The lines are parallel if $k = -\frac{35}{4}$.

b) The lines are perpendicular if the dot product of the direction vectors is zero.

$[4, -5] \cdot [7, k] = 0$

$4(7) + (-5)k = 0$

$k = \frac{28}{5}$

The lines are perpendicular when $k = \frac{28}{5}$.

Chapter 8 Section 1 **Question 22 Page 438**

The direction vectors are scalar multiples of each other:

$$-2[3, -1, 4] = 2[-3, 1, -4]$$
$$= [-6, 2, -8]$$

The lines are at least parallel.

Does (–13, 6, –10) lie on ℓ_1?

Substitute the coordinates into the vector equation.

$[-13, 6, -10] = [11, -2, 17] + t[3, -1, 4]$

Equate the x-coordinates.

$-13 = 11 + 3t$

$t = -8$

Equate the y-coordinates.

$6 = -2 - t$

$t = -8$

Equate the z-coordinates.

$-10 = 17 + 4t$

$t = -\frac{27}{4}$

Since the t values are not equal, the point (–13, 6, –10) does not lie on the line ℓ_1 and so ℓ_1 and ℓ_2 are distinct lines.

Does (–4, 3, –3) lie on ℓ_1?
Substitute the coordinates into the vector equation.
$[-4, 3, -3] = [11, -2, 17] + t[3, -1, 4]$

Equate the x-coordinates.
$-4 = 11 + 3t$
$t = -5$

Equate the y-coordinates.
$3 = -2 - t$
$t = -5$

Equate the z-coordinates.
$-3 = 17 + 4t$
$t = -5$

Since the t values are equal, the point (–4, 3, –3) does lie on the line ℓ_1 and so ℓ_1 and ℓ_3 are different representations of the same line.

Chapter 8 Section 1 **Question 23 Page 439**

a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
$= [-5, 4] - [3, -2]$
$= [-8, 6]$

This is a direction vector for the line.

b) A possible vector equation is $[x, y] = [3, -2] + t[-8, 6],\ t \in \mathbb{R}$.
Possible parametric equations are $x = 3 - 8t$ and $y = -2 + 6t,\ t \in \mathbb{R}$.

c) Choose a vector that makes a zero dot product with $[-8, 6]$. For example, $[3, 4]$.
The scalar equation is of the form $3x + 4y + C = 0$.

Point A is on the line. Substitute its coordinates and solve for C.
$3(3) + 4(-2) + C = 0$
$C = -1$

The scalar equation is $3x + 4y - 1 = 0$

d) Choose convenient values for t and substitute in the parametric equations.
If $t = -1$, (11, –8) is a point on the line.
If $t = -2$, (19, –14) is a point on the line.
If $t = 2$, (–13, 10) is a point on the line.
If $t = -\frac{1}{2}$, (7, –5) is a point on the line.

e) Substitute the coordinates for (35, –26) into the vector equation.

$[35, -26] = [3, -2] + t[-8, 6]$

Equate the x-coordinates.
$35 = 3 - 8t$
$t = -4$

Equate the y-coordinates.
$-26 = -2 + 6t$
$t = -4$

Since the t values are equal, the point (35, –26) does lie on the line.

Substitute the coordinates for (−9, 8) into the vector equation.

$[-9, 8] = [3, -2] + t[-8, 6]$

Equate the x-coordinates.
$-9 = 3 - 8t$
$t = \frac{3}{2}$

Equate the y-coordinates.
$8 = -2 + 6t$
$t = \frac{5}{3}$

Since the t values are not equal, the point (−9, 8) does not lie on the line.

Chapter 8 Section 1 **Question 24 Page 439**

a) The vectors will be perpendicular to the line if their dot product with the direction vector for the line is zero.

i) $[2, -3, -1] \cdot [1, -1, 5] = 2(1) + (-3)(-1) + (-1)(5)$
$= 0$

This vector is perpendicular to the line.

ii) $[2, -3, -1] \cdot [2, 2, 2] = 2(2) + (-3)(2) + (-1)(2)$
$= -4$

This vector is not perpendicular to the line.

iii) $[2, -3, -1] \cdot [-4, -7, 13] = 2(-4) + (-3)(-7) + (-1)(13)$
$= 0$

This vector is perpendicular to the line.

b) Choose vectors that have a dot product of zero with the direction vector for the line. Choose arbitrary values for x and y, then calculate z so that the dot product is zero. Be careful not to choose scalar multiples of the vectors in part i) or iii) above.

If $x = 1$ and $y = 1$, then [1, 1, –1] is a vector perpendicular to the line.
If $x = 5$ and $y = 2$, then [5, 2, 4], is a vector perpendicular to the line.
If $x = 2$ and $y = 0$, then [2, 0, 4], is a vector perpendicular to the line.

Chapter 8 Section 1 **Question 25 Page 439**

a) First find the direction vectors for the sides.

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA}\\ &= [4, 3] - [7, 4]\\ &= [-3, -1]\end{aligned}$$

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{OC} - \overrightarrow{OA}\\ &= [6, -3] - [7, 4]\\ &= [-1, -7]\end{aligned}$$

$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{OC} - \overrightarrow{OB}\\ &= [6, -3] - [4, 3]\\ &= [2, -6]\end{aligned}$$

Vector equations for the sides are:

$$\begin{aligned}AB &: [x, y] = [7, 4] + t[-3, -1],\ 0 \le t \le 1,\ t \in \mathbb{R}\\ AC &: [x, y] = [7, 4] + s[-1, -7],\ 0 \le s \le 1,\ s \in \mathbb{R}\\ BC &: [x, y] = [4, 3] + v[2, -6],\ 0 \le v \le 1,\ v \in \mathbb{R}\end{aligned}$$

Note that the restrictions on the parameters limit the points on the lines to just those within the triangle.

b) First find the direction vectors for the sides.

$$\begin{aligned}\overrightarrow{DE} &= \overrightarrow{OE} - \overrightarrow{OD}\\ &= [1, -1, 8] - [1, -3, 2]\\ &= [0, 2, 6]\end{aligned}$$

$$\begin{aligned}\overrightarrow{DF} &= \overrightarrow{OF} - \overrightarrow{OD}\\ &= [5, -17, 0] - [1, -3, 2]\\ &= [4, -14, -2]\end{aligned}$$

$$\begin{aligned}\overrightarrow{EF} &= \overrightarrow{OF} - \overrightarrow{OE}\\ &= [5, -17, 0] - [1, -1, 8]\\ &= [4, -16, -8]\end{aligned}$$

It is possible to choose simpler (scalar multiples) for the direction vectors.

Vector equations for the sides are:

$\text{DE}: [x, y, z] = [1, -3, 2] + t[0, 1, 3],\ 0 \le t \le 2,\ t \in \mathbb{R}$

$\text{DF}: [x, y, z] = [1, -3, 2] + s[2, -7, -1],\ 0 \le s \le 2,\ s \in \mathbb{R}$

$\text{EF}: [x, y, z] = [1, -1, 8] + v[1, -4, -2],\ 0 \le v \le 4,\ v \in \mathbb{R}$

Chapter 8 Section 1 **Question 26 Page 439**

a) Let $x = 0$.

$0 = 4 + 3t$

$t = -\frac{4}{3}$

Let $y = 0$.

$0 = 1 + t$

$t = -1$

Let $z = 0$.

$0 = -2 - 5t$

$t = -\frac{2}{5}$

For $x = 0$ and $y = 0$, the t-values are different. So, there is no z-intercept.
For $x = 0$ and $z = 0$, the t-values are different. So, there is no y-intercept.
For $y = 0$ and $z = 0$, the t-values are different. So, there is no x-intercept.

This line has no x-, y-, or z-intercepts.

b) A line parallel to the given line would be of the form $[x, y, z] = [a, b, c] + t[3, 1, -5]$.

i) To have an x-intercept, setting $y = 0$ and $z = 0$ must lead to consistent t-values.

Let $y = 0$.

$0 = b + t$

$t = -b$

Let $z = 0$.

$0 = c - 5t$

$t = \frac{1}{5}c$

$\therefore c = -5b$ is the necessary condition.

ii) To have a y-intercept, setting $x = 0$ and $z = 0$ must lead to consistent t-values.

Let $x = 0$.

$0 = a + 3t$

$t = -\frac{1}{3}a$

Let $z = 0$.

$0 = c - 5t$

$t = \frac{1}{5}c$

$\therefore 3c = -5a$ is the necessary condition.

iii) To have a z-intercept, setting $x = 0$ and $y = 0$ must lead to consistent t-values.

Let $x = 0$.

$0 = a + 3t$

$t = -\frac{1}{3}a$

Let $y = 0$.

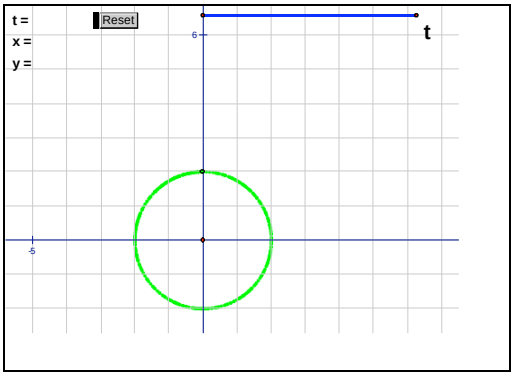
$0 = b + t$

$t = -b$

$\therefore a = 3b$ is the necessary condition.

c) The line needs to contain the origin since a line intersecting two axes can only intersect the third axis if the intersections occur at the origin. The equation will be $[x, y, z] = [0, 0, 0] + t[3, 1, -5],\ t \in \mathbb{R}$.

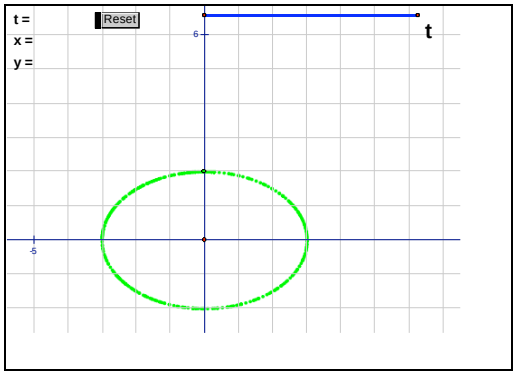
Chapter 8 Section 1 **Question 27 Page 439**



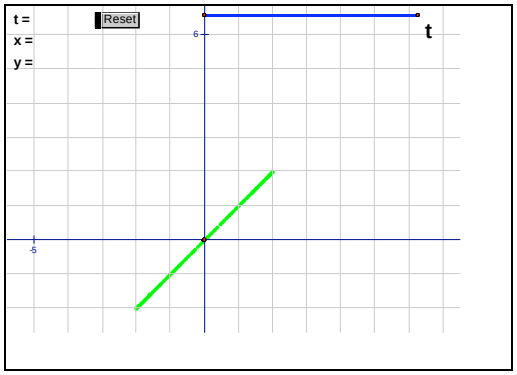
a) The graph is a circle with radius 2 units.

b) The value 2 is the radius of the circle.

c) If the coefficients are changed the curve becomes an ellipse, with the coefficient on x becoming the semi-x-axis for the ellipse and the coefficient of y becoming the semi-y-axis for the ellipse. The example shows the coefficients as 3 and 2 for x and y respectively.

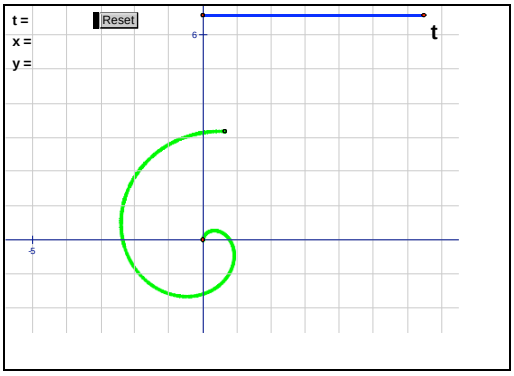


d) The resulting curve is the line segment defined by $y = x$, restricted so that $-2 \le x,\ y \le 2$.

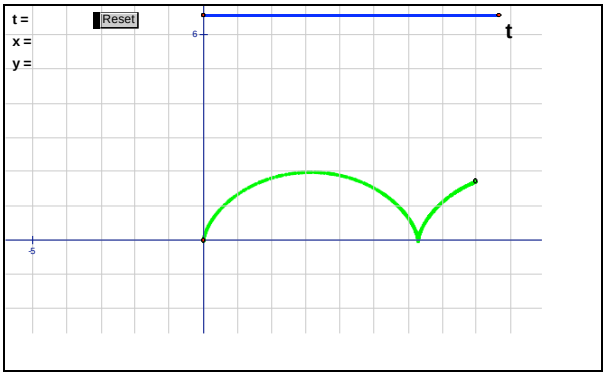


Chapter 8 Section 1 **Question 28 Page 439**

The curve is a spiral (an Archimedean spiral).



Chapter 8 Section 1 **Question 29 Page 439**



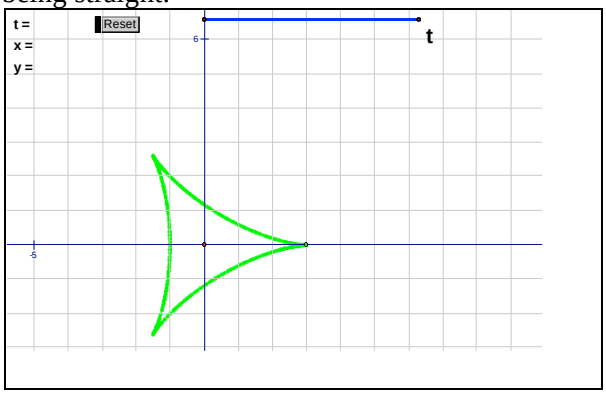
a) The curve is periodic, repeating itself every 6.28 (2π) along the x-axis.

b) The maximum y-value is 2, as determined by the coefficients of y in this example. (cos (t) varies between –1 and +1).

Chapter 8 Section 1 **Question 30 Page 439**

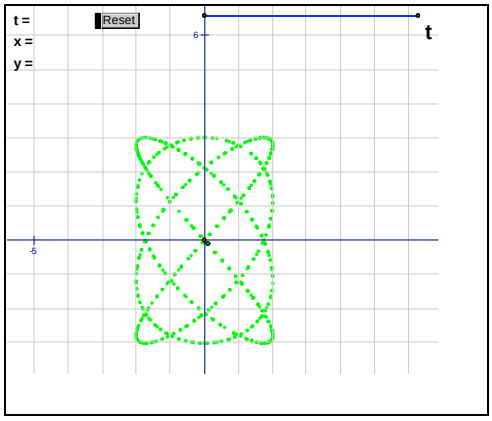
a) Parametric equations for a tricuspoid are of the form: $x = a(2\cos t + \cos 2t)$; $y = a(2\sin t - \sin 2t)$

The curve below is with $a = 1$. This curve was first discovered by Euler in 1745. The curve is sometimes called a deltoid. It looks like an equilateral triangle with sides that curve inwards instead of being straight.



b) Parametric equations for a lissajous curve are of the form: $x = \mathrm{A}\cos\left(\omega_x t - \delta_x\right)$; $y = \mathrm{B}\cos\left(\omega_y t - \delta_y\right)$

The curve below has A = 2, B = 3, $\omega_x = 4$, $\omega_y = 3$, $\delta_x = \frac{\pi}{2}$, and $\delta_y = \frac{\pi}{2}$.

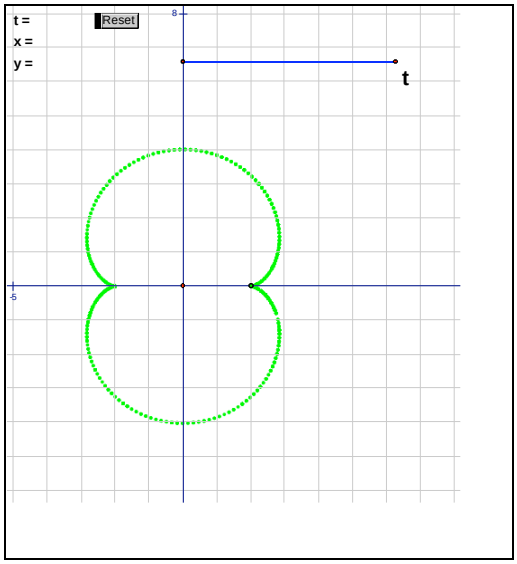


This family of curves models simple harmonic motion. They were investigated first by Nathaniel Bowditch in 1815 and more extensively by J.A. Lissajous in 1857. The curves are described as like a knot in three-space.

c) Parametric equations for a epicycloid are of the form**:**

$x = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right)$; $y = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\theta\right)$.

The curve below has $a = 2$ and $b = 1$.



Some of the early mathematicians exploring this family of curves were Galileo and Mersenne (1599). The shape of the curve is somewhat gear like.

Chapter 8 Section 1 **Question 31 Page 439**

a) Neither. Since $[4, -6, -15] \neq k[-8, 12, 20]$ for any $k \in \mathbb{R}$, the direction vectors are not scalar multiples of each other. So the lines are not parallel. Also:

$$[4, -6, -15] \cdot [-8, 12, 20] = 4(-8) + (-6)(12) + (-15)(20)$$
$$= -404$$

Since this dot product is not zero, the direction vectors and the lines are not perpendicular.

b) Since $[5, 1, -5] \neq k[1, 5, 2]$ for any $k \in \mathbb{R}$, the direction vectors are not scalar multiples of each other. So the lines are not parallel. However:

$$[5, 1, -5] \cdot [1, 5, 2] = 5(1) + 1(5) + (-5)(2)$$
$$= 0$$

Since the dot product is zero, the direction vectors and the lines are perpendicular.

Chapter 8 Section 1 **Question 32 Page 440**

a) $x = 3 + 2t$

$y = 4 + 5t,\ t \in \mathbb{R}$

b) $t = \dfrac{x-3}{2}$

$t = \dfrac{y-4}{5}$

c) $\dfrac{x-3}{2} = \dfrac{y-4}{5}$

d) The components of the direction vector become the denominators of each fraction, with the x-value of the vector under the fraction involving x and the y-value of the vector under the fraction involving y. As well, the x-value of the position vector exists with an opposite sign beside the x in the numerator of corresponding fraction and the
y-value of the position vector exists with an opposite sign beside the y in the numerator of the other fraction.

e) i) $\dfrac{x-1}{3} = \dfrac{y-7}{8}$

ii) $x - 4 = \dfrac{y+2}{9}$

iii) $\dfrac{x-(-5)}{-3} = \dfrac{y-2}{-4}$

This can also be written as $\dfrac{x+5}{-3} = \dfrac{2-y}{4}$.

Chapter 8 Section 1 **Question 33 Page 440**

a) $\vec{m} = [2, 7],\ \vec{r}_0 = [6, 9]$

The vector equation is $[x, y] = [6, 9] + t[2, 7],\ t \in \mathbb{R}$.

For the scalar equation, start with the symmetric equation and then simplify.

$$\frac{x-6}{2} = \frac{y-9}{7}$$

$$7x - 42 = 2y - 18$$

$$7x - 2y - 24 = 0$$

b) $\vec{m} = [4, -5]$, $\vec{r}_0 = [-3, -9]$

The vector equation is $[x, y] = [-3, -9] + t[4, -5]$, $t \in \mathbb{R}$.

For the scalar equation, start with the symmetric equation and then simplify.

$$\frac{x-(-3)}{4} = \frac{y-(-9)}{-5}$$

$$-5x - 15 = 4y + 36$$

$$5x + 4y + 51 = 0$$

c) $\vec{m} = [-7, 1]$, $\vec{r}_0 = [4, -10]$

The vector equation is $[x, y] = [4, -10] + t[-7, 1]$, $t \in \mathbb{R}$.

For the scalar equation, start with the symmetric equation and then simplify.

$$\frac{x-4}{-7} = \frac{y-(-10)}{1}$$

$$x - 4 = -7y - 70$$

$$x + 7y + 66 = 0$$

Chapter 8 Section 1 **Question 34 Page 440**

a) i) $\frac{x-1}{5} = \frac{y-3}{4} = \frac{z-9}{2}$

ii) $\frac{x+4}{-2} = \frac{y+1}{8} = z - 7$

iii) $5 - x = \frac{1-y}{3} = \frac{z+9}{11}$

b) i) $[x, y, z] = [4, 12, 15] + t[8, 5, 2]$, $t \in \mathbb{R}$

ii) $[x, y, z] = [6, -1, -5] + t[1, 7, -3]$, $t \in \mathbb{R}$

iii) $[x, y, z] = [5, -3, 0] + t[-6, -10, 11]$, $t \in \mathbb{R}$

Chapter 8 Section 1 **Question 35 Page 440**

a) $\cos\theta = \dfrac{\vec{m}_1 \cdot \vec{m}_2}{\left|\vec{m}_1\right|\left|\vec{m}_2\right|}$

$\cos\theta = \dfrac{[1, 5]\cdot[-2, 7]}{\left|[1, 5]\right|\left|[-2, 7]\right|}$

$\cos\theta = \dfrac{1(-2)+5(7)}{\sqrt{1^2+5^2}\sqrt{(-2)^2+7^2}}$

$\cos\theta \doteq \dfrac{33}{37.1214}$

$\theta \doteq \cos^{-1}\left(\dfrac{33}{37.1214}\right)$

$\theta \doteq 27.3^\circ$

b) $\cos\theta = \dfrac{\vec{m}_1 \cdot \vec{m}_2}{\left|\vec{m}_1\right|\left|\vec{m}_2\right|}$

$\cos\theta = \dfrac{[2, -2, 3]\cdot[1, -3, 5]}{\left|[2, -2, 3]\right|\left|[1, -3, 5]\right|}$

$\cos\theta = \dfrac{2(1)+(-2)(-3)+3(5)}{\sqrt{2^2+(-2)^2+3^2}\sqrt{1^2+(-3)^2+5^2}}$

$\cos\theta \doteq \dfrac{23}{24.3926}$

$\theta \doteq \cos^{-1}\left(\dfrac{23}{24.3926}\right)$

$\theta \doteq 19.5^\circ$

Chapter 8 Section 1 **Question 36 Page 440**

To find a direction vector perpendicular to both lines, calculate $\vec{m}_1 \times \vec{m}_2$.

$[3, -1, 1]\times[1, -3, 7] = [-1(7)-(-3)(1), 1(1)-7(3), 3(-3)-1(-1)]$

$= [-4, -20, -8]$

Using a simpler vector for this direction, $[1, 5, 2]$, the parametric equations are:

$x = 6 + t$

$y = -2 + 5t$

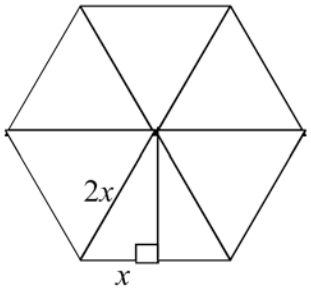
$z = 1 + 2t,\ t \in \mathbb{R}$

Chapter 8 Section 1 **Question 37 Page 440**

Answers may vary. For example:

$$l_1 : [x, y, z] = [3, 1, -1] + t[1, 0, 0],\ t \in \mathbb{R}$$
$$l_2 : [x, y, z] = [3, 1, -1] + s[0, 1, 0],\ s \in \mathbb{R}$$

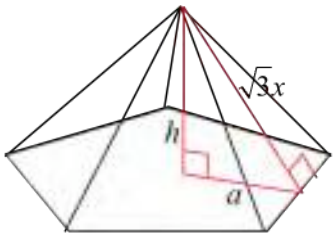
Chapter 8 Section 1 **Question 38 Page 440**



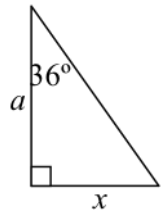
Consider one triangle (area 6 cm^2) in the hexagon. Drawing an altitude creates a right angled triangle. The altitude is $\sqrt{3}x$.

Then,

$$A = \frac{1}{2}(2x)\left(\sqrt{3}x\right)$$
$$6 = \sqrt{3}x^2$$
$$x^2 = \sqrt{\frac{6}{\sqrt{3}}}$$
$$x \doteq 1.86$$



The value of *a* is found by examining a regular pentagon, finding the distance from the centre to one of its sides. The pentagon can be divided into 10 triangles meeting at its centre. Each triangle will have a 36º at the centre of the pentagon.



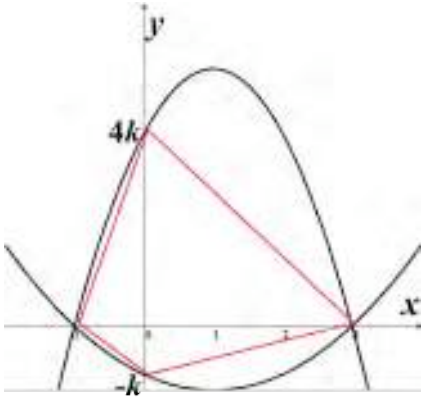
$$\frac{x}{a} = \tan 36^\circ$$
$$a \doteq \frac{1.86}{\tan 36^\circ}$$
$$a \doteq 2.56$$

Finally, to find h, use the Pythagorean theorem with the red triangle above.

$$h^2 + (2.56)^2 = \left(\sqrt{3}(1.86)\right)^2$$
$$h^2 + 6.5536 = 10.3788$$
$$h = \sqrt{3.8252}$$
$$\doteq 1.96$$

The height of the pyramid will be about 1.96 cm.

Chapter 8 Section 1 **Question 39 Page 440**



Thee area of the quadrilateral can be found by considering an upper triangle and a lower triangle, separated by the x-axis.

$$30 = \frac{1}{2}(4)(4k) + \frac{1}{2}(4)(k)$$
$$30 = 10k$$
$$k = 3$$

Chapter 8 Section 2 **Equations of Planes**

Chapter 8 Section 2 **Question 1 Page 451**

Answers may vary. For example:

a) (8, 1, 2), (8, –2, 5), (8, 7, 6)
All points must have the first coordinate 8.

b) (1, 0, 3), (5, 9, 0), (2, 6, 1)
Choose any value for z and then solve for y; x can be any value.

c) (0, 1, –1), (1, 0, –11), (1, 2, 3)
Choose any values for x and y, then solve for z.

d) (0, 0, –2), (4, 0, 0), (2, 0, –1)
Choose any values for x and y, then solve for z.

Chapter 8 Section 2 **Question 2 Page 451**

Answers may vary. For example:
Find vectors between the points found in question 1.

a) $[8, -2, 5] - [8, 1, 2] = [0, -3, 3]$
$[8, 7, 6] - [8, 1, 2] = [0, 6, 4]$

b) $[5, 9, 0] - [1, 0, 3] = [4, 9, -3]$
$[2, 6, 1] - [1, 0, 3] = [1, 6, -2]$

c) $[1, 0, -11] - [0, 1, -1] = [1, -1, -10]$
$[1, 2, 3] - [1, 0, -11] = [0, 2, 14]$

d) $[2, 0, -1] - [0, 0, -2] = [2, 0, 1]$
$[4, 0, 0] - [0, 0, -2] = [4, 0, 2]$

Chapter 8 Section 2 **Question 3 Page 451**

Substitute the coordinates for each point into the equation $4x + 3y - 5z = 10$.

a) L.S.$= 4(1) + 3(2) - 5(0)$ R.S. = 10
$= 10$
L.S. = R.S.

Point A(1, 2, 0) lies on the plane.

b) L.S.$=4(-7)+3(6)-5(4)$ R.S. = 10

$=-30$

L.S. ≠ R.S.

Point B(–7, 6, 4) does not lie on the plane.

c) L.S.$=4(-2)+3(1)-5(-3)$ R.S. = 10

$=10$

LS = RS

Point C(–2, 1, –3) lies on the plane.

d) L.S.$=4(1.2)+3(-2.4)-5(6.2)$ R.S. = 10

$=-33.4$

L.S. ≠ R.S.

Point D(1.2, –2.4, 6.2) does not lie on the plane.

Chapter 8 Section 2 **Question 4 Page 451**

a) Let $y = 0$ and $z = 0$. Let $x = 0$ and $z = 0$. Let $x = 0$ and $y = 0$.

$3x-2(0)+4(0)=12$ $3(0)-2y+4(0)=12$ $3(0)-2(0)+4z=12$

$x = 4$ $y = -6$ $z = 3$

The x-intercept is 4. The y-intercept is –6, and the z-intercept is 3.

b) Let $y = 0$ and $z = 0$. Let $x = 0$ and $z = 0$. Let $x = 0$ and $y = 0$.

$x+5(0)-6(0)=30$ $0+5y-6(0)=30$ $0+5(0)-6z=30$

$x = 30$ $y = 6$ $z = -5$

The x-intercept is 30. The y-intercept is 6, and the z-intercept is –5.

c) Let $y = 0$ and $z = 0$. Let $x = 0$ and $z = 0$. Let $x = 0$ and $y = 0$.

$4x+2(0)-7(0)+14=0$ $4(0)+2y-7(0)+14=0$ $4(0)+2(0)-7z+14=0$

$x=-\frac{7}{2}$ $y = -7$ $z = 2$

The x-intercept is $-\frac{7}{2}$. The y-intercept is –7, and the z-intercept is 2.

d) Let $y = 0$ and $z = 0$. Let $x = 0$ and $z = 0$. Let $x = 0$ and $y = 0$.

$3x+6(0)+18=0$ $3(0)+6(0)+18=0$ $3(0)+6z+18=0$

$x = -6$ No values possible for y. $z = -3$

The x-intercept is –6 and the z-intercept is –3. There is no y-intercept.

Chapter 8 Section 2 **Question 5 Page 451**

$s, t \in \mathbb{R}$ for all equations.

a) $x = 1 - 3s + 9t$
$y = 3 + 4s + 2t$
$z = -2 - 5s - t$

b) $x = s$
$y = -4 + 10s + 3t$
$z = 1 - s + 4t$

c) $x = t$
$y = 3s$
$z = 5 + 5t$

Chapter 8 Section 2 **Question 6 Page 451**

a) $[x, y, z] = [9, 4, -1] + s[3, -7, -5] + t[-2, 1, -4],\ s, t \in \mathbb{R}$

b) $[x, y, z] = [2, 0, 11] + s[1, 12, 6] + t[7, -8, 0],\ s, t \in \mathbb{R}$

c) $[x, y, z] = [-6, 0, 5] + s[0, 8, 0] + t[0, 0, -13],\ s, t \in \mathbb{R}$

Chapter 8 Section 2 **Question 7 Page 451**

a) If P(10, –19, 15) lies on the plane there exists a single set of t- and s-values that satisfy the equation.

$10 = 6 + s + 2t$ ①
$-19 = -7 + 3s - 2t$ ②
$15 = 10 - s + t$ ③

Solve ① and ② for s and t.

$4 = s + 2t$ ①
$-12 = 3s - 2t$ ②
$-8 = 4s$ ①+②
$s = -2$
$4 = -2 + 2t$ ①
$t = 3$

Now check if these values satisfy equation ③.

L.S. = 15 R.S. $= 10 - (-2) + 3$
$= 15$
L.S. = R.S.

Thus, P(10, –19, 15) lies on the plane.

b) If P(–4, –13, 10) lies on the plane there exists a single set of t- and s-values that satisfy the equation.

$-4 = 6 + s + 2t$ ①

$-13 = -7 + 3s - 2t$ ②

$10 = 10 - s + t$ ③

Solve ① and ② for s and t.

$-10 = s + 2t$ ①

$-6 = 3s - 2t$ ②

$-16 = 4s$ ①+②

$s = -4$

$-10 = -4 + 2t$ ①

$t = -3$

Now check if these values satisfy equation ③.

L.S. = 10 R.S. $= 10 - (-4) + (-3)$

$= 11$

L.S. ≠ R.S.

Thus, P(–4, –13, 10) does not lie on the plane

c) If P(8.5, –3.5, 9) lies on the plane there exists a single set of t- and s-values that satisfy the equation.

$8.5 = 6 + s + 2t$ ①

$-3.5 = -7 + 3s - 2t$ ②

$9 = 10 - s + t$ ③

Solve ① and ② for s and t.

$2.5 = s + 2t$ ①

$3.5 = 3s - 2t$ ②

$6 = 4s$ ①+②

$s = 1.5$

$2.5 = 1.5 + 2t$ ①

$t = 0.5$

Now check if these values satisfy equation ③.

L.S. = 9 R.S. $= 10 - 1.5 + 0.5$

$= 9$

L.S. = R.S.

Thus, P(8.5, –3.5, 9) lies on the plane.

Chapter 8 Section 2 **Question 8 Page 451**

Answers may vary. For example:

Let $s = 1$ and $t = -1$. P(5, –2, 8) is a point on the plane.
Let $s = 1$ and $t = 0$. P(7, –4, 9) is a point on the plane.
Let $s = 0$ and $t = 0$. P(6, –7, 10) is a point on the plane.

Chapter 8 Section 2 **Question 9 Page 451**

a) To find the x-intercept, let $y = 0$ and $z = 0$. Solve for s and t.

$x = 1 + s + 2t$ ①

$0 = 8 - 12s + 4t$ ②

$0 = 6 - 12s - 3t$ ③

Solve ② and ③ for s and t.

$-8 = -12s + 4t$ ②

$-6 = -12s - 3t$ ③

$-2 = 7t$ ②–③

$t = -\frac{2}{7}$

$-6 = -12s + \frac{6}{7}$ ③

$s = \frac{4}{7}$

Now substitute in equation ①.

$x = 1 + \frac{4}{7} + 2\left(-\frac{2}{7}\right)$

$= 1$

The x-intercept is 1.

To find the y-intercept, let $x = 0$ and $z = 0$. Solve for s and t.

$0 = 1 + s + 2t$ ①

$y = 8 - 12s + 4t$ ②

$0 = 6 - 12s - 3t$ ③

Solve ① and ③ for s and t.

$-1 = s + 2t$ ①

$-6 = -12s - 3t$ ③

$-18 = 21t$ 12①+③

$t = -\frac{6}{7}$

$-6 = -12s + \frac{18}{7}$ ③

$s = \frac{5}{7}$

Now substitute in equation ②.

$y = 8 - 12\left(\frac{5}{7}\right) + 4\left(-\frac{6}{7}\right)$

$= -4$

The y-intercept is -4.

To find the z-intercept, let $x = 0$ and $y = 0$. Solve for s and t.

$0 = 1 + s + 2t$ ①

$0 = 8 - 12s + 4t$ ②

$z = 6 - 12s - 3t$ ③

Solve ① and ② for s and t.

$-1 = s + 2t$ ①

$-8 = -12s + 4t$ ②

$6 = 14s$ 2①−②

$s = \frac{3}{7}$

$-1 = \frac{3}{7} + 2t$ ①

$t = -\frac{5}{7}$

Now substitute in equation ③.

$z = 6 - 12\left(\frac{3}{7}\right) - 3\left(-\frac{5}{7}\right)$

$= 3$

The z-intercept is 3.

b) To find the x-intercept, let $y = 0$ and $z = 0$. Solve for s and t.

$x = 6 + s + 3t$ ①

$0 = -9 - 4s + 3t$ ②

$0 = -8 - 4s + 8t$ ③

Solve ② and ③ for s and t.

$9 = -4s + 3t$ ②

$\underline{8 = -4s + 8t}$ ③

$1 = -5t$ ②–③

$t = -\frac{1}{5}$

$8 = -4s + 8\left(-\frac{1}{5}\right)$ ③

$s = -\frac{12}{5}$

Now substitute in equation ①.

$x = 6 + \left(-\frac{12}{5}\right) + 3\left(-\frac{1}{5}\right)$

$= 3$

The x-intercept is 3.

To find the y-intercept, let $x = 0$ and $z = 0$. Solve for s and t.

$0 = 6 + s + 3t$ ①

$y = -9 - 4s + 3t$ ②

$0 = -8 - 4s + 8t$ ③

Solve ① and ③ for s and t.

$-6 = s + 3t$ ①

$\underline{8 = -4s + 8t}$ ③

$-16 = 20t$ 4①+③

$t = -\frac{4}{5}$

$8 = -4s + 8\left(-\frac{4}{5}\right)$ ③

$s = -\frac{18}{5}$

Now substitute in equation ②.

$$y = -9 - 4\left(-\frac{18}{5}\right) + 3\left(-\frac{4}{5}\right)$$
$$= 3$$

The y-intercept is 3.

To find the z-intercept, let $x = 0$ and $y = 0$. Solve for s and t.

$0 = 6 + s + 3t$ ①

$0 = -9 - 4s + 3t$ ②

$z = -8 - 4s + 8t$ ③

Solve ① and ② for s and t.

$-6 = s + 3t$ ①

$9 = -4s + 3t$ ②

$-15 = 5s$ ①–②

$s = -3$

$-6 = -3 + 3t$ ①

$t = -1$

Now substitute in equation ③.

$z = -8 - 4(-3) + 8(-1)$
$= -4$

The z-intercept is –4.

Chapter 8 Section 2 **Question 10 Page 451**

$s, t \in \mathbb{R}$ for all equations.

a) Vector: $[x, y, z] = [6, -1, 0] + s[2, 0, -5] + t[1, -3, 1]$

Parametric: $x = 6 + 2s + t$

$y = -1 - 3t$

$z = -5s + t$

b) If the plane is parallel to a line, then it can have the same direction vector as the line.
In this case, [1, –1, 1] and [–6, 2, 5] are direction vectors for the plane.

Vector: $[x, y, z] = [9, 1, -2] + s[1, -1, 1] + t[-6, 2, 5]$

Parametric: $x = 9 + s - 6t$

$y = 1 - s + 2t$

$z = -2 + s + 5t$

c) Direction vectors for the plane can be:

For $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$: $[3, -9, 7] - [1, 3, -2] = [2, -12, 9]$.

For $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$: $[4, -4, 5] - [1, 3, -2] = [3, -7, 7]$.

Vector: $[x, y, z] = [1, 3, -2] + s[2, -12, 9] + t[3, -7, 7]$

Parametric: $x = 1 + 2s + 3t$

$y = 3 - 12s - 7t$

$z = -2 + 9s + 7t$

d) Points on the plane include A(8, 0, 0), B(0, –3, 0), and C(0, 0, 2).
Direction vectors for the plane can be:

For $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$: $[8, 0, 0] - [0, -3, 0] = [8, 3, 0]$.

For $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$: $[8, 0, 0] - [0, 0, 2] = [8, 0, -2]$.

Vector: $[x, y, z] = [8, 0, 0] + s[8, 3, 0] + t[8, 0, -2]$

Parametric: $x = 8 + 8s + 8t$

$y = 3s$

$z = -2t$

Chapter 8 Section 2 Question 11 Page 451

a) This plane is parallel to the *yz*-plane with all points having an *x*-coordinate of 5.

b) This plane is parallel to the *xz*-plane with all points having a *y*-coordinate of –7.

c) This plane is parallel to the *xy*-plane with all points having a *z*-coordinate of 10.

d) This plane is parallel to the *z*-axis and contains points where the *x* and *y* values add to 8. The *z*-coordinate can be any real number.

e) This plane is parallel to the *y*-axis and contains points where the sum of the *x* value and two times the *z* value is 4. The *y*-coordinate can be any real number.

f) This plane is parallel to the *x*-axis and contains points where the result of two times the *z*-value subtracted from three times the *y*-value is 12. The *x*-coordinate can be any real number.

g) This plane contains points where the *x*-, *y*-, and *z*-values add to 0. The plane passes through the origin.

h) This plane has an *x*-intercept of 2, a *y*-intercept of 3, and a *z*-intercept of –6.

i) This plane has an *x*-intercept of –5, a *y*-intercept of 4, and a *z*-intercept of –10.

Chapter 8 Section 2 **Question 12 Page 452**

Answers for part e) to part g) may vary.

a) $y = 1$

b) $z = k,\ k \in \mathbb{R}$

c) First find two direction vectors for the plane.

For $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$: $[5, -3, 2] - [2, 1, 1] = [3, -4, 1]$.

For $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$: $[0, -1, 4] - [2, 1, 1] = [-2, -2, 3]$.

A vector equation is $[x, y, z] = [2, 1, 1] + s[3, -4, 1] + t[-2, -2, 3],\ s, t \in \mathbb{R}$.

d) Direction vectors for the plane are perpendicular to $\vec{a}$ and so have a dot product of zero with $\vec{a}$.
Two possibilities are [5, –4, 0] and [1, 0, 2].
A possible vector equation is $[x, y, z] = [1, 0, 0] + s[5, -4, 0] + t[1, 0, 2],\ s, t \in \mathbb{R}$.

Verify that $P_0(1, 0, 0)$ does not create a plane passing through the origin.

$0 = 1 + 5s + t$ ①

$0 = -4s$ ②

$0 = 2t$ ③

Solve ② and ③ for s and t.

$s = 0$ ②

$t = 0$ ③

Now check if these values satisfy equation ①.

L.S. = 0 R.S. $= 1 + 5(0) + (0)$

$= 1$

L.S. ≠ R..S.

The origin, (0, 0, 0) is not on the plane.

e) Direction vectors for this plane include [4, 1, –3] and [1, 0, 0]. Any point on the x-axis is on the plane. Use P(3, 0, 0).
A possible vector equation is $[x, y, z] = [3, 0, 0] + s[4, 1, -3] + t[1, 0, 0],\ s, t \in \mathbb{R}$.

f) $3x + z = 10$ is a possible scalar equation. (In general $Ax + Bz = C$ where none of $A,\ B,\ C \neq 0$.)

g) $x + 2y = 1$ is a possible scalar equation. (In general $Ax + By = C$ where none of $A,\ B,\ C \neq 0$.)

Chapter 8 Section 2 **Question 13 Page 452**

a) The points (0, –3, 0), R(2, –3, 4), and S(–2, –3, –4) are collinear.
There are many planes passing through any one line.

b) $\overrightarrow{AB} = [5, -15, 7] - [2, 0, 1] = [3, -15, 6]$

$\overrightarrow{AC} = [0, 10, -3] - [2, 0, 1] = [-2, 10, -4]$

$\overrightarrow{AB} = -\frac{3}{2}\overrightarrow{AC}$

Therefore A, B, and C are collinear and many planes pass through the points.

c) Since $[8, -12, 4] = -\frac{4}{3}[-6, 9, -3]$, the direction vectors are parallel.
The two direction vectors cannot be parallel if the plane is to be unique.

d) Check to see if the point is on the line.
Substitute the coordinates into the vector equation.
$[3, -1, 4] = [-5, 3, 10] + t[4, -2, -3]$

Equate the *x*-coordinates.	Equate the *y*-coordinates.	Equate the *y*-coordinates.
$3 = -5 + 4t$	$-1 = 3 - 2t$	$4 = 10 - 3t$
$t = 2$	$t = 2$	$t = 2$

Since the *t* values are equal, the point P(3, –1, 4) does lie on the line.
There are many planes passing through any one line.

e) Since $-2[2, -3, 1] = [-4, 6, -2]$, the direction vectors are parallel which does not define a unique plane.

f) Check to see if the lines are identical.
Since $[-1, 5, -2] = -[1, -5, 2]$, the lines are at least parallel.

Check if the point $P_0(0, 1, 1)$ is on the second line.
Substitute the coordinates into the vector equation.
$[0, 1, 1] = [-1, 6, -1] + t[1, -5, 2]$

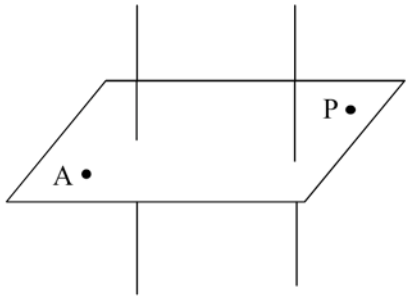
Equate the *x*-coordinates.	Equate the *y*-coordinates.	Equate the *y*-coordinates.
$0 = -1 + t$	$1 = 6 - 5t$	$1 = -1 + 2t$
$t = 1$	$t = 1$	$t = 1$

Since the *t* values are equal, the point $P_0(0, 1, 1)$ lies on the second line.
There are many planes passing through any one line.

Chapter 8 Section 2 **Question 14 Page 452**

The equation for the front wall is $x = 8$ and for the back wall is $x = 0$.
The equation for the left wall is $y = 0$ and for the right wall is $y = 6$.
The equation for the floor is $z = 0$ and for the ceiling is $z = 4$.

Chapter 8 Section 2 **Question 15 Page 452**



If the plane is perpendicular to the lines, then the lines are perpendicular to the plane.
If A is on the plane, then AP lies in the plane and $\overrightarrow{AP}$ will be perpendicular to the lines.

$$\begin{aligned}\overrightarrow{AP} &= \overrightarrow{OP} - \overrightarrow{OA} \\ &= [-1, 4, -2] - [7, 10, 16] \\ &= [-8, -6, -18]\end{aligned}$$

Check perpendicularity.

$$\begin{aligned}[-8, -6, -18] \cdot [1, 2, -1] &= -8(1) + (-6)(2) + (-18)(-1) \\ &= -2\end{aligned}$$

Since the dot product is not zero, $\overrightarrow{AP}$ is not perpendicular to the first line and therefore A does not lie on the plane.

Chapter 8 Section 2 **Question 16 Page 452**

First find direction vectors.

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= [2, -3, 1] - [3, 0, 4] \\ &= [-1, -3, -3]\end{aligned}$$

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{OC} - \overrightarrow{OA} \\ &= [-5, 8, -4] - [3, 0, 4] \\ &= [-8, 8, -8]\end{aligned}$$

The plane containing A, B, and C has vector equation:
$[x, y, z] = [3, 0, 4] + s[1, 3, 3] + t[1, -1, 1],\ s, t \in \mathbb{R}$

Check if D(1, 4, 3) lies on the plane.
Substitute the coordinates in the vector equation

$1 = 3 + s + t$ ①

$4 = 3s - t$ ②

$3 = 4 + 3s + t$ ③

Solve ① and ② for s and t.

$-2 = s + t$ ①

$4 = 3s - t$ ②

$2 = 4s$ ①+②

$s = \frac{1}{2}$

$-2 = \frac{1}{2} + t$ ①

$t = -\frac{5}{2}$

Now check if these values satisfy equation ③.

L.S. = 3

$\text{R.S.} = 4 + 3\left(\frac{1}{2}\right) + \left(-\frac{5}{2}\right)$

$= 3$

L.S. = R.S.

The point D lies on the same plane as A, B, and C.

Chapter 8 Section 2 **Question 17 Page 452**

First find direction vectors.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= [0, 1, 0] - [4, -2, 6]$

$= [-4, 3, -6]$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= [1, 0, -5] - [4, -2, 6]$

$= [-3, 2, -11]$

$\overrightarrow{AB} \times \overrightarrow{AC}$ gives a vector perpendicular to the plane.

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= [-4, 3, -6] \times [-3, 2, -11] \\ &= [3(-11) - 2(-6), (-6)(-3) - (-11)(-4), (-4)(2) - (-3)(3)] \\ &= [-21, -26, 1]\end{aligned}$$

If D lies on the plane:

$$\begin{aligned}\overrightarrow{BD} &= \overrightarrow{OD} - \overrightarrow{OB} \\ &= [1, k, -2] - [0, 1, 0] \\ &= [1, k-1, -2]\end{aligned}$$

Then BD must be perpendicular to $\overrightarrow{AB} \times \overrightarrow{AC}$.

The dot product of these two vectors must be zero.

$$\begin{aligned}[-21, -26, 1] \cdot [1, k-1, -2] &= 0 \\ -21 - 26(k-1) - 2 &= 0 \\ -26k &= -3 \\ k &= \frac{3}{26}\end{aligned}$$

Chapter 8 Section 2 **Question 18 Page 452**

Answers may vary. For example:

a) Choose arbitrary values for two of the variables and solve for the third.
Let $x = 0$ and $y = 0$. The resulting point is J$(0, 0, -D)$,
Let $y = 0$ and $z = 0$. The resulting point is K$(D, 0, 0)$,
Let $y = 1$ and $z = 0$. The resulting point is L$(D - 2, 1, 0)$,
Let $y = 0$ and $z = 1$. The resulting point is M$(D + 1, 0, 1)$

b)

$$\begin{aligned}\overrightarrow{JK} &= \overrightarrow{OK} - \overrightarrow{OJ} \\ &= [D, 0, 0] - [0, 0, -D] \\ &= [D, 0, D]\end{aligned}$$

$$\begin{aligned}\overrightarrow{JL} &= \overrightarrow{OL} - \overrightarrow{OJ} \\ &= [D-2, 1, 0] - [0, 0, -D] \\ &= [D-2, 1, D]\end{aligned}$$

$$\begin{aligned}\overrightarrow{JM} &= \overrightarrow{OM} - \overrightarrow{OJ} \\ &= [D+1, 0, 1] - [0, 0, -D] \\ &= [D+1, 0, D+1]\end{aligned}$$

c) $\overrightarrow{JK} \cdot \overrightarrow{JL} \times \overrightarrow{JM} = [D, 0, D] \cdot [D-2, 1, D] \times [D+1, 0, D+1]$

$= [D, 0, D] \cdot [1(D+1) - 0(D),\ D(D+1) - (D+1)(D-2),\ (D-2)(0) - (D+1)(1)]$

$= [D, 0, D] \cdot [D+1,\ D^2 + D - D^2 + D + 2,\ -D-1]$

$= [D, 0, D] \cdot [D+1,\ 2D+2,\ -D-1]$

$= D(D+1) + 0(2D+2) + D(-D-1)$

$= D^2 + D - D^2 - D$

$= 0$

Since the triple scalar product is zero, the volume of the parallelepiped defined by the three vectors is zero, which implies that the three vectors are coplanar.

d) Find the cross product of two of the vectors.

$\overrightarrow{JK} \times \overrightarrow{JL} = [D, 0, D] \times [D-2, 1, D]$

$= [0(D) - 1(D),\ D(D-2) - D(D),\ D(1) - (D-2)(0)]$

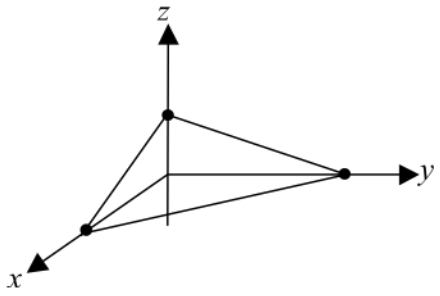
$= [-D, -2D, D]$

Choose any vector parallel to this vector. A simple vector is [1, 2, –1]. Note the similarity to the coefficients in the original scalar equation.

Chapter 8 Section 2 **Question 19 Page 453**

Answers may vary. For example:

Yes, as an example, a plane defined by 3x + 2y + 4z = 12 can be written as $\frac{x}{4} + \frac{y}{6} + \frac{z}{3} = 1$. The plane has an *x*-intercept of 4, a *y*-intercept of 6 and a *z*-intercept of 3.



Chapter 8 Section 2 **Question 20 Page 453**

To find the number, create placeholders for the nine digits (ddddddddd) and replace them when you can.
The fifth digit must be 5 or 0 to get divisibility by 5. Not using 0, so: dddd5dddd.
The second, fourth, sixth, and eighth digits must be even. Thus, the other digits are odd: OEOE5EOEO.

Numbers are divisible by 4 if the last two digits are divisible by 4. Since the third and seventh digits are odd, the fourth and eighth digits must then be 2 and 6 in some order since O4 and O8 cannot produce multiples of 4. Therefore, the second and sixth digits are 4 and 8 in some order. This gives dAdB5AdBd0 where A = 4 or 8 and B = 2 or 6.

Since the sixth number is even, the seventh and eighth digits form a number divisible by 8.

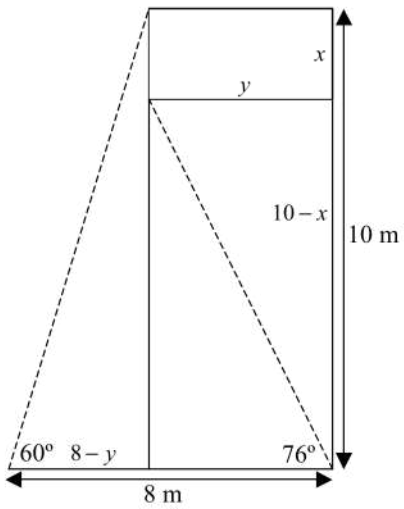
The first three digits must add to a multiple of three to get divisibility by 3, and likewise so must the next three digits and the last three digits.
Consider the second set of three, where the first digit is 2 or 6, the next is 5, and the last is 4 or 8. From these combinations, only 654 and 258 work. Each of these forces the second and eighth digits. This gives either d8d654d2d0 or d4d258d6d.
Now, in the 654 case, the seventh digit must be 3 or 7 to give divisibility by 8, and in the 258 case, the seventh digit must by 1 or 9 (or 5, but that is taken).

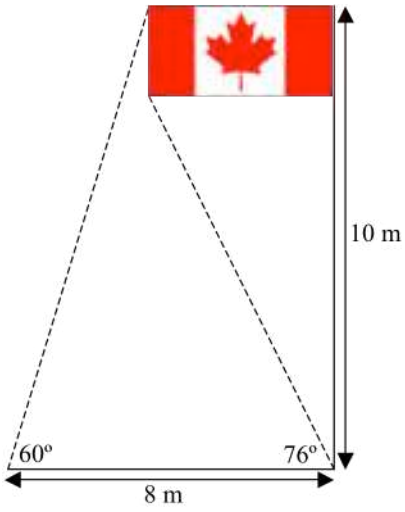
What numbers can go in the first and third positions? Some pair of 1, 3, 7, 9 must go there such that the first three add up to a number divisible by three, and a valid number must be left for the seventh position.
For 654, these begining numbers are divisible by three: 183, 189, 381, 387, 783, 789, 981, 987. Must discard 387 and 783 because the seventh digit must be 3 or 7 in this case.
For 258, there are only two possible begining numbers: 147, 741.

Now for the remaining eight cases, fill in the one of the two possible seventh digits and test the numbers formed by the first seven digits for divisibility by 7.
1836547, 1896543, and 1896547 are not divisible by 7.
3816547 works, so the answer might be 381654729.
7896543, 9816543, 9816547, and 9876543 are not divisible by 7.
1472589 and 7412589 are not divisible by 7.

Divisibility by 9 is guaranteed since the digits 1 to 9 have a sum of 45 which is divisible by 9.
Therefore, the number is 381 654 729.

Chapter 8 Section 2 **Question 21 Page 453**





Let the dimensions of the flag be x and y as shown.
From the 60º and 30º right-angled triangle:

$$\tan 60^\circ = \frac{10}{8-y}$$

$$8 - y = \frac{10}{\tan 60^\circ}$$

$$y \doteq 2.23$$

From the 76º and 14º right-angled triangle:

$$\tan 14^\circ = \frac{y}{10-x}$$

$$10 - x = \frac{2.23}{\tan 14^\circ}$$

$$x \doteq 1.06$$

$$2.23 \times 1.06 \doteq 2.36$$

The area of the flag is approximately 2.36 m^2.

Chapter 8 Section 3 **Properties of Planes**

Chapter 8 Section 3 **Question 1 Page 459**

Substitute the coordinates in the equation $x + 2y - 3z - 5 = 0$.

a) L.S. $= 5 + 2(-3) - 3(-2) - 5$ R.S. $= 0$
$= 0$
L.S. = R.S.

Therefore, M lies on the plane.

b) L.S. $= 3 + 2(2) - 3(-1) - 5$ R.S. $= 0$
$= 5$
L.S. $\neq$ R.S.

Therefore, N does not lie on the plane.

c) L.S. $= -7 + 2(0) - 3(-4) - 5$ R.S. $= 0$
$= 0$
L.S. = R.S.

Therefore, P lies on the plane.

d) L.S. $= 6 + 2(1) - 3(1) - 5$ R.S. $= 0$
$= 0$
L.S. = R.S.

Therefore, Q lies on the plane.

e) L.S. $= 0 + 2(0) - 3(5) - 5$ R.S. $= 0$
$= -20$
L.S. $\neq$ R.S.

Therefore, R does not lie on the plane.

f) L.S. $= 1 + 2(2) - 3(-3) - 5$ R.S. $= 0$
$= 9$
L.S. $\neq$ R.S.

Therefore, S does not lie on the plane.

Chapter 8 Section 3 **Question 2 Page 459**

Answers may vary. For example:

a) $\vec{n}_1 = [1, 2, 2]; \vec{n}_2 = [-2, -4, -4]$

b) $\vec{n}_1 = [6, -1, 4]; \vec{n}_2 = [12, -2, 8]$

c) $\vec{n}_1 = [5, 0, 2]; \vec{n}_2 = [15, 0, 6]$

d) $\vec{n}_1 = [0, 5, 0]; \vec{n}_2 = [0, 1, 0]$

e) $\vec{n}_1 = [3, 4, 0]; \vec{n}_2 = [-6, -8, 0]$

f) $\vec{n}_1 = [-1, -3, 1]; \vec{n}_2 = [1, 3, -1]$

Chapter 8 Section 3 **Question 3 Page 459**

Answers may vary. Choose any vector that has a dot product of zero with $\vec{n}$. For example:

a) [–2, 0, 1]

b) [1, 2, –1]

c) [–2, 0, 5]

d) [2, 0, –1]

e) [4, –3, 10]

f) [0, 1, 3]

Chapter 8 Section 3 **Question 4 Page 459**

a) The scalar equation has the form $x - y + z + D = 0$.

P lies on the plane. Substitute its coordinates and solve for *D*.

$2 - (-1) + 8 + D = 0$

$D = -11$

The scalar equation is $x - y + z - 11 = 0$.

b) The scalar equation has the form $3x + 7y + z + D = 0$.

P lies on the plane. Substitute its coordinates and solve for D.

$3(3) + 7(-6) + 4 + D = 0$

$D = 29$

The scalar equation is $3x + 7y + z + 29 = 0$.

c) The scalar equation has the form $2x - 5z + D = 0$.

P lies on the plane. Substitute its coordinates and solve for D.

$2(1) - 5(-3) + D = 0$

$D = -17$

The scalar equation is $2x - 5z - 17 = 0$.

d) The scalar equation has the form $-9x + D = 0$.

P lies on the plane. Substitute its coordinates and solve for D.

$-9(-2) + D = 0$

$D = -18$

The scalar equation is $-9x - 18 = 0$ or $x + 2 = 0$.

e) The scalar equation has the form $4x - 3y + 2z + D = 0$.

P lies on the plane. Substitute its coordinates and solve for D.

$4(6) - 3(3) + 2(-4) + D = 0$

$D = -7$

The scalar equation is $4x - 3y + 2z - 7 = 0$.

f) The scalar equation has the form $4x - 3y + 4z + D = 0$.

P lies on the plane. Substitute its coordinates and solve for D.

$4(-2) - 3(5) + 4(3) + D = 0$

$D = 11$

The scalar equation is $4x - 3y + 4z + 11 = 0$.

Chapter 8 Section 3 **Question 5 Page 459**

Answers for part b) and part c) may vary.

a) A possible normal vector is $\vec{n} = [-1, 4, 2]$.

b) Let $x = 0$ and $y = 0$.

$$-(0) + 4(0) + 2z + 6 = 0$$
$$z = -3$$

S(0, 0, –3) is one point.

Let $y = 0$ and $z = 0$.

$$-x + 4(0) + 2(0) + 6 = 0$$
$$x = 6$$

T(6, 0, 0) is another point.

c) $\overrightarrow{ST} = \overrightarrow{OT} - \overrightarrow{OS}$

$$= [6, 0, 0] - [0, 0, -3]$$
$$= [6, 0, 3]$$

d) $\overrightarrow{ST} \cdot \vec{n} = [6, 0, 3] \cdot [-1, 4, 2]$

$$= 6(-1) + 0(4) + 3(2)$$
$$= 0$$

Therefore, $\overrightarrow{ST} \perp \vec{n}$.

Chapter 8 Section 3 **Question 6 Page 459**

a) Find a normal vector to the plane.

$$\vec{n} = [1, 2, -1] \times [1, -2, 3]$$
$$= [2(3) - (-2)(-1), (-1)(1) - 3(1), 1(-2) - 1(2)]$$
$$= [4, -4, -4]$$

Use [1, –1, –1]. The scalar equation has the form $x - y - z + D = 0$.

Use the point (3, 7, –5) to determine D.

$$3 - 7 - (-5) + D = 0$$
$$D = -1$$

A scalar equation of the plane is $x - y - z - 1 = 0$.

b) Find a normal vector to the plane.

$$\begin{aligned}\vec{n} &= [3, -2, 4] \times [5, -2, 6] \\ &= [-2(6) - (-2)(4), 4(5) - 6(3), 3(-2) - 5(-2)] \\ &= [-4, 2, 4]\end{aligned}$$

Use [2, –1, –2]. The scalar equation has the form $2x - y - 2z + D = 0$.

Use the point (5, –2, 3) to determine D.

$$\begin{aligned}2(5) - (-2) - 2(3) + D &= 0 \\ D &= -6\end{aligned}$$

A scalar equation of the plane is $2x - y - 2z - 6 = 0$.

c) Find a normal vector to the plane.

$$\begin{aligned}\vec{n} &= [2, -1, -1] \times [1, 3, 3] \\ &= [-1(3) - 3(-1), (-1)(1) - 3(2), 2(3)(3) - 1(-1)] \\ &= [0, -7, 7]\end{aligned}$$

Use [0, 1, –1]. The scalar equation has the form $y - z + D = 0$.

Use the point (6, 8, 2) to determine D.

$$\begin{aligned}8 - 2 + D &= 0 \\ D &= -6\end{aligned}$$

A scalar equation of the plane is $y - z - 6 = 0$.

d) Find a normal vector to the plane.

$$\begin{aligned}\vec{n} &= [6, 5, 2] \times [3, -3, 1] \\ &= [5(1) - (-3)(2), 2(3) - 1(6), 6(-3) - 3(5)] \\ &= [11, 0, -33]\end{aligned}$$

Use [1, 0, –3]. The scalar equation has the form $x - 3z + D = 0$.

Use the point (9, 1, –8) to determine D.

$$\begin{aligned}9 - 3(-8) + D &= 0 \\ D &= -33\end{aligned}$$

A scalar equation of the plane is $x - 3z - 33 = 0$.

e) Find a normal vector to the plane.

$$\begin{aligned}\vec{n} &= [0, 1, 0] \times [0, 0, -1] \\ &= [1(-1) - 0(0), 0(0) - (-1)(0), 0(0) - 0(1)] \\ &= [-1, 0, 0]\end{aligned}$$

Use [1, 0, 0]. The scalar equation has the form $x + D = 0$.

Use the point (0, 0, 1) to determine D.

$$\begin{aligned}0 + D &= 0 \\ D &= 0\end{aligned}$$

A scalar equation of the plane is $x = 0$.

f) Find a normal vector to the plane.

$$\begin{aligned}\vec{n} &= [2, 0, 3] \times [3, 0, 2] \\ &= [0(2) - 0(3), 3(3) - 2(2), 2(0) - 3(0)] \\ &= [0, 5, 0]\end{aligned}$$

Use [0, 1, 0]. The scalar equation has the form $y + D = 0$.

Use the point (3, 2, 1) to determine D.

$$\begin{aligned}2 + D &= 0 \\ D &= -2\end{aligned}$$

A scalar equation of the plane is $y - 2 = 0$.

Chapter 8 Section 3 **Question 7 Page 459**

a) Find a normal vector to the plane.

$$\begin{aligned}\vec{n} &= [-2, 3, -1] \times [2, 1, -2] \\ &= [3(-2) - 1(-1), -1(2) - (-2)(-2), -2(1) - 2(3)] \\ &= [-5, -6, -8]\end{aligned}$$

Use [5, 6, 8]. The scalar equation has the form $5x + 6y + 8z + D = 0$.

Use the point (3, 1, 5) to determine D.

$$\begin{aligned}5(3) + 6(1) + 8(5) + D &= 0 \\ D &= -61\end{aligned}$$

A scalar equation of the plane is $5x + 6y + 8z - 61 = 0$.

b) Find a normal vector to the plane.

$$\begin{aligned}\vec{n} &= [0, -1, 1] \times [5, -2, 0] \\ &= [-1(0) - (-2)(1), 1(5) - 0(0), 0(-2) - 5(-1)] \\ &= [2, 5, 5]\end{aligned}$$

Use [2, 5, 5]. The scalar equation has the form $2x + 5y + 5z + D = 0$.

Use the point (–1, 3, –2) to determine D.

$$\begin{aligned}2(-1) + 5(3) + 5(2) + D &= 0 \\ D &= -3\end{aligned}$$

A scalar equation of the plane is $2x + 5y + 5z - 3 = 0$.

c) Find a normal vector to the plane.

$$\begin{aligned}\vec{n} &= [-1, -1, 3] \times [2, 4, 1] \\ &= [-1(1) - 4(3), 3(2) - 1(-1), -1(4) - 2(-1)] \\ &= [-13, 7, -2]\end{aligned}$$

Use [13, –7, 2]. The scalar equation has the form $13x - 7y + 2z + D = 0$.

Use the point (–1, 1, 2) to determine D.

$$\begin{aligned}13(-1) - 7(1) + 2(2) + D &= 0 \\ D &= 16\end{aligned}$$

A scalar equation of the plane is $13x - 7y + 2z + 16 = 0$.

d) Find a normal vector to the plane.

$$\begin{aligned}\vec{n} &= [1, 2, 4] \times [1, 0, 2] \\ &= [2(2) - 0(4), 4(1) - 2(1), 1(0) - 1(2)] \\ &= [4, 2, -2]\end{aligned}$$

Use [2, 1, –1]. The scalar equation has the form $2x + y - z + D = 0$.

Use the point (2, 3, 5) to determine D.

$$\begin{aligned}2(2) + 3 - 5 + D &= 0 \\ D &= -2\end{aligned}$$

A scalar equation of the plane is $2x + y - z - 2 = 0$.

e) Find a normal vector to the plane.

$$\vec{n} = [5, 3, -1] \times [2, 0, 2]$$
$$= [3(2) - 0(-1), -1(2) - 2(5), 5(0) - 2(3)]$$
$$= [6, -12, -6]$$

Use [1, –2, –1]. The scalar equation has the form $x - 2y - z + D = 0$.

Use the point (–3, 2, –1) to determine D.

$$-3 - 2(2) - (-1) + D = 0$$
$$D = 6$$

A scalar equation of the plane is $x - 2y - z + 6 = 0$.

f) Find a normal vector to the plane.

$$\vec{n} = [3, -1, 2] \times [1, -5, 3]$$
$$= [-1(3) - (-5)(2), 2(1) - 3(3), 3(-5) - 1(-1)]$$
$$= [7, -7, -14]$$

Use [1, –1, –2]. The scalar equation has the form $x - y - 2z + D = 0$.

Use the point (0, 2, 1) to determine D.

$$-2 - 2(1) + D = 0$$
$$D = 4$$

A scalar equation of the plane is $x - y - 2z + 4 = 0$.

Chapter 8 Section 3 **Question 8 Page 460**

Answers may vary. For example:

a) The direction vector $[3, 5, -3]$ for the line can be the normal vector for the plane.

The scalar equation has the form $3x + 5y - 3z + D = 0$

Use the point (4, –2, 7) to determine D.

$$3(4) + 5(-2) - 3(7) + D = 0$$
$$D = 19$$

A scalar equation of the plane is $3x + 5y - 3z + 19 = 0$.

For direction vectors for the plane, choose any two non-collinear vectors whose dot product with $[3, 5, -3]$ is zero.

Two such vectors are $[1, 0, 1]$ and $[5, -3, 0]$.

A vector equation for the plane is $[x, y, z] = [4, -2, 7] + s[1, 0, 1] + t[5, -3, 0]$, $s, t \in \mathbb{R}$.

b) Since the plane is parallel to the *yz*-plane, its equation is of the form $x + D = 0$.
Since the point (–1, –2, 5) is on the plane, a scalar equation of the plane is $x + 1 = 0$.

Since the plane is parallel to the *yz*-plane, the vectors $\vec{j}$ and $\vec{k}$ are direction vectors.
A vector equation for the plane is $[x, y, z] = [-1, -2, 5] + s[0, 0, 1] + t[0, 1, 0]$, $s, t \in \mathbb{R}$.

c) Parallel planes have the same normals. Here $\vec{n} = [3, -9, 1]$.
The scalar equation of the plane has the form $3x - 9y + z + D = 0$

Use the point (–3, 7, 1) to determine *D*.
$3(-3) - 9(7) + 1 + D = 0$
$D = 71$

A scalar equation of the plane is $3x - 9y + z + 71 = 0$.

For direction vectors for the plane, choose any two non-collinear vectors whose dot product with $\vec{n} = [3, -9, 1]$ is zero.
Two such vectors are $[3, 1, 0]$ and $[1, 0, -3]$.
A vector equation for the plane is $[x, y, z] = [-3, 7, 1] + s[3, 1, 0] + t[1, 0, -3]$, $s, t \in \mathbb{R}$.

d) Since the lines are in the plane, their points and direction vectors apply to the plane.
However, the direction vectors are parallel. Find the vector between the two position vectors for the lines.
$\vec{m} = [-2, 3, 12] - [1, -4, 4]$
$= [-3, 7, 8]$

A vector equation for the plane is $[x, y, z] = [1, -4, 4] + s[-2, 1, 5] + t[-3, 7, 8]$, $s, t \in \mathbb{R}$.

Need a normal vector for a scalar equation. Find the cross product of the vectors in the plane.
$[-2, 1, 5] \times [-3, 7, 8] = [1(8) - 7(5), 5(-3) - 8(-2), (-2)(7) - (-3)(1)]$
$= [-27, 1, -11]$

Use $[27, -1, 11]$. The scalar equation of the plane has the form $27x - y + 11z + D = 0$

Use the point (1, –4, 4) to determine *D*.
$27(1) - (-4) + 11(4) + D = 0$
$D = -75$

A scalar equation of the plane is $27x - y + 11z - 75 = 0$.

Chapter 8 Section 3 **Question 9 Page 460**

Answer may vary. For example:

In general, it was easier to determine the scalar equation of each line. However, in part d) it was easier to write the vector equation.

Chapter 8 Section 3 **Question 10 Page 460**

a) $\vec{n}_1 = \left[4, -5, 1\right]; \vec{n}_2 = \left[2, -9, 1\right]$

$$\begin{aligned}\vec{n}_1 \cdot \vec{n}_2 &= 4(2) + (-5)(-9) + 1(1)\\ &= 54\\ &\neq 0\end{aligned}$$

$$\begin{aligned}\vec{n}_1 \times \vec{n}_2 &= \left[-5(1) - (-9)(1), 1(2) - 1(4), 4(-9) - 2(-5)\right]\\ &\neq \vec{0}\end{aligned}$$

Since neither the dot product not the cross product is zero (vector), the planes are neither parallel nor perpendicular.

b) $\vec{n}_1 = \left[5, -6, 2\right]; \vec{n}_2 = \left[2, -5, -20\right]$

$$\begin{aligned}\vec{n}_1 \cdot \vec{n}_2 &= 5(2) + (-6)(-5) + 2(-20)\\ &= 0\end{aligned}$$

Since the dot product is zero, the normals are perpendicular and, therefore, the planes are perpendicular.

c) $\vec{n}_1 = \left[4, -2, 1\right]; \vec{n}_2 = \left[-2, 1, -3\right]$

$$\begin{aligned}\vec{n}_1 \cdot \vec{n}_2 &= 4(-2) + (-2)(1) + 1(-3)\\ &= -13\\ &\neq 0\end{aligned}$$

$$\begin{aligned}\vec{n}_1 \times \vec{n}_2 &= \left[-2(-3) - 1(1), 1(-2) - (-3)(4), 4(1) - (-2)(-2)\right]\\ &\neq \vec{0}\end{aligned}$$

Since neither the dot product nor the cross product is zero (vector), the planes are neither parallel nor perpendicular.

Chapter 8 Section 3 **Question 11 Page 460**

Since the line and plane are perpendicular, the direction vector of the line will be the same as the normal for the plane.

The vector equation is $\left[x, y, z\right] = \left[3, 9, -2\right] + t\left[3, -7, 3\right],\ s, t \in \mathbb{R}$.

Chapter 8 Section 3 **Question 12 Page 460**

a) Find a point on the plane. Let $x = 5$ and $z = 2$. P(5, 3, 2) is a point on the plane.
Choose two direction vectors by choosing 2 non-collinear vectors whose dot product with $\vec{n} = [0, 1, 0]$ is zero.
Two possible vectors are $[4, 0, 1]$ and $[1, 0, -5]$.
The vector equation of the plane is $[x, y, z] = [5, 3, 2] + s[4, 0, 1] + t[1, 0, -5]$, $s, t \in \mathbb{R}$.

b) Find a point on the plane. Let $x = 0$ and $z = 0$. P(0, –8, 0) is a point on the plane.
Choose two direction vectors by choosing 2 non-collinear vectors whose dot product with $\vec{n} = [1, 1, 0]$ is zero.
Two possible vectors are $[1, -1, 1]$ and $[1, -1, 2]$.
The vector equation of the plane is $[x, y, z] = [0, 8, 0] + s[1, -1, 1] + t[1, -1, 2]$, $s, t \in \mathbb{R}$.

c) Find a point on the plane. Let $x = 0$ and $y = 0$. P(0, 0, 10) is a point on the plane.
Choose two direction vectors by choosing 2 non-collinear vectors whose dot product with $\vec{n} = [1, 1, 1]$ is zero.
Two possible vectors are $[2, 1, -3]$ and $[2, -1, -1]$.
The vector equation of the plane is $[x, y, z] = [0, 0, 10] + s[2, 1, -3] + t[2, -1, -1]$, $s, t \in \mathbb{R}$.

d) Find a point on the plane. Let $y = 0$ and $z = 0$. P(1, 0, 0) is a point on the plane.
Choose two direction vectors by choosing 2 non-collinear vectors whose dot product with $\vec{n} = [4, -1, 8]$ is zero.
Two possible vectors are $[1, -4, -1]$ and $[2, 0, -1]$.
The vector equation of the plane is $[x, y, z] = [1, 0, 0] + s[1, -4, -1] + t[2, 0, -1]$, $s, t \in \mathbb{R}$.

e) Find a point on the plane. Let $y = 0$ and $z = 0$. P(4, 0, 0) is a point on the plane.
Choose two direction vectors by choosing 2 non-collinear vectors whose dot product with $\vec{n} = [3, 2, -1]$ is zero.
Two possible vectors are $[1, -2, -1]$ and $[4, -1, 10]$.
The vector equation of the plane is $[x, y, z] = [4, 0, 0] + s[1, -2, -1] + t[4, -1, 10]$, $s, t \in \mathbb{R}$.

f) Find a point on the plane. Let $y = 0$ and $z = 0$. P(15, 0, 0) is a point on the plane.
Choose two direction vectors by choosing 2 non-collinear vectors whose dot product with $\vec{n} = [2, -5, -3]$ is zero.
Two possible vectors are $[5, 2, 0]$ and $[3, 0, 2]$.
The vector equation of the plane is $[x, y, z] = [15, 0, 0] + s[5, 2, 0] + t[3, 0, 2]$, $s, t \in \mathbb{R}$.

Chapter 8 Section 3 **Question 13 Page 460**

Use the 3D Grapher or other software to verify your answers.

Chapter 8 Section 3 **Question 14 Page 460**

a) The bottom plane of the ramp has scalar equation $z = 0$.
The right side plane has equation $y = 0$.
The left side plane has equation $y = 2$.
The back plane of the ramp has equation $x = 5$.
The slanted top plane of the ramp has equation $3x - 5z = 0$.

b) The line passes through the point (5, 0, 3) and has the direction vector parallel to the *y*-axis.
The vector equation of the line is $[x, y, z] = [5, 0, 3] + t[0, 1, 0],\ t \in \mathbb{R}$.
To be more precise, the equation for the line segment along the top of the ramp is:
$[x, y, z] = [5, 0, 3] + t[0, 1, 0],\ 0 \le t \le 2$.

Chapter 8 Section 3 **Question 15 Page 460**

Answers may vary. For example:

a) All of the resulting planes are parallel since they all have the normal $\vec{n} = [3, 6, -4]$. When $k = 0$, the plane passes through the origin. As *k* increases in absolute value, the three intercepts increase in their distance from the origin.

b) The constant term does not affect the slant of the plane; it only affects the position of the intercepts of the graph. Larger (absolute) values of *k* lead to planes that are further from the origin.

c) They all pass through the origin.

Chapter 8 Section 3 **Question 16 Page 460**

Solutions for Achievement Checks are shown in the Teacher Resource.

Chapter 8 Section 3 **Question 17 Page 461**

a) $\cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{\left|\vec{n}_1\right|\left|\vec{n}_2\right|}$

$\cos\theta = \dfrac{[3, 2, 5]\cdot[4, -3, -2]}{\left|[3, 2, 5]\right|\left|[4, -3, -2]\right|}$

$\cos\theta = \dfrac{3(4) + 2(-3) + 5(-2)}{\sqrt{3^2 + 2^2 + 5^2}\sqrt{4^2 + (-3)^2 + (-2)^2}}$

$\cos\theta = \dfrac{-4}{33.1964}$

$\theta = \cos^{-1}\left(\dfrac{-4}{33.1964}\right)$

$\theta \doteq 96.9^\circ$

b) $\cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{\left|\vec{n}_1\right|\left|\vec{n}_2\right|}$

$\cos\theta = \dfrac{[2, 0, 3]\cdot[-1, 6, 3]}{\left|[2, 0, 3]\right|\left|[-1, 6, 3]\right|}$

$\cos\theta = \dfrac{2(-1) + 0(6) + 3(3)}{\sqrt{2^2 + 0^2 + 3^2}\sqrt{(-1)^2 + 6^2 + 3^2}}$

$\cos\theta = \dfrac{7}{24.4540}$

$\theta = \cos^{-1}\left(\dfrac{7}{24.4540}\right)$

$\theta \doteq 73.4^\circ$

c) $\cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{\left|\vec{n}_1\right|\left|\vec{n}_2\right|}$

$\cos\theta = \dfrac{[4, 1, -2]\cdot[3, -2, 5]}{\left|[4, 1, -2]\right|\left|[3, -2, 5]\right|}$

$\cos\theta = \dfrac{4(3) + 1(-2) + (-2)(5)}{\sqrt{4^2 + 1^2 + (-2)^2}\sqrt{3^2 + (-2)^2 + 5^2}}$

$\cos\theta = \dfrac{0}{28.2489}$

$\theta = \cos^{-1}(0)$

$\theta \doteq 90^\circ$

Chapter 8 Section 3 **Question 18 Page 461**

$\vec{n}_1 = [2, 6, -2]$ and $\vec{n}_2 = [5, 15, k]$

a) The planes are parallel if the normals vectors are parallel. This occurs only if one vector is a scalar multiple of the other. $[2, 6, -2] = p[5, 15, k]$

$2 = 5p$ $\quad$ $6 = 15p$ $\quad$ $-2 = pk$

$p = 0.4$ $\quad$ $p = 0.4$ $\quad$ $-2 = 0.4k$

$k = -5$

The planes are parallel if $k = -5$.

b) The planes are perpendicular if their normals are perpendicular. This occurs if the dot product of the 2 normal vectors is zero.

$[2, 6, -2] \cdot [5, 15, k] = 0$

$2(5) + 6(15) + (-2)k = 0$

$-2k = -100$

$k = 50$

The planes are perpendicular if $k = 50$.

Chapter 8 Section 3 **Question 19 Page 461**

Answers may vary. For example:

One plane that has an x-intercept of 4 and is parallel to the *z*-axis would be one that is also parallel to the *y*-axis (ie parallel to the *yz*-plane). A second plane that has the given *x*-intercept would be one that contains the *z*-axis.

There are an infinite number of planes that are parallel to the *z*-axis and have an x-intercept of 4. This can be seen by creating a plane that passes through (4, 0, 0) and oriented such that it is parallel to the z-axis, and rotating this plane around (4, 0, 0), keeping it parallel to the z-axis.

Algebraically, the family of such planes is determined by the equation $\frac{x}{4} + \frac{y}{k} = 1$. Each plane in the family has
x-intercept 4, *y*-intercept *k*, and no *z*-intercept (hence it is parallel to the *z*-axis). Since $k \in \mathbb{R}$, there are an infinite number of planes that satisfy the conditions. (See also question 19 in section 8.2, student textbook page 453.)

Chapter 8 Section 3 **Question 20 Page 461**

Answers may vary. For example:

a) Choose convenient points such as the three intercepts, since they cannot be collinear.
Let $y = 0$ and $z = 0$. One point on the plane is A(6, 0, 0),
Let $x = 0$ and $y = 0$. A second point is B(0, 0, 3) .
Let $x = 0$ and $z = 0$. A third point is C(0, –4.5, 0).

b) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
$= [0, 0, 3] - [6, 0, 0]$
$= [-6, 0, 3]$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$
$= [0, -4.5, 0] - [6, 0, 0]$
$= [-6, -4.5, 0]$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$
$= [0, -4.5, 0] - [0, 0, 3]$
$= [0, -4.5, -3]$

c) **i)** $\overrightarrow{AB} + \overrightarrow{AC} = [-6, 0, 3] + [-6, -4.5, 0]$
$= [-12, -4.5, 3]$

ii) $\overrightarrow{AC} + \overrightarrow{BC} = [-6, -4.5, 0] + [0, -4.5, -3]$
$= [-6, -9, -3]$

iii) $2\overrightarrow{AC} - 4\overrightarrow{AB} = 2[-6, -4.5, 0] - 4[-6, 0, 3]$
$= [-12, -9, 0] - [-24, 0, 12]$
$= [12, -9, -12]$

iv) $3\overrightarrow{AB} + 5\overrightarrow{BC} = 3[-6, 0, 3] + 5[0, -4.5, -3]$
$= [-18, 0, 9] + [0, -22.5, -15]$
$= [-18, -22.5, -6]$

d) If the vectors are parallel to the plane, then their dot product with $\vec{n} = [3, -4, 6]$ will be zero.

i) $[-12, -4.5, 3] \cdot [3, -4, 6] = -12(3) + (-4.5)(-4) + 3(6)$
$= 0$

ii) $[-6, -9, -3] \cdot [3, -4, 6] = -6(3) + (-9)(-4) + (-3)(6)$
$= 0$

iii) $[12, -9, -12] \cdot [3, -4, 6] = 12(3) + (-9)(-4) + (-12)(6)$
$= 0$

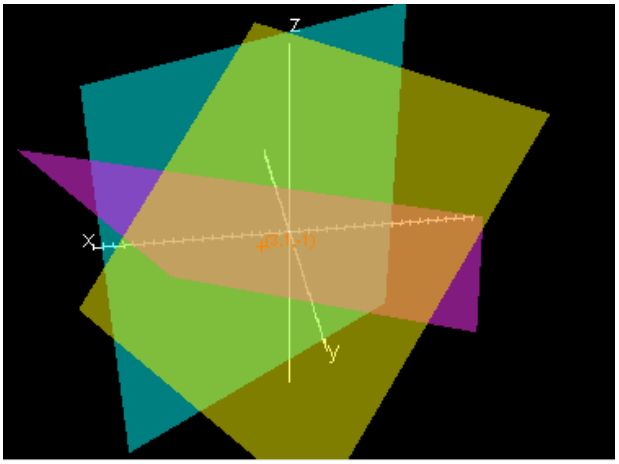
iv) $[-18, -22.5, -6] \cdot [3, -4, 6] = -18(3) + (-22.5)(-4) + (-6)(6)$
$= 0$

Therefore, all vectors in part c) are parallel to the plane.

Chapter 8 Section 3 **Question 21 Page 461**

The required plane is perpendicular to the given planes. Therefore the normal vectors $\vec{n}_1 = [2, -3, 0]$ and $\vec{n}_2 = [1, 2, -2]$ are direction vectors for the required plane.

The vector equation of the required plane is $[x, y, z] = [3, 1, -1] + s[2, -3, 0] + t[1, 2, -2]$, $s, t \in \mathbb{R}$.



The yellow plane is the required plane in the image from 3D Grapher shown above.

Chapter 8 Section 3 **Question 22 Page 461**

The required plane is perpendicular to the given planes. Therefore the normal vectors $\vec{n}_1 = [3, -2, 1]$ and $\vec{n}_2 = [4, 6, -5]$ are direction vectors for the required plane.

The normal to the required plane is necessary for the scalar equation.

$$\begin{aligned}\vec{n} &= \vec{n}_1 \times \vec{n}_2 \\ &= [3, -2, 1] \times [4, 6, -5] \\ &= [-2(-5) - 6(1), 1(4) - (-5)(3), 3(6) - 4(-2)] \\ &= [4, 19, 26]\end{aligned}$$

The scalar equation is of the form $4x + 19y + 26z + D = 0$.

A is a point on the plane. Determine the value of D.

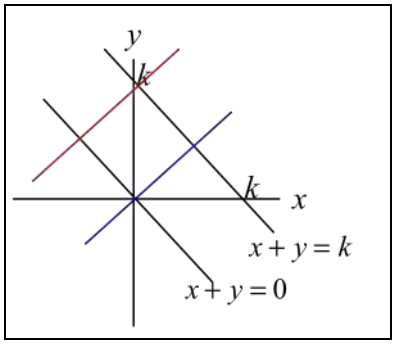
$$\begin{aligned}4(2) + 19(1) + 26(-5) + D &= 0 \\ D &= 103\end{aligned}$$

A scalar equation for the plane is $4x + 19y + 26z + 103 = 0$.

Chapter 8 Section 3 **Question 23 Page 461**

a) The first two planes are parallel to the z-axis and perpendicular to the xy-plane. Call this direction vertical.

The diagram below shows a top view of the situation.



The other two vertical planes will be of the form $x - y = k$. In particular choose, $x - y = 0$ (blue) and $x - y = k$ (red)

The remaining plane is the top of the box. It will be of the form $z = q$ where q is the length of one side of the box.

From the diagram:

$$q^2 + q^2 = k^2$$
$$2q^2 = k^2$$
$$q^2 = \frac{k^2}{2}$$
$$q = \frac{k}{\sqrt{2}}$$

The required equations can be $x - y = 0$, $x - y = k$, and $z = \frac{k}{\sqrt{2}}$. Note that these planes are not uniquely defined.

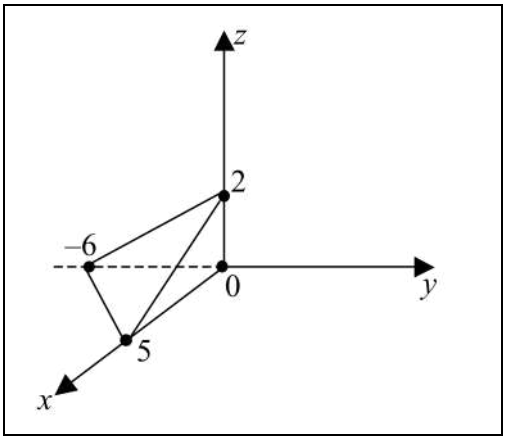
b) The sides of the cube are $\frac{k}{\sqrt{2}}$ units long.

Chapter 8 Section 3 **Question 24 Page 461**

Answers may vary. For example:

The simplest answer is to choose three planes parallel to the coordinate axes: $x = 3$, $y = -1$, and $z = -2$. Clearly these planes are not parallel (in fact, they are mutually perpendicular). It is also clear that the coordinates of A(3, –1, –2) satisfy each of the equations.

Chapter 8 Section 3 **Question 25 Page 461**



From the diagram, three of the faces are coordinate planes: $x = 0$, $y = 0$, and $z = 0$.
The fourth plane has x-, y-, and z-intercepts of 5, –6, and 2 respectively and so has equation $\frac{x}{5} + \frac{y}{-6} + \frac{z}{2} = 1$, which can also be written as $6x - 5y + 15z - 30 = 0$.

Chapter 8 Section 3 **Question 26 Page 461**

Answers may vary. For example:

a) If the plane is perpendicular to the x-axis, its equation is of the form $x = k$. In this case, $B = C = 0$ and $A, D \in \mathbb{R}$.

b) The equation is $\frac{x}{3} + \frac{y}{5} + \frac{z}{6} = 1$ or $10x + 6y + 5z - 30 = 0$. In this case, the previous equation can be multiplied by any constant (on both sides of the equation).
Therefore $A = 10k$, $B = 6k$, $C = 5k$, and $D = -30k$ for any $k \in \mathbb{R}$.

c) If the plane is parallel to the z-axis, it has no z-intercept and its equation is of the form $Ax + BY + D = 0$. In this case, $C = 0$ and $A, B, D \in \mathbb{R}$.

d) Perpendicular planes have perpendicular normals. The normals for these two planes are $[A, B, C]$ and $[1, 4, -7]$.

$$[A, B, C] \cdot [1, 4, -7] = 0$$
$$A + 4B - 7C = 0$$

Therefore A, B, and C must satisfy the equation $A + 4B - 7C = 0$ and D can be any real number.

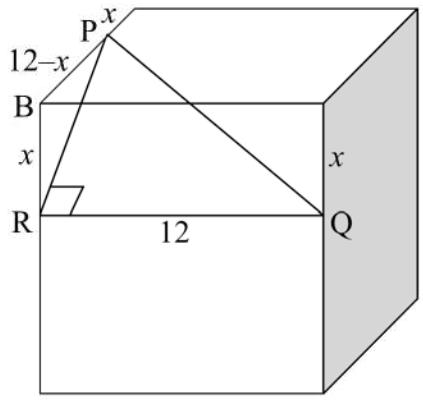
Chapter 8 Section 3 **Question 27 Page 461**

Answers may vary. For example:

Solution 1
Both equations have no y-term and so they have no y-intercept. They are parallel to the y-axis. Therefore their normals are perpendicular to the y-axis. Any plane having these two vectors as direction vectors will be perpendicular to the
y-axis.

Solution 2
The cross product of the normals to the given planes is [0, 31, 0] which is parallel to [0, 1, 0] which would be the normal to the family of planes that would be perpendicular to the y-axis.

Chapter 8 Section 3 **Question 28 Page 461**



Let x be the distance the bugs have travelled from A and C, arriving at P and Q respectively.
Triangles PBR and PQR are right angled.
From ΔPBR:

$$PR^2 = x^2 + (12 - x)^2$$

From ΔPQR:

$$\begin{aligned} PQ^2 &= x^2 + (12 - x)^2 + 12^2 \\ &= 2x^2 - 24x + 288 \end{aligned}$$

To find the minimum distance, differentiate and set the derivative equal to zero.

$$\begin{aligned} 4x - 24 &= 0 \\ x &= 6 \end{aligned}$$

The minimum distance occurs when $x = 6$. At this time:

$$\begin{aligned} PQ &= \sqrt{2(6)^2 - 24(6) + 288} \\ &= \sqrt{216} \\ &= 6\sqrt{6} \\ &\doteq 14.7 \end{aligned}$$

The minimum distance between the bugs is approximately 14.7 cm.

Chapter 8 Section 3 **Question 29 Page 461**

$\log_9 x = \log_3 y + 1$

Convert all logarithms to the same base. Use base 9, so $1 = \log_9 9$.

$\log_3 y = p$

$3^p = y$

$9^{\frac{1}{2}p} = y$

$\log_9 y = \frac{1}{2}p$

$p = 2\log_9 y$

$\log_3 y = \log_9 y^2$

The equation then becomes:

$\log_9 x = \log_9 y^2 + \log_9 9$

$\log_9 x = \log_9 9y^2$

$x = 9y^2$

$y^2 = \frac{1}{9}x$

$y = \frac{\sqrt{x}}{3}$

Note that logarithm functions have domain of the positive real numbers only.

Chapter 8 Section 4 **Intersections of Lines in Two-Space and Three-Space**

Chapter 8 Section 4 **Question 1 Page 471**

a) One, as the two lines have different direction vectors (i.e. different slopes).

b) One, as the two lines have different direction vectors (i.e. different slopes).

c) Zero solutions, as the two lines are parallel and distinct.

d) One, as the two lines have different direction vectors (i.e. different slopes).

Chapter 8 Section 4 **Question 2 Page 471**

a) Use elimination.

$x + y = 8$ ①

$x - y = 13$ ②

$2y = -5$ ①–②

$y = -\frac{5}{2}$

Substitute $y = -\frac{5}{2}$ into equation ①.

$x - \frac{5}{2} = 8$

$x = \frac{21}{2}$

The point of intersection is $\left(\frac{21}{2}, -\frac{5}{2}\right)$.

b) Use elimination.

$3.5x - 2.1y = 14$ ①

$1.5x - 0.3y = 8$ ②

$-7x = -42$ ①–7②

$x = 6$

Substitute $x = 6$ into equation ①.

$3.5(6) - 2.1y = 14$

$2.1y = -14 + 21$

$y = \frac{7}{2.1}$

$y = \frac{70}{21}$ or $\frac{10}{3}$

The point of intersection is $\left(6, \frac{10}{3}\right)$.

c) Use elimination.

$-12+8s=2+3t$ ①

$-7-5s=-1-2t$ ②

$16s-6t=28$ 2①

$-15s+6t=18$ 3②

$s=46$ 2①+3②

$-14+8(46)=3t$ Substitute in ①.

$t=118$

Substitute $s=46$ and $t=118$ in one of the original vector equations.
The point of intersection is (356, –237).

d) Use elimination.

$16+5s=-7-7t$ ①

$1+s=12+3t$ ②

$5s+7t=-23$ ①

$5s-15t=55$ 5②

$22t=-78$ ①–5②

$t=-\frac{39}{11}$

$s=11+3\left(-\frac{39}{11}\right)$ Substitute in ②.

$s=\frac{4}{11}$

Substitute $s=\frac{4}{11}$ and $t=-\frac{39}{11}$ in one of the original vector equations.

The point of intersection is $\left(\frac{196}{11}, \frac{15}{11}\right)$.

Chapter 8 Section 4 **Question 3 Page 471**

a) The lines are not parallel since the slopes are $-\frac{1}{4}$ and -2.

Therefore the lines will intersect and there will be one solution.

b) The lines are parallel since the slopes are $\frac{12}{21}$ and $\frac{8}{14}$, which are equal. Multiplying the second equation by $\frac{3}{2}$ gives the first equation except for the constant terms.

Therefore the lines are distinct and there will be no solutions.

c) The lines are parallel since the direction vectors are scalar multiple ($-2[3, -2] = [-6, 4]$). Letting $s = 1$ in the first equation produces the point (4, 4).
Therefore the lines are coincident and there are an infinite number of solutions.

d) The lines are not parallel since the slopes are $-\frac{3}{8}$ and $\frac{8}{3}$.
Therefore the lines will intersect and there will be one solution.

e) Multiplying the first equation by –5 and the second equation by 3 produces the same result.
Therefore the lines are coincident and there are an infinite number of solutions.

f) The lines are not parallel since the direction vectors [2, 7] and [1, 4] are not scalar multiples of each other.
Therefore the lines will intersect and there will be one solution.

Chapter 8 Section 4 **Question 4 Page 471**

a) Substitute the coordinates of $[5, 1, 3]$ into the second vector equation.

$[5, 1, 3] = [2, 3, 9] + t[2, 1, 7]$

Equate the *x*-coordinates.
$5 = 2 + 2t$
$t = \frac{3}{2}$

Equate the *y*-coordinates.
$1 = 3 + t$
$t = -2$

Equate the *z*-coordinates.
$3 = 9 + 7t$
$t = -\frac{6}{7}$

Since the *t* values are not equal, the point (5, 1, 3) does not lie on the second line and the lines are distinct.

b) Substitute the coordinates of $[4, 1, 0]$ into the second vector equation.

$[4, 1, 0] = [13, -14, 18] + t[-3, 5, -6]$

Equate the *x*-coordinates.
$4 = 13 - 3t$
$t = 3$

Equate the *y*-coordinates.
$1 = -14 + 5t$
$t = 3$

Equate the *z*-coordinates.
$0 = 18 - 6t$
$t = 3$

Since the *t* values are equal, the point (4, 1, 0) lies on the second line and the lines are coincident.

c) Substitute the coordinates of $[5, 1, 3]$ into the second vector equation.

$[5, 1, 3] = [3, -7, 5] + t[-2, -8, 2]$

Equate the *x*-coordinates.
$5 = 3 - 2t$
$t = -1$

Equate the *y*-coordinates.
$1 = -7 - 8t$
$t = -1$

Equate the *z*-coordinates.
$3 = 5 + 2t$
$t = -1$

Since the *t* values are equal, the point (5, 1, 3) lies on the second line and the lines are coincident.

d) Substitute the coordinates of $[4, -8, 0]$ into the second vector equation.

$[4, -8, 0] = [25, 55, -42] + t[-8, -24, 16]$

Equate the x-coordinates. Equate the y-coordinates. Equate the z-coordinates.

$4 = 25 - 8t$ $-8 = 55 - 24t$ $0 = -42 + 16t$

$t = \frac{21}{8}$ $t = \frac{63}{24} = \frac{21}{8}$ $t = \frac{42}{16} = \frac{21}{8}$

Since the t values are equal, the point (4, –8, 0) lies on the second line and the lines are coincident.

Chapter 8 Section 4 **Question 5 Page 471**

a) The direction vectors are not parallel since the direction vectors are not scalar multiples of each other.
Equate the expressions for like coordinates.

$6 - s = 11 + 4t$ $5 + s = 0 - t$ $-14 + 3s = -17 - 6t$

$s + 4t = -5$ ① $s + t = -5$ ② $3s + 6t = -3$ ③

Solve equations ① and ② for s and t.

$s + 4t = -5$ ①

$\underline{s + t = -5}$ ②

$3t = 0$ ①–②

$t = 0$

Substituting in ②.

$s + 0 = -5$

$s = -5$

Check that s and t satisfy ③.

L.S. $= 3(-5) + 6(0)$ R.S. $= -3$

$= -15$

L.S. ≠ R.S.

Therefore the lines do not intersect.

b) The direction vectors are not parallel since the direction vectors are not scalar multiples of each other.
Equate the expressions for like coordinates.

$3 - s = 1$ $-2 - 2s = 2t$ $2 = -1 - 3t$

$s = 2$ ① $s + t = -1$ ② $t = -1$ ③

Solve equations ① and ③ for s and t.

$s = 2$

$t = -1$

Check that s and t satisfy ②.

L.S. $= 2 + (-1)$ R.S. $= -1$

$= 1$

L.S. $\neq$ R.S.

Therefore the lines do not intersect.

c) The direction vectors are parallel since the direction vectors are scalar multiples of each other (actually equal).

Solve to find a point of intersection.

Equate the expressions for like coordinates.

$7 + 2s = -7 + 2t$ $s = -7 + t$ $-15 - 5s = 20 - 5t$

$s - t = -7$ ① $s - t = -7$ ② $s - t = -7$ ③

Since the equations ①, ②, and ③ are identical, multiple values for s and t will satisfy them.

Therefore the lines intersect at infinitely many points.

The lines must be coincident.

d) The direction vectors are not parallel since the direction vectors are not scalar multiples of each other.

Equate the expressions for like coordinates.

$8 + 2s = 1 + t$ $-1 - 3s = 20 - 5t$ $8 = 3t$

$2s - t = -7$ ① $3s - 5t = -21$ ② $t = \frac{8}{3}$ ③

Solve equations ① and ③ for s and t.

$2s - \left(\frac{8}{3}\right) = -7$ ①

$2s = -\frac{13}{3}$

$s = -\frac{13}{6}$

Check that s and t satisfy ②.

L.S. $= 3\left(-\frac{13}{6}\right) - 5\left(\frac{8}{3}\right)$ R.S. $= -21$

$= -\frac{119}{6}$

L.S. $\neq$ R.S.

Therefore the lines do not intersect.

Chapter 8 Section 4 **Question 6 Page 471**

Try to find an intersection point in each case.

a) Equate the expressions for like coordinates.

$4-2s=1+4t$ $\quad 7+s=3-t$ $\quad -1+2s=-1+2t$

$2s+4t=3$ ① $\quad s+t=-4$ ② $\quad s-t=0$ ③

Solve equations ① and ② for s and t.

$2s+4t=3$ ①

$\underline{2s+2t=-8}$ 2②

$2t=11$ ①–2②

$t=\frac{11}{2}$

Substituting in ②.

$s+\frac{11}{2}=-4$

$s=-\frac{19}{2}$

Check that s and t satisfy ③.

$\text{L.S.}=-\frac{19}{2}-\frac{11}{2}$ $\quad \text{R.S.}=0$

$=-15$

$\text{L.S.}\neq\text{R.S.}$

Therefore the lines do not intersect. They are skew lines.

b) Equate the expressions for like coordinates.

$6+6s=7+6t$ $\quad 2+18s=13-t$ $\quad 1-6s=1-2t$

$6s-6t=1$ ① $\quad 18s+t=11$ ② $\quad 3s-t=0$ ③

Solve equations ② and ③ for s and t.

$18s+t=11$ ②

$\underline{3s-t=0}$ ③

$21s=11$ ②+③

$s=\frac{11}{21}$

Substituting in ③.

$3\left(\frac{11}{21}\right)-t=0$

$t=\frac{11}{7}$

Check that s and t satisfy ①.

$\text{L.S.} = 6\left(\frac{11}{21}\right) - 6\left(\frac{11}{7}\right)$ $\qquad$ R.S. = 1

$= -\frac{132}{21}$

L.S. ≠ R.S.

Therefore the lines do not intersect. They are skew lines.

c) Equate the expressions for like coordinates.

$2 + 2s = 4 - 2t$ $\qquad$ $4 + s = 3 + t$ $\qquad$ $2 - 5s = 7 - 5t$

$s + t = 1$ ① $\qquad$ $s - t = -1$ ② $\qquad$ $s - t = -1$ ③

Solve equations ① and ② for s and t.

$s + t = 1$ ①

$\underline{s - t = -1}$ ②

$2s = 0$ ①+②

$s = 0$

Substituting in ①.

$0 + t = 1$

$t = 1$

Clearly s and t satisfy③ since it is identical to ②.
Let $s = 0$ to find the point of intersection, which is (2, 4, 2).
The lines are not skew.

d) Equate the expressions for like coordinates.

$-6 + 2s = 8 - 2t$ $\qquad$ $12 + s = -9 + t$ $\qquad$ $8 - 5s = -7 - 5t$

$s + t = 7$ ① $\qquad$ $s - t = -21$ ② $\qquad$ $s - t = 3$ ③

Clearly equations ② and ③ are contradictory and there are no solutions for s and t.
Therefore the lines do not intersect. They are skew lines.

e) Equate the expressions for like coordinates.

$5 + 6s = 2 + t$ $\qquad$ $-4 + 4s = -3 + 2t$ $\qquad$ $1 - 2s = 4 - 3t$

$6s - t = -3$ ① $\qquad$ $4s - 2t = 1$ ② $\qquad$ $2s - 3t = -3$ ③

Solve equations ① and ② for s and t.

$12s - 2t = -6$ $\quad 2$①

$\underline{4s - 2t = 1}$ $\quad$ ②

$8s = -7$ $\quad 2$①–②

$s = -\frac{7}{8}$

Substituting in ②.

$$4\left(-\frac{7}{8}\right)-2t=1$$

$$t=-\frac{9}{4}$$

Check that s and t satisfy ③.

$$\text{L.S.}=2\left(-\frac{9}{4}\right)-3\left(-\frac{7}{8}\right) \qquad \text{R.S.} = -3$$

$$=-\frac{15}{8}$$

$$\text{L.S.} \neq \text{R.S.}$$

Therefore the lines do not intersect. They are skew lines.

f) Equate the expressions for like coordinates.

$$1+2s=1+3t \qquad -1-s=2+2t \qquad 3s=3+4t$$

$$s-t=0 \quad ① \qquad s+2t=-3 \quad ② \qquad 3s-4t=3 \quad ③$$

Solve equations ① and ② for s and t.

$$2s-3t=0 \qquad ①$$

$$\underline{2s+4t=-6} \qquad 2②$$

$$-7t=6 \qquad ①-2②$$

$$t=-\frac{6}{7}$$

Substituting in ①.

$$2s-3\left(-\frac{6}{7}\right)=0$$

$$s=\frac{9}{7}$$

Check that s and t satisfy ③.

$$\text{L.S.}=3\left(\frac{9}{7}\right)-4\left(-\frac{6}{7}\right) \qquad \text{R.S.} = 3$$

$$=\frac{51}{7}$$

$$\text{L.S.} \neq \text{R.S.}$$

Therefore the lines do not intersect. They are skew lines.

Chapter 8 Section 4 **Question 7 Page 472**

Use 3D Grapher or other software to verify results.

Chapter 8 Section 4 **Question 8 Page 472**

First examine the direction vectors. $\vec{m}_1 = [3, 4, -2]$, $\vec{m}_2 = [9, 12, -6]$, and $\vec{m}_3 = [9, 12, 6]$.
Line l_3 has a direction vector that is not a scalar multiple of the other two lines, so it is a unique line.
The first two lines have direction vectors that are scalar multiples of each other: $\vec{m}_2 = 3\vec{m}_1$.
These lines are parallel and may be coincident. Check if (2, –8, 1) is on ℓ_2.

$$[2, -8, 1] = [4, -16, 2] + s[9, 12, -6]$$

Equate the *x*-coordinates.
$2 = 4 + 9s$
$s = -\frac{2}{9}$

Equate the *y*-coordinates.
$-8 = -16 + 12s$
$s = \frac{2}{3}$

Equate the *z*-coordinates.
$1 = 2 - 6s$
$s = \frac{1}{6}$

Since the *s* values are not equal, the point (2, –8, 1) does not lie on ℓ_2.
Lines ℓ_1 and ℓ_2 are distinct but parallel lines and ℓ_3 is a line not parallel to either of the others.

Chapter 8 Section 4 **Question 9 Page 472**

For the distance between skew lines, $d = \frac{|\overrightarrow{P_1P_2} \cdot \vec{n}|}{|\vec{n}|}$ where $P_1 \in \ell_1$, $P_2 \in \ell_2$, and $\vec{n} = \vec{m}_1 \times \vec{m}_2$

a) Let $P_1(3, 1, 0)$, $P_2(-4, 2, 1)$, $\vec{m}_1 = [1, 8, 2]$, and $\vec{m}_2 = [-1, -2, 1]$.

$$\overrightarrow{P_1P_2} = [-7, 1, 1]$$

$$\vec{n} = \vec{m}_1 \times \vec{m}_2$$
$$= [12, -3, 6]$$

$$d = \frac{|[-7, 1, 1] \cdot [12, -3, 6]|}{|[12, -3, 6]|}$$
$$d = \frac{81}{\sqrt{189}}$$
$$d \doteq 5.89$$

b) Let $P_1(1, -5, 6)$, $P_2(0, 7, 2)$, $\vec{m}_1 = [3, 1, -4]$, and $\vec{m}_2 = [2, -1, 5]$.

$\overrightarrow{P_1P_2} = [-1, 12, -4]$

$\vec{n} = \vec{m}_1 \times \vec{m}_2$

$= [1, -23, -5]$

$$d = \frac{\left|[-1, 12, -4] \cdot [1, -23, -5]\right|}{\left|[1, -23, -5]\right|}$$

$$d = \frac{257}{\sqrt{555}}$$

$d \doteq 10.9$

c) Let $P_1(2, 0, 8)$, $P_2(1, 1, 1)$, $\vec{m}_1 = [0, 3, 2]$, and $\vec{m}_2 = [4, 0, -1]$.

$\overrightarrow{P_1P_2} = [-1, 1, -7]$

$\vec{n} = \vec{m}_1 \times \vec{m}_2$

$= [-3, 8, -12]$

$$d = \frac{\left|[-1, 1, -7] \cdot [-3, 8, -12]\right|}{\left|[-3, 8, -12]\right|}$$

$$d = \frac{95}{\sqrt{217}}$$

$d \doteq 6.45$

d) Let $P_1(5, 2, -3)$, $P_2(-1, -4, -4)$, $\vec{m}_1 = [5, 5, 1]$, and $\vec{m}_2 = [7, -2, -2]$.

$\overrightarrow{P_1P_2} = [-6, -6, -1]$

$\vec{n} = \vec{m}_1 \times \vec{m}_2$

$= [-8, 17, -45]$

$$d = \frac{\left|[-6, -6, -1] \cdot [-8, 17, -45]\right|}{\left|[-8, 17, -45]\right|}$$

$$d = \frac{9}{\sqrt{2378}}$$

$d \doteq 0.2$

Chapter 8 Section 4 **Question 10 Page 472**

a) Equate the expressions for like coordinates for ℓ_1 and ℓ_2.

$-1+5r=7+3s$ $\quad -1-r=-10-8s$

$5r-3s=8$ ① $\quad r-8s=9$ ②

Solve equations ① and ② for s and t.

$5r-3s=8$ ①

$\underline{5r-40s=45}$ 5②

$37s=-37$ ①–5②

$s=-1$

Substituting in ①.

$r-8(-1)=9$

$r=1$

Use $r = 1$. The intersection point for ℓ_1 and ℓ_2 is A(4, –2).

Equate the expressions for like coordinates for ℓ_1 and ℓ_3.

$-1+5r=3+2t$ $\quad -1-r=13+7t$

$5r-2t=4$ ① $\quad r+7t=-14$ ②

Solve equations ① and ② for s and t.

$5r-2t=4$ ①

$\underline{5r+35t=-70}$ 5②

$-37t=74$ ①–5②

$t=-2$

Substituting in ①.

$5r-2(-2)=4$

$r=0$

Use $r = 0$. The intersection point for ℓ_1 and ℓ_3 is B(–1, –1).

Equate the expressions for like coordinates for ℓ_2 and ℓ_3

$7+3s=3+2t$ $\quad -10-8s=13+7t$

$3s-2t=-4$ ① $\quad 8s+7t=-23$ ②

Solve equations ① and ② for s and t.

$21s-14t=-28$ 7①

$\underline{16s+14t=-46}$ 2②

$37s=-74$ ①+5②

$s=-2$

Substituting in ①.

$3(-2) - 2t = -4$

$t = -1$

Use $s = -2$. The intersection point for ℓ_2 and ℓ_3 is C(1, 6).

b) $\overrightarrow{AB} = [-5, 1]$, $\overrightarrow{AC} = [-3, 8]$, and $\overrightarrow{BC} = [2, 7]$

$\sqrt{(-5)^2 + 1^2} + \sqrt{(-3)^2 + 8^2} + \sqrt{2^2 + 7^2} = \sqrt{26} + \sqrt{73} + \sqrt{53}$

$\doteq 20.9$

The perimeter is 20.9 units.

Chapter 8 Section 4 **Question 11 Page 472**

Answers may vary. For example:

There are four possibilities.

- The lines are identical. The equation of the y-axis is $[x, y, z] = [0, 1, 0] + t[0, 1, 0]$, $t \in \mathbb{R}$.
- The lines are parallel but distinct. The equation of a second line could be $[x, y, z] = [1, 2, 3] + t[0, 1, 0]$, $t \in \mathbb{R}$.
- The lines are skew. The equation of a second line could be $[x, y, z] = [1, 2, 3] + t[1, 0, 0]$, $t \in \mathbb{R}$.
- The lines intersect at a single point. The equation of a second line could be $[x, y, z] = [1, 0, 0] + t[1, 0, 0]$, $t \in \mathbb{R}$. This is the x-axis and the two lines intersect at the origin.

Chapter 8 Section 4 **Question 12 Page 472**

a) Find the intersection point of the two lines.

Equate the expressions for like coordinates for the two lines.

$-82.2 + 0.8s = -84.4 + 3t$ $\qquad$ $43 + 2.2s = 45.5 - 0.3t$

$0.8s - 3t = -2.2$ ① $\qquad$ $2.2s + 0.3t = 2.5$ ②

Solve equations ① and ② for s and t.

$0.8s - 3t = -2.2$ $\qquad$ ①

$22s + 3t = 25$ $\qquad$ 10②

$22.8s = 22.8$ $\qquad$ ①+10②

$s = 1$

Substituting in ①.

$0.8 - 3t = -2.2$

$t = 1$

Use $s = 1$. The intersection point is (–81.4, 45.2).

The position of the Cove Island buoy is (–81.4, 45.2).

b) Find the intersection point of the two lines.
Equate the expressions for like coordinates for the two lines.

$-82.2 + 0.2s = -84.4 + 2.0t$ $\qquad$ $43 + 2s = 44.5 - 0.5t$

$0.2s - 2t = -2.2$ ① $\qquad$ $2s + 0.5t = 1.5$ ②

Solve equations ① and ② for s and t.

$2s - 20t = -22$ $\qquad$ 10①

$\underline{2s + 0.5t = 1.5}$ $\qquad$ ②

$-20.5t = -23.5$ $\qquad$ 10①−②

$t \doteq 1.15$

Substituting in ①.

$0.2s - 2(1.15) = -2.2$

$s \doteq -0.05$

Use $t = 1.15$. The intersection point is (–82.12, 43.93).
The position of the new weather buoy is (–82.12, 43.93).

c) To get from (–81.4, 45.2) to (–82.12, 43.93), the resultant vector is
$\left[-82.12 - \left(-81.4\right), 43.93 - 45.2\right] = \left[-0.7, -1.27\right]$.

Chapter 8 Section 4 $\qquad$ **Question 13 Page 473**

No. The formula for the distance between skew lines requires a vector $\vec{n}$ that is perpendicular to both lines. In the case of parallel lines, there is no such unique direction vector. In fact, the method to find $\vec{n}$ involves $\vec{m}_1 \times \vec{m}_2$ which would give the result $\vec{0}$ here since $\vec{m}_1$ is parallel to $\vec{m}_2$.

It is possible to calculate the distance from a point to a line (and hence the distance between parallel lines). Let P_1 be on one line and P_2 on the other. The distances $\left|\text{proj}_{\vec{m}} \overrightarrow{P_1P_2}\right|$ and $\left|\overrightarrow{P_1P_2}\right|$ can be calculated easily; the required distance is then the third side of a right-angled triangle.

Chapter 8 Section 4 **Question 14 Page 473**

a) Find the (possible) intersection points of the two lines.
Equate the expressions for like coordinates for the two lines.

$8s = 60 + 2t$ $\quad 4s = -20 + 6t$ $\quad s = 22 - t$

$4s - t = 30$ ① $\quad 4s - 6t = -20$ ② $\quad s + t = 22$ ③

Solve equations ① and ② for s and t.

$$\begin{aligned} 4s - t &= 30 && ① \\ \underline{4s - 6t} &= \underline{-20} && ② \\ 5t &= 50 && ①-② \\ t &= 10 \end{aligned}$$

Substituting in ①.

$$\begin{aligned} 4s - 10 &= 30 \\ s &= 10 \end{aligned}$$

Check that s and t satisfy ③.

L.S. $= 10 + 10$ $\quad$ R.S. = 22

$= 20$

L.S. $\neq$ R.S.

Therefore the flight paths do not intersect.

b) Find the shortest distance between the (skew) lines.

For the distance between skew lines, $d = \dfrac{\left|\overrightarrow{P_1P_2} \cdot \vec{n}\right|}{\left|\vec{n}\right|}$ where $P_1 \in \ell_1$, $P_2 \in \ell_2$, and $\vec{n} = \vec{m}_1 \times \vec{m}_2$.

Let $P_1(0, 0, 0)$, $P_2(60, -20, 22)$, $\vec{m}_1 = [8, 4, 1]$, and $\vec{m}_2 = [2, 6, -1]$.

$\overrightarrow{P_1P_2} = [60, -20, 22]$

$$\begin{aligned} \vec{n} &= \vec{m}_1 \times \vec{m}_2 \\ &= [-10, 10, 40] \text{ or } [-1, 1, 4] \end{aligned}$$

$$\begin{aligned} d &= \frac{\left|[60, -20, 22] \cdot [-1, 1, 4]\right|}{[-1, 1, 4]} \\ &= \frac{8}{\sqrt{18}} \\ &\doteq 1.89 \end{aligned}$$

c) Not necessarily. They would only meet if the s and t values at the intersection point represented the same moment in time, not just the same location in space.

Chapter 8 Section 4 **Question 15 Page 473**

a) The lines are not parallel since $\vec{m}_1 \neq k\vec{m}_2$ for any value of k.

b) $l_1 : \begin{cases} x = 7 + s \\ y = 7 + 2s, \quad s \in \mathbb{R} \\ z = -3 - 3s \end{cases}$

$l_2 : \begin{cases} x = 10 + 2t \\ y = 7 + 2t, \quad t \in \mathbb{R} \\ z = -t \end{cases}$

Find the (possible) intersection points of the two lines.
Equate the expressions for like coordinates for the two lines.

$7 + s = 10 + 2t$ $\qquad$ $7 + 2s = 7 + 2t$ $\qquad$ $-3 - 3s = -t$

$s - 2t = 3$ ① $\qquad$ $s - t = 0$ ② $\qquad$ $3s - t = -3$ ③

Solve equations ① and ② for s and t.

$$\begin{array}{rll} s - 2t = 3 & & ① \\ \underline{s - t = 0} & & ② \\ -t = 3 & & ①-② \\ t = -3 & & \end{array}$$

Substituting in ①.

$s - 2(-3) = 3$

$s = -3$

Check that s and t satisfy ③.

L.S. $= 3(-3) - (-3)$ $\qquad$ R.S. $= -3$

$= -6$

L.S. $\neq$ R.S.

Therefore the lines do not intersect.

c) Fnd the least distance between the (skew) lines.

For the distance between skew lines, $d = \dfrac{\left|\overrightarrow{P_1P_2} \cdot \vec{n}\right|}{\left|\vec{n}\right|}$ where $P_1 \in \ell_1$, $P_2 \in \ell_2$, and $\vec{n} = \vec{m}_1 \times \vec{m}_2$.

Let $P_1(7, 7, -3)$, $P_2(10, 7, 0)$, $\vec{m}_1 = [1, 2, -3]$, and $\vec{m}_2 = [2, 2, -1]$.

$\overrightarrow{P_1P_2} = [3, 0, 3]$

$\vec{n} = \vec{m}_1 \times \vec{m}_2$

$= [4, -5, -2]$

$d = \dfrac{\left|[3, 0, 3] \cdot [4, -5, -2]\right|}{[4, -5, -2]}$

$= \dfrac{6}{\sqrt{45}}$

$\doteq 0.9$

Chapter 8 Section 4 **Question 16 Page 473**

Answers may vary. For example:

a) $[x, y, z] = [1, 2, 2] + t[0, -9, -1]$, $t \in \mathbb{R}$ and $[x, y, z] = [3, -1, 1] + s[1, 3, 0]$, $s \in \mathbb{R}$.
Let $t = 1$ and $s = -2$ to verify intersection point.

b) $[x, y, z] = [1, 3, -4] + t[1, 1, 1]$, $t \in \mathbb{R}$ and $[x, y, z] = [-2, 6, -5] + s[2, -1, 1]$, $s \in \mathbb{R}$
Let $t = 1$ and $s = 2$ to verify intersection point.

Chapter 8 Section 4 **Question 17 Page 473**

a) Many methods are possible to find the centroid X.

Method 1: Use geometric and Cartesian vectors.
From Chapter 6:

$\overrightarrow{OX} = \dfrac{1}{3}\left(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}\right)$

$= \dfrac{1}{3}\left([2, 6] + [10, 9] + [9, 3]\right)$

$= \dfrac{1}{3}[21, 18]$

$= [7, 6]$

Method 2: Use equations of lines.
The midpoint of BC is D(9.5, 6) and the midpoint of AB is E(6, 7.5).
The equation of line AD is $[x, y] = [2, 6] + t[7.5, 0]$.
The equation of line CE is $[x, y] = [9, 3] + s[-3, 4.5]$.
Find the intersection point of these two lines.

Equate the expressions for like coordinates for the two lines.

$$2 + 7.5t = 9 - 3s \qquad 6 = 3 + 4.5s$$

$$3s + 7.5t = -7 \quad ① \qquad s = \frac{2}{3} \quad ②$$

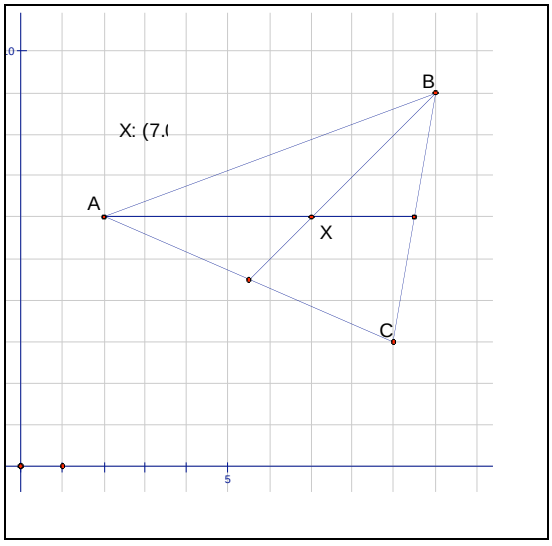
Solve equations ① and ② for s and t.
Substituting for s in ①.

$$3\left(\frac{2}{3}\right) + 7.5t = 7$$

$$t = \frac{2}{3}$$

Use $s = \frac{2}{3}$. The intersection point is X(7, 6).

Method 3: Use *The Geometer's Sketchpad*®.
Plot points…
Construct midpoints…
Construct segments…
Construct Intersection Points….
Measure Coordinates.
The intersection point is X(7, 6).



b) See Method 3 in part a).

c) Use Method 1 from part a).

$$\begin{aligned}\overrightarrow{OX} &= \frac{1}{3}\left(\overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OF}\right)\\ &= \frac{1}{3}\left(\left[2, -6, 8\right] + \left[9, 0, 2\right] + \left[-1, 3, -2\right]\right)\\ &= \frac{1}{3}\left[10, -3, 8\right]\\ &= \left[\frac{10}{3}, -1, \frac{8}{3}\right]\end{aligned}$$

Chapter 8 Section 4 **Question 18 Page 473**

a) Solve by elimination.

$$\begin{aligned} ax + by &= c && ①\\ dx + ey &= f && ②\\ aex + eby &= ec && e①\\ bdx + eby &= bf && b②\\ \left(ae - bd\right)x &= ec - bf && e① - b②\end{aligned}$$

$$x = \frac{ec - bf}{ae - bd}$$

Substitute for x in ①.

$$a\left(\frac{ec - bf}{ae - bd}\right) + by = c$$

$$\begin{aligned} by &= c\frac{ae - bd}{ae - bd} - \frac{a\left(ec - bf\right)}{ae - bd}\\ y &= \frac{1}{b}\frac{aec - bcd - aec + abf}{ae - bd}\\ y &= \frac{b\left(af - cd\right)}{b\left(ae - bd\right)}\\ y &= \frac{af - cd}{ae - bd}\end{aligned}$$

Note that the algebra assumes that no denominators are equal to zero.

Therefore, $\left(x, y\right) = \left(\frac{ec - bf}{ae - bd}, \frac{af - cd}{ae - bd}\right)$.

b) Equate the expressions for like coordinates for the two lines.

$a + cs = e + gt$ $\qquad$ $b + ds = f + ht$

$cs - gt = e - a$ ① $\qquad$ $ds - ht = f - b$ ②

Solve equations ① and ② for s and t.

$chs - ght = eh - ah$ $\qquad h$①

$\underline{dgs - ght = fg - bg}$ $\qquad g$②

$(ch - dg)s = eh - ah - fg + bg$ $\qquad h$①$-g$②

$$s = \frac{eh - ah - fg + bg}{ch - dg}$$

Substituting in ①.

$$c\left(\frac{eh - ah - fg + bg}{ch - dg}\right) - gt = e - a$$

$$gt = c\left(\frac{eh - ah - fg + bg}{ch - dg}\right) - (e - a)\frac{ch - dg}{ch - dg}$$

$$gt = \frac{\cancel{ceh} - \cancel{ach} - cfg + bcg - \cancel{ceh} + deg + \cancel{ach} - adg}{ch - dg}$$

$$t = \frac{-cf + bc + de - ad}{ch - dg}$$

Use the value for s to find the intersection points.

$$x = a + \left(\frac{eh - ah - fg + bg}{ch - dg}\right)c$$

$$= \frac{a(ch - dg)}{ch - dg} + \frac{c(eh - ah - fg + bg)}{ch - dg}$$

$$= \frac{\cancel{ach} - adg + ceh - \cancel{ach} - cfg + bcg}{ch - dg}$$

$$= \frac{-adg + ceh - cfg + bcg}{ch - dg}$$

$$y = b + \left(\frac{eh - ah - fg + bg}{ch - dg}\right)d$$

$$= \frac{b(ch - dg)}{ch - dg} + \frac{d(eh - ah - fg + bg)}{ch - dg}$$

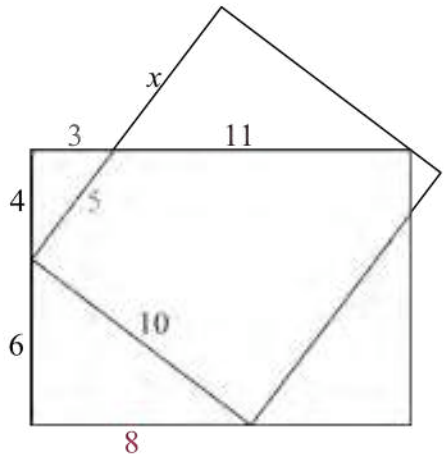
$$= \frac{bch - \cancel{bdg} + deh - adh - dfg + \cancel{bdg}}{ch - dg}$$

$$= \frac{bch + deh - adh - dfg}{ch - dg}$$

Therefore, $(x, y) = \left(\frac{-adg + ceh - cfg + bcg}{ch - dg}, \frac{bch + deh - adh - dfg}{ch - dg}\right)$.

Chapter 8 Section 4 **Question 19 Page 473**

Examine the diagram carefully. There are 5 small right-angled triangles created by the overlap of the rectangles. They are all similar.



Use the Pythagorean theorem in the lower left triangle to obtain the length 8.
The triangle in the upper left is similar and so must have sides 3 and 5.

Label the unknown part of the length of the second rectangle as x. The missing length of the part of the top of the first rectangle is 11. The two triangles, upper and upper left, are similar.

Therefore,

$$\frac{x}{11} = \frac{3}{5}$$

$$x = \frac{33}{5}$$

$$x = 6.6$$

$5 + 6.6 = 11.6$

The length of the second rectangle is 11.6 cm.

Chapter 8 Section 4 **Question 20 Page 473**

If the angle between two planes is θ, then $\cos\ \theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{\left|\vec{n}_1\right|\left|\vec{n}_2\right|}$.

Therefore:

$$\cos\ 60^\circ = \frac{[1, 1, 2]\cdot[2, -1, k]}{\left|[1, 1, 2]\right|\left|[2, -1, k]\right|}$$

$$\frac{1}{2} = \frac{2-1+2k}{\sqrt{6}\sqrt{5+k^2}}$$

$$2+4k = \sqrt{30+6k^2}$$

$$4+16k+16k^2 = 30+6k^2$$

$$10k^2+16k-26 = 0$$

$$5k^2+8k-13 = 0$$

$$(5k+13)(k-1) = 0$$

So, $k = 1$ or $k = -\dfrac{13}{5}$ or -2.6.

Chapter 8 Section 5 **Intersections of Lines and Planes**

Chapter 8 Section 5 **Question 1 Page 479**

Substitute the parametric equations into the scalar equation of the plane and solve for t.

$$(4+t)+5(2-2t)+(6+3t)-8=0$$
$$4+t+10-10t+6+3t-8=0$$
$$-6t=-12$$
$$t=2$$

Substitute this value of t in the parametric equations.

$x = 4+2$ $\quad y = 2-2(2)$ $\quad z = 6+3(2)$

$= 6$ $\quad = -2$ $\quad = 12$

The point of intersection of the line and the plane is (6, –2, 12).

Chapter 8 Section 5 **Question 2 Page 479**

The line and the plane are parallel if $\vec{n}\cdot\vec{m}=0$.

a) $\vec{n}\cdot\vec{m}=[3,5,1]\cdot[2,-1,-1]$

$=3(2)+5(-1)+1(-1)$

$=0$

So, the line and the plane are parallel.

Check coincidence by checking if the point (1, 2, –8) on the line satisfies the plane equation.

L.S. $=3(1)+5(2)+(-8)-5$ $\quad$ R.S. = 0

$=0$

L.S. = R.S.

Therefore, the line and the plane are coincident.

b) $\vec{n}\cdot\vec{m}=[4,-1,6]\cdot[7,-14,-7]$

$=4(7)+(-1)(-14)+6(-7)$

$=0$

So, the line and the plane are parallel.

Check coincidence by checking if the point (4, 3, 10) on the line satisfies the plane equation.

L.S. $=4(4)-(3)+6(10)-12$ $\quad$ R.S. = 0

$=61$

L.S. $\neq$ R.S.

Therefore, the line and the plane are parallel and distinct.

c) $\vec{n} \cdot \vec{m} = [0, 3, 10] \cdot [2, -10, 3]$

$= 0(2) + 3(-10) + 10(3)$

$= 0$

So, the line and the plane are parallel.

Check coincidence by checking if the point (7, 1, –9) on the line satisfies the plane equation.

L.S. $= 3(1) + 10(-9) + 1$ R.S. $= 0$

$= -86$

L.S. $\neq$ R.S.

Therefore, the line and the plane are parallel and distinct.

d) $\vec{n} \cdot \vec{m} = [1, 2, -5] \cdot [1, 2, 1]$

$= 1(1) + 2(2) + (-5)(1)$

$= 0$

So, the line and the plane are parallel.

Check coincidence by checking if the point (10, 3, 4) on the line satisfies the plane equation.

L.S. $= 10 + 2(3) - 5(4) + 4$ R.S. $= 0$

$= 0$

L.S. = R.S.

Therefore the line and the plane are coincident.

Chapter 8 Section 5 Question 3 Page 479

a) Substitute the parametric equations into the scalar equation of the plane and solve for t.

$3(3 + 7t) - (-11t) + 4(5 - 8t) - 8 = 0$

$9 + 21t + 11t + 20 - 32t - 8 = 0$

$0t = -21$

There are no values of t that make this equation true. The line and the plane do not intersect.

b) Substitute the parametric equations into the scalar equation of the plane and solve for t.

$-2(5 + t) + 6(-1 - 2t) + 4(4 + 3t) - 4 = 0$

$-10 - 2t - 6 - 12t + 16 + 12t - 4 = 0$

$-2t = 4$

$t = -2$

Substitute this value of t in the parametric equations.

$x = 5 + (-2)$ $y = -1 - 2(-2)$ $z = 4 + 3(-2)$

$= 3$ $= 3$ $= -2$

The point of intersection of the line and the plane is (3, 3, –2).

c) Substitute the parametric equations into the scalar equation of the plane and solve for t.

$$5(4+t)+3(1+2t)+4(5+3t)-20=0$$
$$20+5t+3+6t+20+12t-20=0$$
$$23t=-23$$
$$t=-1$$

Substitute this value of t in the parametric equations.

$x=4+(-1)$ $\quad y=1+2(-1)$ $\quad z=5+3(-1)$

$=3$ $\quad =-1$ $\quad =2$

The point of intersection of the line and the plane is $(3, -1, 2)$.

d) Substitute the parametric equations into the scalar equation of the plane and solve for t.

$$5(10+2t)-3(-5+t)+7(-2t)+7=0$$
$$50+10t+15-3t-14t+7=0$$
$$-7t=-72$$
$$t=\frac{72}{7}$$

Substitute this value of t in the parametric equations.

$x=10+2\left(\frac{72}{7}\right)$ $\quad y=-5+\left(\frac{72}{7}\right)$ $\quad z=-2\left(\frac{72}{7}\right)$

$=\frac{214}{7}$ $\quad =\frac{37}{7}$ $\quad =-\frac{144}{7}$

The point of intersection of the line and the plane is $\left(\frac{214}{7}, \frac{37}{7}, -\frac{144}{7}\right)$.

e) Substitute the parametric equations into the scalar equation of the plane and solve for t.

$$9(4+2t)-6(t)+12(-1-t)-24=0$$
$$36+18t-6t-12-12t-24=0$$
$$0t=0$$

This equation is true for all values of t. Any point on the line is a solution.
There are an infinite number of solutions.

f) Substitute the parametric equations into the scalar equation of the plane and solve for t.

$$6(4+2t)-2(12-3t)+3(-19+5t)+6=0$$
$$24+12t-24+6t-57+15t+6=0$$
$$33t=51$$
$$t=\frac{17}{11}$$

Substitute this value of t in the parametric equations.

$$x=4+2\left(\frac{17}{11}\right) \qquad y=12-3\left(\frac{17}{11}\right) \qquad z=-19+5\left(\frac{17}{11}\right)$$
$$=\frac{78}{11} \qquad =\frac{81}{11} \qquad =-\frac{124}{11}$$

The point of intersection of the line and the plane is $\left(\frac{78}{11}, \frac{81}{11}, -\frac{124}{11}\right)$.

Chapter 8 Section 5 **Question 4 Page 479**

a) $[3, -2, 4]\cdot[3, 6, 2]=5$

Therefore, the line and the plane are not parallel and so they intersect.

b) $[3, -2, 4]\cdot[-2, 1, 2]=0$

Therefore, the line and the plane are parallel and may not intersect.

Check if $(-3, -5, 1)$ is on the plane.

L.S. $=3(-3)-2(-5)+4(1)$ R.S. $=5$

$=5$

L.S. = R.S.

The line and the plane are coincident and so they do intersect.

c) $[3, -2, 4]\cdot[4, 4, 1]=8$

Therefore, the line and the plane are not parallel and so they intersect.

Chapter 8 Section 5 **Question 5 Page 480**

First find the normal vector to the plane.

$$\begin{aligned}\vec{n} &= \vec{m}_1 \times \vec{m}_2 \\ &= [1, -3, 1] \times [2, 3, 1] \\ &= [(-3)(1) - 3(1), 1(2) - 1(1), 1(3) - 2(-3)] \\ &= [-6, 1, 9]\end{aligned}$$

a) $\vec{n} \cdot \vec{m} = [-6, 1, 9] \cdot [1, -12, 2]$

$= 0$

Therefore, the line and the plane are parallel and may not intersect.

Check if (5, –9, 3) is on the plane. Substitute in the equation for the plane.

$5 = 4 + s + 2t$ $\quad -9 = -15 - 3s + 3t$ $\quad 3 = -8 + s + t$

$s + 2t = 1$ ① $\quad s - t = -2$ ② $\quad s + t = 11$ ③

Solve equations ① and ② for s and t.

$$\begin{array}{rll} s + 2t = 1 & & ① \\ \underline{s - t = -2} & & ② \\ 3t = 3 & & ① - ② \\ t = 1 & & \end{array}$$

Substituting in ①.

$s + 2(1) = 1$

$s = -1$

Check that s and t satisfy ③.

L.S. $= -1 + 1$ $\quad$ R.S. $= -2$

$= 0$

L.S. $\neq$ R.S.

Therefore (5, –9, 3) is not on the plane. The line and the plane are parallel and distinct and do not intersect.

b) $\vec{n} \cdot \vec{m} = [-6, 1, 9] \cdot [2, -5, 4]$

$= 19$

Therefore, the line and the plane are not parallel.

They intersect at one point, so there is one solution.

c) $\vec{n} \cdot \vec{m} = [-6, 1, 9] \cdot [1, 4, -2]$

$= -20$

Therefore, the line and the plane are not parallel.

They intersect at one point, so there is one solution.

d) $\vec{n} \cdot \vec{m} = [-6, 1, 9] \cdot [2, -1, 1]$

$= -4$

Therefore, the line and the plane are not parallel.
They intersect at one point, so there is one solution.

e) $\vec{n} \cdot \vec{m} = [-6, 1, 9] \cdot [-1, 3, -1]$

$= 0$

Therefore, the line and the plane are parallel and may not intersect.

Check if (2, –3, 0) is on the plane. Substitute in the equation for the plane.

$2 = 4 + s + 2t$ $\quad -3 = -15 - 3s + 3t$ $\quad 0 = -8 + s + t$

$s + 2t = -2$ ① $\quad s - t = -4$ ② $\quad s + t = 8$ ③

Solve equations ① and ② for s and t.

$s + 2t = -2$ ①

$s - t = -4$ ②

$3t = 2$ ①–②

$t = \frac{2}{3}$

Substituting in ①.

$s + 2\left(\frac{2}{3}\right) = 1$

$s = -\frac{1}{3}$

Check that s and t satisfy ③.

L.S. $= -\frac{1}{3} + \frac{2}{3}$ $\quad$ R.S. = 8

$= \frac{1}{3}$

L.S. ≠ R.S.

Therefore (2, –3, 0) is not on the plane. The line and the plane are parallel and distinct and do not intersect.

f) $\vec{n} \cdot \vec{m} = [-6, 1, 9] \cdot [-2, 2, 4]$

$= 50$

Therefore, the line and the plane are not parallel.
They intersect at one point, so there is one solution.

Chapter 8 Section 5 **Question 6 Page 480**

The distance between a point P and a plane is given by $d = \dfrac{\left|\vec{n} \cdot \overrightarrow{PQ}\right|}{\left|\vec{n}\right|}$ where Q is any point on the plane with normal $\vec{n}$.

a) $\overrightarrow{PQ} = [2, 0, 0] - [2, 0, 1]$
$= [0, 0, -1]$

$$d = \frac{\left|[2, -1, 2] \cdot [0, 0, -1]\right|}{\sqrt{2^2 + (-1)^2 + 2^2}}$$
$$= \frac{2}{3}$$
$$\doteq 0.67$$

b) $\overrightarrow{PQ} = [4, 0, 0] - [2, -1, -1]$
$= [2, 1, 1]$

$$d = \frac{\left|[1, -1, 1] \cdot [2, 1, 1]\right|}{\sqrt{1^2 + (-1)^2 + 1^2}}$$
$$= \frac{2}{\sqrt{3}}$$
$$\doteq 1.16$$

c) $\overrightarrow{PQ} = [2, 0, 0] - [0, -1, 1]$
$= [2, 1, -1]$

$$d = \frac{\left|[2, -3, 0] \cdot [2, 1, 1]\right|}{\sqrt{2^2 + (-3)^2 + 0^2}}$$
$$= \frac{1}{\sqrt{13}}$$
$$\doteq 0.28$$

d) $\overrightarrow{PQ} = [3, 1, 1] - [1, 5, 1]$

$= [2, -4, 0]$

$$d = \frac{|[11, -24, -5] \cdot [2, -4, 0]|}{\sqrt{11^2 + (-4)^2 + (-5)^2}}$$

$$= \frac{118}{\sqrt{722}}$$

$\doteq 4.39$

e) $\overrightarrow{PQ} = \left[0, 0, -\frac{1}{14}\right] - [2, -1, 0]$

$= \left[-2, 1, -\frac{1}{14}\right]$

$$d = \frac{\left|[0, -14, -14] \cdot \left[-2, 1, -\frac{1}{14}\right]\right|}{\sqrt{0^2 + (-14)^2 + (-14)^2}}$$

$$= \frac{13}{\sqrt{392}}$$

$\doteq 0.66$

f) $\overrightarrow{PQ} = [0, -2, 0] - [3, 8, 1]$

$= [-3, -10, -1]$

$$d = \frac{|[8, -6, -13] \cdot [-3, -10, -1]|}{\sqrt{8^2 + (-6)^2 + (-13)^2}}$$

$$= \frac{49}{\sqrt{269}}$$

$\doteq 2.99$

Chapter 8 Section 5 **Question 7 Page 480**

The distance between a point P and a plane is given by $d = \dfrac{\left|\vec{n} \cdot \overrightarrow{PQ}\right|}{\left|\vec{n}\right|}$ where Q is any point on the plane with normal $\vec{n}$.

a) $\overrightarrow{PQ} = [0, 0, -1] - [0, 0, -4]$
$= [0, 0, 3]$

$$d = \frac{\left|[2, -1, -1] \cdot [0, 0, 3]\right|}{\sqrt{2^2 + (-1)^2 + (-1)^2}}$$
$$= \frac{3}{\sqrt{6}}$$
$$\doteq 1.22$$

b) $\overrightarrow{PQ} = [3, 0, 0] - [1, 0, 0]$
$= [2, 0, 0]$

$$d = \frac{\left|[1, 3, -2] \cdot [2, 0, 0]\right|}{\sqrt{1^2 + 3^2 + (-2)^2}}$$
$$= \frac{2}{\sqrt{14}}$$
$$\doteq 0.53$$

c) $\overrightarrow{PQ} = [3, 0, 0] - [2, 0, 0]$
$= [1, 0, 0]$

$$d = \frac{\left|[2, -3, 1] \cdot [1, 0, 0]\right|}{\sqrt{2^2 + (-3)^2 + 1^2}}$$
$$= \frac{2}{\sqrt{14}}$$
$$\doteq 0.53$$

d) The planes are identical. The distance is zero.

e) $\overrightarrow{PQ} = [0, -3.5, 0] - [0, 2, 0]$
$= [0, -5.5, 0]$

$$d = \frac{|[3, -1, 12] \cdot [0, 5.5, 0]|}{\sqrt{3^2 + (-1)^2 + (-12)^2}}$$
$$= \frac{5.5}{\sqrt{154}}$$
$$\doteq 0.44$$

f) $\overrightarrow{PQ} = [-4, 0, 0] - [0, 0, -0.5]$
$= [-4, 0, 0.5]$

$$d = \frac{|[1, -6, -3] \cdot [-4, 0, 0.5]|}{\sqrt{1^2 + (-6)^2 + (-3)^2}}$$
$$= \frac{5.5}{\sqrt{46}}$$
$$\doteq 0.81$$

Chapter 8 Section 5 **Question 8 Page 480**

The distance between a point P and a plane is given by $d = \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{|\vec{n}|}$ where Q is any point on the plane with normal $\vec{n}$.

a) $\overrightarrow{PQ} = [0, 0, -1] - [2, 1, 6]$
$= [-2, -1, -7]$

$$d = \frac{|[3, 9, -1] \cdot [-2, -1, -7]|}{\sqrt{3^2 + 9^2 + (-1)^2}}$$
$$= \frac{8}{\sqrt{91}}$$
$$\doteq 0.84$$

b) $\overrightarrow{PQ} = [10, 0, 0] - [-5, 0, 1]$
$= [15, 0, -1]$

$$d = \frac{\left|[1, 2, 6] \cdot [15, 0, -1]\right|}{\sqrt{1^2 + 2^2 + 6^2}}$$
$$= \frac{9}{\sqrt{41}}$$
$$\doteq 1.41$$

c) $\overrightarrow{PQ} = [0, 0, -3] - [1, 4, -9]$
$= [-1, -4, 6]$

$$d = \frac{\left|[4, -2, -7] \cdot [-1, -4, 6]\right|}{\sqrt{4^2 + (-2)^2 + (-7)^2}}$$
$$= \frac{38}{\sqrt{69}}$$
$$\doteq 4.57$$

d) $\overrightarrow{PQ} = [0, 0, -2] - [-2, 3, -3]$
$= [2, -3, 1]$

$$d = \frac{\left|[2, -5, 3] \cdot [2, -3, 1]\right|}{\sqrt{2^2 + (-5)^2 + 3^2}}$$
$$= \frac{22}{\sqrt{38}}$$
$$\doteq 3.57$$

e) $\overrightarrow{PQ} = [0, 4, 0] - [5, -3, 2]$
$= [-5, 7, -2]$

$$d = \frac{\left|[2, -1, 5] \cdot [-5, 7, -2]\right|}{\sqrt{2^2 + (-1)^2 + 5^2}}$$
$$= \frac{27}{\sqrt{30}}$$
$$\doteq 4.93$$

f) $\overrightarrow{PQ} = [-3, 0, 0] - [-4, -5, 3]$

$= [1, 5, -3]$

$$d = \frac{|[-3, -3, 5] \cdot [1, 5, -3]|}{\sqrt{(-3)^2 + (-3)^2 + 5^2}}$$

$$= \frac{33}{\sqrt{43}}$$

$\doteq 5.03$

Chapter 8 Section 5 **Question 9 Page 480**

Answers may vary. For example:

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= [2, 4, -8]$

$\vec{m} = [2, 4, -8]$ is one direction vector for the plane.

$\vec{m} = [1, 2, -3]$ is also a direction vector for the plane.

The vector equation is $[x, y, z] = [2, -3, 6] + s[1, 2, -3] + t[2, 4, -8],\ s, t \in \mathbb{R}$

Chapter 8 Section 5 **Question 10 Page 480**

First find the direction vector for the line.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= [4, 0, 2] - [2, 3, 2]$

$= [2, -3, 0]$

Check:

$\vec{m} \cdot \vec{n} = [2, -3, 0] \cdot [2, 1, -3]$

$= 1$

Since the dot product does not equal zero, the line and plane are not parallel and so they must intersect.

Chapter 8 Section 5 **Question 11 Page 480**

Find the intersection of the line passing through A and B and the plane $x = 0$.

$$\overrightarrow{AB} = [1.95, 2.48, 1.2] - [2, 2.5, 1.3]$$
$$= [-0.05, -0.02, -0.1]$$

The vector equation of the line is $[x, y, z] = [2, 2.5, 1.3] + t[-0.05, -0.02, -0.1]$, $t \in \mathbb{R}$,

To find the intersection point, let $x = 0$ and solve for t.

$0 = 2 - 0.05t$

$t = 40$

The intersection point occurs when $t = 40$.

$(0, 2.5 - 0.02(40), 1.3 - 0.1(40)) = (0, 1.7, -2.7)$

The point is (0, 1.7, –2.7).

Chapter 8 Section 5 **Question 12 Page 481**

Answers may vary. For example:

Lines that pass through the given point can have the form $[x, y, z] = [1, 4, -1] + t[k, l, m]$ where k, l, and m can have any real values.

Planes passing through the given point can have the form $Ax + By + Cz + D = 0$ where $D = -A - 4B + C$ and A, B, and C can have any real values.

One possible equation pair is $x + 2y + 5z - 4 = 0$ and $[x, y, z] = [-1, 3, 0] + t[2, 1, -1]$, $t \in \mathbb{R}$

Chapter 8 Section 5 **Question 13 Page 481**

a) Yes. Two skew lines can always lie in parallel planes.
Consider the points A and B on each plane that indicate the points of closest approach of the planes. This line is perpendicular to both lines, which guarantees that the distance AB is the shortest. Construct two planes through A and B that are perpendicular to AB. These (unique) planes will be parallel to each other and contain the given lines.

b) Yes. Two skew lines can always lie in non-parallel planes.
Every line lies in a family containing an infinite number of planes, each having different normals. They have the line as their common line of intersection.

Choose a plane from each family so that the chosen planes are non-parallel.

Chapter 8 Section 5 **Question 14 Page 481**

Answers may vary. For example:

a) Clearly the lines are not parallel since the direction vectors, $[3, 1, 1]$ and $[2, -1, 1]$ are not scalar multiples of each other. Now, check for possible intersection points of the two lines.
Equate the expressions for like coordinates for the two lines.

$3s = 1 + 2t$ $\quad 2 + s = -3 - t$ $\quad 1 + s = t$

$3s - 2t = 1$ ① $\quad s + t = -5$ ② $\quad s - t = -1$ ③

Solve equations ② and ③ for s and t.

$s + t = -5$ ②

$\underline{s - t = -1}$ ③

$2s = -6$ ②+③

$s = -3$

Substituting in ②.

$-3 + t = -5$

$t = -2$

Check that s and t satisfy ①.

L.S. $= 3(-3) - 2(-2)$ $\quad$ R.S. $= 1$

$= -5$

L..S $\neq$ R.S.

Therefore, the lines do not intersect.

Note: Could also have found the non-zero distance between the lines to verify that they do not intersect.

Therefore, the lines are skew.

b) Refer to the answer to question 13, part a).
Find $\vec{n} = \vec{m}_1 \times \vec{m}_2$ which will be the normal vector for our planes.

$\vec{n} = [3, 1, 1] \times [2, -1, 1]$

$= [2, -1, -5]$

The parallel planes will be of the form $2x - y - 5z + D = 0$.
To find values for D, choose a point on each line.

$(0, 2, 1) \in \ell_1$ $\quad (1, -3, 0) \in \ell_2$

$\therefore -2 - 5(1) + D = 0$ $\quad \therefore 2(1) - (-3) - 5(0) + D = 0$

$D = 7$ $\quad D = -5$

Therefore, the two planes have equations $2x - y - 5z + 7 = 0$ and $2x - y - 5z - 5 = 0$.

Chapter 8 Section 5 **Question 15 Page 481**

Answers may vary. For example:
Refer to the answer to question 13, part b).

Choose two points on the first plane. Use (5, 0, 0) and (4, 1, 2). A line can be defined by the vector between these points and one of the points.
A possible vector equation for the first line is $l_1 : [x, y, z] = [4, 1, 2] + t[1, -1, -2],\ t \in \mathbb{R}$.

Now choose two points on the second plane. Use (–8, 0, 0) and (0, 2, –1).
A possible vector equation for the second line is $l_2 : [x, y, z] = [0, 2, -1] + s[8, 2, -1], s \in \mathbb{R}$.

Verify that the direction vectors are not scalar multiples of each other. They are not. (If they were, then choose different points on one plane.)

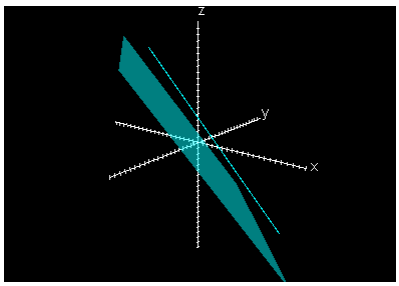
Chapter 8 Section 5 **Question 16 Page 481**

Answers may vary. For example:
Use the plane $Ax + By + Cy + D = 0$ and the line $[x, y, z] = [a, b, c] + t[d, e, f],\ t \in \mathbb{R}$.

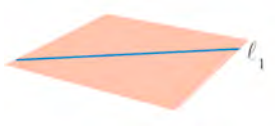
a) In general, the situation looks like this:

$[A, B, C] \cdot [d, e, f] = 0$, and (a, b, c) is not on the plane. $Aa + Bb + Cc + D \neq 0$.

For a particular example, see the sketch below using 3D Grapher. It shows the plane $x + y + z + 1 = 0$ and the line $[x, y, z] = [5, 3, -4] + t[-1, 0, 1],\ t \in \mathbb{R}$

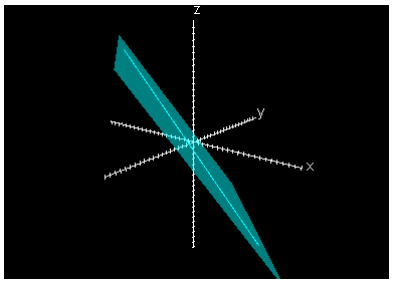


b) In general the situation looks like this:

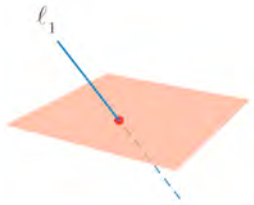


$[A, B, C] \cdot [d, e, f] = 0$. However (a, b, c) is not a point on the plane. $Aa + Bb + Cc + D \neq 0$.

For a particular example, see the sketch below using 3D Grapher. It shows the plane $x + y + z + 1 = 0$ and the line $[x, y, z] = [5, -2, -4] + t[-1, 0, 1],\ t \in \mathbb{R}$

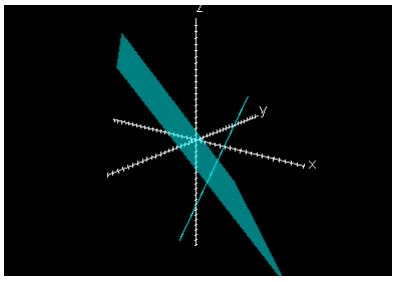


c) In general the situation looks like this:



$[A, B, C] \cdot [d, e, f] \neq 0$.

For a particular example, see the sketch below using 3D Grapher. It shows the plane $x + y + z + 1 = 0$ and the line $[x, y, z] = [5, -2, -4] + t[-1, 3, 1],\ t \in \mathbb{R}$



Chapter 8 Section 5 **Question 17 Page 481**

Answers may vary. For example:

a) Since the plane contains ℓ_1, the position vector and the direction for ℓ_1 can be used for the plane. Also the direction vector for ℓ_2 is also a direction vector for the plane. Therefore a possible vector equation for the plane is:
$[x, y, z] = [1, -2, 4] + s[1, 1, -3] + t[2, 3, 1], \quad s, t \in \mathbb{R}$.

To find the scalar equation of this plane, first find the normal vector.
$\vec{n} = \vec{m}_1 \times \vec{m}_2$.
$= [1, 1, -3] \times [2, 3, 1]$
$= [10, -7, 1]$

The equation is of the form $10x - 7y + z + D = 0$.

Substitute the point (1, –2, 4) to determine D.
$10(1) - 7(-2) + 4 + D = 0$
$D = -28$

The scalar equation of the plane is $10x - 7y + z - 28 = 0$.

b) If ℓ_2 lies in the plane then (4, –2, k) must lie in the plane.
$10(4) - 7(-2) + k - 28 = 0$
$k = -26$

c) The line ℓ_2 and the plane are parallel. For the line to be 10 units away, consider *any* point on the line.

The distance between a point P and a plane with normal $\vec{n}$ is given by $d = \dfrac{\left|\vec{n} \cdot \overrightarrow{PQ}\right|}{\left|\vec{n}\right|}$ where Q is any point on the plane.

In this case, use P(4, –2, k), Q(1, –2, 4), and $\vec{n} = [10, -7, 1]$.
$10 = \dfrac{\left|[10, -7, 1] \cdot [-3, 0, 4 - k]\right|}{\sqrt{10^2 + (-7)^2 + 1^2}}$
$10\sqrt{150} = \left|-30 - 4k\right|$

If $-30-4k \geq 0$,

$$10\sqrt{150} = -30-4k$$
$$k = \frac{-10\sqrt{150}-30}{4}$$
$$= \frac{-5\sqrt{150}-15}{2}$$
$$= \frac{-25\sqrt{6}-15}{2}$$
$$\doteq -38.1$$

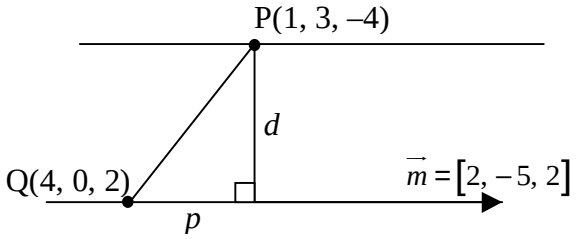
If $-30-4k < 0$,

$$10\sqrt{150} = 30+4k$$
$$k = \frac{+10\sqrt{150}-30}{4}$$
$$= \frac{5\sqrt{150}-15}{2}$$
$$= \frac{25\sqrt{6}-15}{2}$$
$$\doteq 23.1$$

It is reasonable that there are two values for k since one would lead to a line 10 units above the plane and the other would lead to a line 10 units below the plane.

Chapter 8 Section 5 **Question 18 Page 481**

a) Draw a diagram.



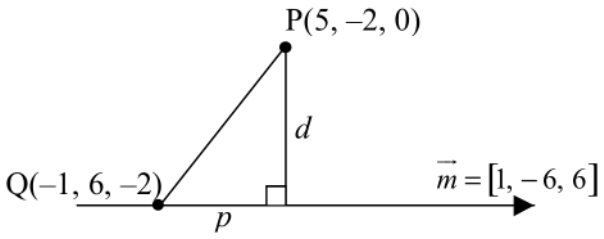
$$\left|\overrightarrow{PQ}\right| = \left|[3, -3, 6]\right|$$
$$= \sqrt{3^2 + (-3)^2 + 6^2}$$
$$= \sqrt{54}$$

$$p = \left|\text{proj}_{\vec{m}}\overrightarrow{PQ}\right|$$
$$= \left|\frac{\overrightarrow{PQ}\cdot\vec{m}}{|\vec{m}|}\right|$$
$$= \left|\frac{[3, -3, 6]\cdot[2, -5, 2]}{|[2, -5, 2]|}\right|$$
$$= \frac{33}{\sqrt{33}}$$

Use the Pythagorean theorem,

$$d^2 = \left|\overrightarrow{PQ}\right|^2 - p^2$$
$$d^2 = 54 - 33$$
$$d^2 = 21$$
$$d = \sqrt{21}$$
$$d \doteq 4.58$$

b) Draw a diagram.



$$\left|\overrightarrow{PQ}\right| = \left|[-8, 7, 5]\right|$$
$$= \sqrt{(-8)^2 + 7^2 + 5^2}$$
$$= \sqrt{138}$$

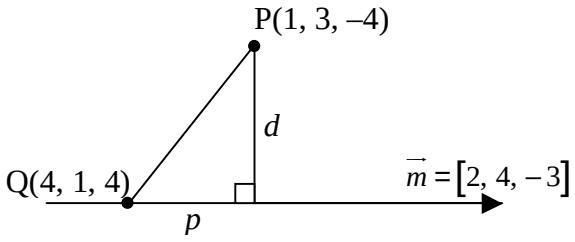
$$p = \left|\text{proj}_{\vec{m}}\overrightarrow{PQ}\right|$$
$$= \left|\frac{\overrightarrow{PQ} \cdot \vec{m}}{\left|\vec{m}\right|}\right|$$
$$= \left|\frac{[-8, 7, 5] \cdot [4, -1, 5]}{\left|[4, -1, 5]\right|}\right|$$
$$= \frac{14}{\sqrt{42}}$$

Use the Pythagorean theorem.

$$d^2 = \left|\overrightarrow{PQ}\right|^2 - p^2$$
$$d^2 = 138 - \frac{14}{3}$$
$$d^2 = \frac{400}{3}$$
$$d = \sqrt{\frac{400}{3}}$$
$$d \doteq 11.55$$

Chapter 8 Section 5 **Question 19 Page 481**

a) Draw a diagram.



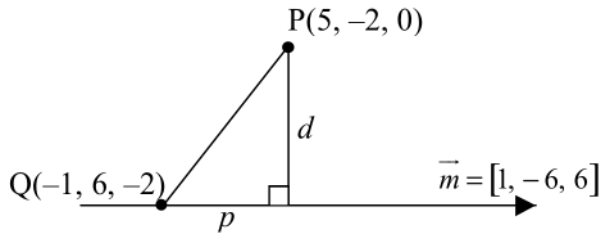
$$\left|\overrightarrow{PQ}\right| = \left|[3, -2, 8]\right|$$
$$= \sqrt{3^2 + (-2)^2 + 8^2}$$
$$= \sqrt{77}$$

$$p = \left|\text{proj}_{\vec{m}} \overrightarrow{PQ}\right|$$
$$= \left|\frac{\overrightarrow{PQ} \cdot \vec{m}}{|\vec{m}|}\right|$$
$$= \left|\frac{[3, -2, 8] \cdot [2, 4, -3]}{\left|[2, 4, -3]\right|}\right|$$
$$= \frac{26}{\sqrt{29}}$$

Use the Pythagorean theorem.

$$d^2 = \left|\overrightarrow{PQ}\right|^2 - p^2$$
$$d^2 = 77 - \frac{676}{29}$$
$$d^2 = \frac{1557}{29}$$
$$d = \sqrt{\frac{1557}{29}}$$
$$d \doteq 7.33$$

b) Draw a diagram.



$$\left|\overrightarrow{PQ}\right| = \left|[-6, 8, -2]\right|$$
$$= \sqrt{(-6)^2 + 8^2 + (-2)^2}$$
$$= \sqrt{104}$$

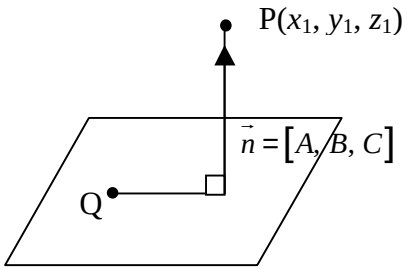
$$p = \left|\text{proj}_{\vec{m}} \overrightarrow{PQ}\right|$$
$$= \left|\frac{\overrightarrow{PQ} \cdot \vec{m}}{|\vec{m}|}\right|$$
$$= \left|\frac{[-6, 8, -2] \cdot [1, -6, 6]}{\left|[1, -6, 6]\right|}\right|$$
$$= \frac{66}{\sqrt{73}}$$

Use the Pythagorean theorem.

$$d^2 = \left|\overrightarrow{PQ}\right|^2 - p^2$$
$$d^2 = 104 - \frac{4356}{73}$$
$$d^2 = \frac{3236}{73}$$
$$d = \sqrt{\frac{3236}{73}}$$
$$d \doteq 6.66$$

Chapter 8 Section 5 **Question 20 Page 481**

Draw a diagram. Let Q(x, y, z) be any point on the plane; that is, any point for which $Ax + By + Cz + D = 0$.



$$\begin{aligned}
\left|\overrightarrow{PQ}\right| &= \left|\text{proj}_{\vec{n}}\overrightarrow{PQ}\right| \\
&= \left|\frac{\overrightarrow{PQ}\cdot\vec{n}}{|\vec{n}|}\right| \\
&= \left|\frac{[x - x_1,\ y - y_1,\ z - z_1]\cdot[A,\ B,\ C]}{\sqrt{A^2 + B^2 + C^2}}\right| \\
&= \left|\frac{Ax - Ax_1 + By - By_1 + Cz - Cz_1}{\sqrt{A^2 + B^2 + C^2}}\right| \\
&= \left|\frac{Ax + By + Cz - Ax_1 - By_1 - Cz_1}{\sqrt{A^2 + B^2 + C^2}}\right| \\
&= \frac{|-Ax_1 - By_1 - Cz_1 - D|}{\sqrt{A^2 + B^2 + C^2}}
\end{aligned}$$

Note that the numerator must be positive. The formula can be more conveniently written as;

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Chapter 8 Section 5 **Question 21 Page 481**

$$[3, 7, -2]\times[a, 3, b] = [-1, k, -5]$$
$$[7b + 6, -2a - 3b, 9 - 7a] = [-1, k, -5]$$

Equating like coefficients,

$7b + 6 = -1$ $\quad$ $9 - 7a = -5$ $\quad$ $-2a - 3b = k$

$7b = -7$ $\quad$ $7a = 14$ $\quad$ $k = -2(2) - 3(-1)$

$b = -1$ $\quad$ $a = 2$ $\quad$ $k = -1$

Chapter 8 Section 5 **Question 22 Page 481**

$k = \log_{10} 2$ and $m = \log_{10} 3$ are given.

$$\begin{aligned} 3 - 2k + m &= 3 - 2\log_{10} 2 + \log_{10} 3 \\ &= \log_{10} 1000 - \log_{10} 4 + \log_{10} 3 \\ &= \log_{10}\left(\frac{1000(3)}{4}\right) \\ &= \log_{10} 750 \end{aligned}$$

Therefore, $n = 750$.

Chapter 8 Section 6 **Intersections of Planes**

Chapter 8 Section 6 **Question 1 Page 491**

a) Use elimination.

$$\begin{aligned} x + 5y - 3z - 8 &= 0 && ① \\ 5y + 10z - 20 &= 0 && 5② \\ \hline x - 13z + 12 &= 0 && ③ \qquad ①–5② \end{aligned}$$

Let $z = t$.
③ becomes $x = 13t - 12$.
② becomes $5y + 10t - 20 = 0$.

The parametric equations for the line are:

$$x = -12 + 13t$$
$$y = 4 - 2t$$
$$z = t$$

The vector equation for the line is $[x, y, z] = [-12, 4, 0] + t[13, -2, 1],\ t \in \mathbb{R}$.

b) Use elimination.

$$\begin{aligned} 5x - 4y + z - 3 &= 0 && ① \\ x + 3y - 9 &= 0 && ② \\ 5x - 4y + z - 3 &= 0 && ① \\ 5x + 15y - 45 &= 0 && 5② \\ \hline -19y + z + 42 &= 0 && ③ \qquad ①–5② \end{aligned}$$

Let $y = t$.
③ becomes $z = 19t - 42$.
② becomes $x + 3t - 9 = 0$.

The parametric equations for the line are:

$$x = 9 - 3t$$
$$y = t$$
$$z = -42 - 19t$$

The vector equation for the line is $[x, y, z] = [9, 0, -42] + t[-3, 1, 19],\ t \in \mathbb{R}$.

c) Use elimination.

$2x - y + z - 22 = 0$ ①

$x - 11y + 2z - 8 = 0$ ②

$2x - y + z - 22 = 0$ ①

$2x - 22y + 4z - 16 = 0$ 2②

$21y - 3z - 6 = 0$ ③ ①–2②

Let $z = t$.

③ becomes:

$21y - 3t - 6 = 0$

$21y = 6 + 3t$

$y = \frac{6 + 3t}{21}$

$y = \frac{2 + t}{7}$

① becomes:

$2x - \left(\frac{2 + t}{7}\right) + t - 22 = 0$

$14x - 2 - t + 7t - 154 = 0$

$x = \frac{156 - 6t}{14}$

The parametric equations for the line are

$x = \frac{78}{7} - \frac{3}{7}t$

$y = \frac{2}{7} + \frac{1}{7}t$

$z = t$

The vector equation for the line is $[x, y, z] = \left[\frac{78}{7}, \frac{2}{7}, 0\right] + t[-3, 1, 7],\ t \in \mathbb{R}$.

Note that the direction vector for the line, $\left[-\frac{3}{7}, \frac{1}{7}, 1\right]$, has been replaced by the (simpler) scalar multiple $[-3, 1, 7]$.

d) Use elimination.

$3x + y - 5z - 7 = 0$ ①

$2x - y - 21z + 33 = 0$ ②

$5x - 26z + 26 = 0$ ③ ①+②

Let $z = t$.

③ becomes:

$5x - 26t + 26 = 0$

$5x = 26t - 26$

$x = \frac{26t - 26}{5}$

① becomes:

$3\left(\frac{26t - 26}{5}\right) + y - 5t - 7 = 0$

$y = 5t + 7 + \frac{78 - 78t}{5}$

$y = \frac{113 - 53t}{5}$

The parametric equations for the line are

$x = -\frac{26}{5} + \frac{26}{5}t$

$y = \frac{113}{5} - \frac{53}{5}t$

$z = t$

The vector equation for the line is $[x, y, z] = \left[-\frac{26}{5}, \frac{113}{5}, 0\right] + t[26, -53, 5],\ t \in \mathbb{R}$.

Note that the direction vector for the line, $\left[\frac{26}{5}, -\frac{53}{5}, 1\right]$, has been replaced by the (simpler) scalar multiple $[26, -53, 5]$.

Chapter 8 Section 6 **Question 2 Page 491**

a) Check the normal vectors.

$2[2, 3, -7] = [4, 6, -14]$. The normals are parallel and so the planes are parallel.

Check if they are coincident or distinct.

Eliminate one of the variables. Eliminate x.

$$2x + 3y - 7z - 2 = 0 \quad ①$$
$$4x + 6y - 14z - 8 = 0 \quad ②$$
$$4 = 0 \quad 2①–②$$

This equation can never be true. There is no solution to the system of equations.

The planes are parallel and distinct.

b) Check the normal vectors.

$4[3, 9, -6] = 3[4, 12, -8]$. The normals are parallel and so the planes are parallel.

Check if they are coincident or distinct.

Eliminate one of the variables. Eliminate x.

$$3x + 9y - 6z - 24 = 0 \quad ①$$
$$4x + 12y - 8z - 32 = 0 \quad ②$$
$$0 = 0 \quad 4①–3②$$

This equation is always true. The planes intersect for every value of x, y, and z that satisfy the equations.

Therefore the planes are coincident.

c) Check the normal vectors.

$6[4, -12, -16] = -4[-6, 18, 24]$. The normals are parallel and so the planes are parallel.

However, notice that 6 times equation ① equals –4 times equation ②. Therefore, the planes are coincident.

d) Check the normal vectors.

$3[1, -2, 2.5] = [3, -6, 7.5]$. The normals are parallel and so the planes are parallel.

However notice that 3 times equation ① equals equation ②. Therefore, the planes are coincident.

Chapter 8 Section 6 **Question 3 Page 491**

a) Substitute the parametric equations for the line in the plane.

$$(10+3t)+2(5+t)-(16+15t)-4=0$$
$$10+3t+10+2t-16-5t-4=0$$
$$0t=0$$

This equation is true for all values of t. therefore all points on the line are also on the plane.

b) Substitute the parametric equations for the line in the plane.

$$9(10+3t)-2(5+t)-5(16+5t)=0$$
$$90+27t-10-2t-80-25t=0$$
$$0t=0$$

This equation is true for all values of t. therefore all points on the line are also on the plane.

Chapter 8 Section 6 **Question 4 Page 491**

a) Eliminate one of the variables from two pairs of equations. Eliminate y.

$$\begin{array}{lll} x+y+z=7 & ① & \\ \underline{2x+y+3z=17} & ② & \\ -x-2z=-10 & ④ & ①-② \end{array}$$

$$\begin{array}{lll} x+y+z=7 & ① & \\ \underline{2x-y-2z=-5} & ③ & \\ 3x-z=2 & ⑤ & ①+③ \end{array}$$

Solve equations ④ and ⑤ for x and z.

$$\begin{array}{ll} -x-2z=-10 & ④ \\ \underline{6x-2z=4} & 2⑤ \\ -7x=-14 & ④-2⑤ \\ x=2 & \end{array}$$

Substitute $x=2$ in ⑤.

$$3(2)-z=2$$
$$z=4$$

Substitute $x=2$ and $z=4$ in ①.

$$2+y+4=7$$
$$y=1$$

The planes intersect at the point $(2,1,4)$.

b) Eliminate one of the variables from two pairs of equations. Eliminate x.

$$\begin{aligned} 2x + y + 4z &= 15 && ① \\ 2x + 3y + z &= -6 && ② \\ \hline -2y + 3z &= 21 && ④ \qquad ①-② \end{aligned}$$

$$\begin{aligned} 2x + y + 4z &= 15 && ① \\ 2x - y + 2z &= 12 && ③ \\ \hline 2y + 2z &= 3 && ⑤ \qquad ①-③ \end{aligned}$$

Solve equations ④ and ⑤ for y and z.

$$\begin{aligned} -2y + 3z &= 21 && ④ \\ 2y + 2z &= 3 && ⑤ \\ \hline 5z &= 24 && ④+⑤ \\ z &= 4.8 \end{aligned}$$

Substitute $z = 4.8$ in ⑤.

$$\begin{aligned} 2y + 2(4.8) &= 3 && ⑤ \\ 2y &= -6.6 \\ y &= -3.3 \end{aligned}$$

Substitute $y = -3.3$ and $z = 4.8$ in ①.

$$\begin{aligned} 2x + (-3.3) + 4(4.8) &= 15 \\ 2x &= -0.9 \\ x &= -0.45 \end{aligned}$$

The planes intersect at the point (–0.45, –3.3, 4.8).

c) Eliminate one of the variables from two pairs of equations. Eliminate y.

$5x - 2y - 7z = 19$ ①

$2x - 2y + 2z = 16$ 2②

$3x - 9z = 3$ ④ ①–2②

$x - 3z = 1$ ④ simplified

$10x - 4y - 14z = 38$ 2①

$3x + 4y + z = 1$ ③

$13x - 13z = 39$ ⑤ 2①+③

$x - z = 3$ ⑤ simplified

Solve equations ④ and ⑤ for y and z.

$x - 3z = 1$ ④

$x - z = 3$ ⑤

$-2z = -2$ ④–⑤

$z = 1$

Substitue z = 1 in ⑤.

$x - 3(1) = 1$ ⑤

$x = 4$

Substitute $x = 4$ and $z = 1$ in ①.

$5(4) - 2y - 7(41) = 19$

$-2y = 6$

$y = -3$

The planes intersect at the point $(4, -3, 1)$.

d) Eliminate one of the variables from two pairs of equations. Eliminate x.

$$\begin{aligned} 2x-5y-z&=9 && ① \\ 2x+4y+4z&=-26 && 2② \\ \hline -9y-5z&=35 && ④ \qquad ①-2② \end{aligned}$$

$$\begin{aligned} 2x-5y-z&=9 && ① \\ 2x+8y+3z&=-19 && ③ \\ \hline -13y-4z&=28 && ⑤ \qquad ①-③ \end{aligned}$$

Solve equations ④ and ⑤ for y and z.

$$\begin{aligned} -36y-20z&=140 && 4④ \\ -65y-20z&=140 && 5⑤ \\ \hline 29y&=0 && 4④-5⑤ \\ y&=0 \end{aligned}$$

Substitute $y = 0$ into ⑤.

$$\begin{aligned} -13(0)-4z&=28 && ⑤ \\ z&=-7 \end{aligned}$$

Substitute $y = 0$ and $z = -7$ in ①.

$$\begin{aligned} 2x-5(0)-(-7)&=9 \\ 2x&=2 \\ x&=1 \end{aligned}$$

The planes intersect at the point $(1, 0, -7)$.

Chapter 8 Section 6 **Question 5 Page 491**

a) Since 2 times equation ① equals equation ③, these planes are identical. Use only equations ① and ②.
Eliminate one of the variables. Eliminate y.

$$\begin{aligned} 4x+2y+2z&=14 && 2① \\ \underline{4x+3y-3z&=13} && ② \\ -y+5z&=1 && ③ \qquad 2①-② \end{aligned}$$

Let $z = t$.
③ becomes $y = -1 + 5t$.
① becomes:

$$\begin{aligned} 2x+(-1+5t)+t&=7 \\ 2x&=8-6t \\ x&=4-3t \end{aligned}$$

The parametric equations for the line are:

$$\begin{aligned} x&=4-3t \\ y&=-1+5t \\ z&=t \end{aligned}$$

The vector equation for the line is $[x, y, z]=[4, -1, 0]+t[-3, 5, 1],\ t \in \mathbb{R}$.

b) Eliminate one of the variables. Eliminate x.

$$\begin{aligned} 3x+9y-3z&=12 && 3① \\ \underline{3x+8y-4z&=4} && ② \\ y+z&=8 && ④ \qquad 3①-② \end{aligned}$$

$$\begin{aligned} x+3y-z&=4 && ① \\ \underline{x+2y-2z&=-4} && ③ \\ y+z&=8 && ⑤ \qquad ①-③ \end{aligned}$$

Since ④ and ⑤ are identical, let $z = t$.
④ becomes $y = 8 - t$.
① becomes:

$$\begin{aligned} x+3(8-t)-t&=4 \\ x&=-20+4t \end{aligned}$$

The parametric equations for the line are:

$$\begin{aligned} x&=-20+4t \\ y&=8-t \\ z&=t \end{aligned}$$

The vector equation for the line is $[x, y, z]=[-20, 8, 0]+t[4, -1, 1],\ t \in \mathbb{R}$.

c) Eliminate one of the variables. Eliminate x.

$x + 9y + 3z = 23$ ①

$x + 15y + 3z = 29$ ②

$-6y = -6$ ①–②

$y = 1$ ④

$4x + 36y + 12z = 92$ 4①

$4x - 13y + 12z = 43$ ③

$49y = 49$ ⑤ 4①–③

$y = 1$

Since ④ and ⑤ are identical, let $z = t$.
① becomes:
$x + 9(1) + 3t = 23$

$x = 14 - 3t$

The parametric equations for the line are:
$x = 14 - 3t$

$y = 1$

$z = t$

The vector equation for the line is $[x, y, z] = [14, 1, 0] + t[-3, 0, 1],\ t \in \mathbb{R}$.

d) Since 2 times equation ① equals equation ③, these planes are identical. Use only equations ① and ②. Eliminate one of the variables. Eliminate x.

$x - 6y + z = -1$ ①

$x - y = 5$ ②

$-5y + z = -6$ ③ ①–②

Let $y = t$.
③ becomes $z = -6 + 5t$
② becomes:
$x - (t) = 5$

$x = 5 + t$

The parametric equations for the line are:
$x = 5 + t$

$y = t$

$z = -6 + 5t$

The vector equation for the line is $[x, y, z] = [5, 0, -6] + t[1, 1, 5],\ t \in \mathbb{R}$.

Chapter 8 Section 6 **Question 6 Page 492**

a) $\vec{n}_1 = [3, 15, -9]$; $\vec{n}_2 = [6, 30, -18]$; $\vec{n}_3 = [5, 25, -15]$

$\vec{n}_1 = \frac{1}{2}\vec{n}_2$

$= \frac{3}{5}\vec{n}_3$

So, the three planes are parallel.

$① = \frac{1}{2} \times ②$

$\neq \frac{3}{5} \times ③$

So, the first two planes are identical and the third plane is parallel but distinct from the other two planes.

There are no points of intersection.

b) $\vec{n}_1 = [2, -1, 4]$; $\vec{n}_2 = [6, -3, 12]$; $\vec{n}_3 = [4, -2, 8]$

$\vec{n}_1 = \frac{1}{3}\vec{n}_2$

$= \frac{1}{2}\vec{n}_3$

So, the three planes are parallel.

$① = \frac{1}{3} \times ②$

$= \frac{1}{2} \times ③$

So, the three planes are coincident.

There is a plane of intersection points.

c) $\vec{n}_1 = [3, 2, -1]$; $\vec{n}_2 = [12, 8, -4]$; $\vec{n}_3 = [18, 12, -6]$

$\vec{n}_1 = \frac{1}{4}\vec{n}_2$

$= \frac{1}{6}\vec{n}_3$

So, the three planes are parallel.

$① \neq \frac{1}{4} \times ②$

$\neq \frac{1}{6} \times ③$

So, all three planes are parallel but distinct.

There are no points of intersection.

d) $\vec{n}_1 = [8, 4, 6]$; $\vec{n}_2 = [12, 6, 9]$; $\vec{n}_3 = [3, 2, 4]$

$\vec{n}_1 = \frac{2}{3}\vec{n}_2$

So, the first two planes are parallel.

$① \neq \frac{2}{3} \times ②$

So, the first two planes are parallel but distinct. They are intersected by a non-parallel third plane.

There are no points of intersection for all three planes.

Chapter 8 Section 6 **Question 7 Page 492**

a) First examine the normals. None are scalar multiples of each other. An intersection point is possible. Eliminate one of the variables from two pair of equations. Eliminate z.

$15x + 5y - 10z = 90$ 5①

$6x - 4y + 10z = -10$ ②

$21x + y = 80$ ④ 5①+②

$6x - 4y + 10z = -10$ ②

$3x - 5y + 10z = 10$ ③

$3x + y = -20$ ⑤ ②–③

Solve equations ④ and ⑤ for x and y.

$21x + y = 80$ ④

$3x + y = -20$ ⑤

$18x = 100$ ④–⑤

$x = \frac{50}{9}$

Substitute $x = \frac{50}{9}$ in ⑤.

$$3\left(\frac{50}{9}\right) + y = -20 \qquad ⑤$$

$$y = -\frac{110}{3}$$

Substitute $x = \frac{50}{9}$ and $y = -\frac{110}{3}$ in ①.

$$3\left(\frac{50}{9}\right) + \left(-\frac{110}{3}\right) - 2z = 18$$

$$2z = \frac{50}{3} - \frac{110}{3} - 18$$

$$z = -19$$

The system of equations is consistent; the planes intersect at the point $\left(\frac{50}{9}, -\frac{110}{3}, -19\right)$.

b) $\vec{n}_1 = [2, 5, -3]$; $\vec{n}_2 = [3, -2, 3]$; $\vec{n}_3 = [4, 10, -6]$

Since $\vec{n}_1 = \frac{1}{2}\vec{n}_3$, but ① $\neq \frac{1}{2}$③, the first and third planes are parallel but distinct.

The system is inconsistent.

c) $\vec{n}_1 = [2, -3, 2]$; $\vec{n}_2 = [5, -15, 5]$; $\vec{n}_3 = [-4, 6, -4]$

Since $\vec{n}_1 = -\frac{1}{2}\vec{n}_3$, but ① $\neq -\frac{1}{2}$③, the first and third planes are parallel but distinct.

The system is inconsistent.

d) $\vec{n}_1 = [3, -5, -5]$; $\vec{n}_2 = [-6, 10, 10]$; $\vec{n}_3 = [15, -25, -25]$

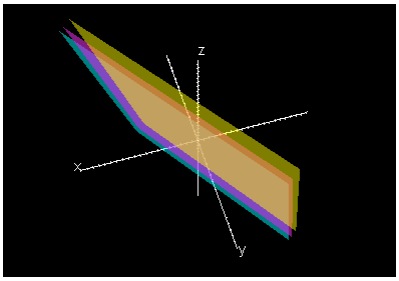
Since $\vec{n}_1 = \frac{1}{5}\vec{n}_3$, but ① $\neq -\frac{1}{5}$③, the first and third planes are parallel but distinct.

Note that the first two planes are coincident since ① $= -\frac{1}{2}$②.

The system is inconsistent.

Chapter 8 Section 6 **Question 8 Page 492**

Answers may vary. Question 6, part c) is verified below using the 3D Grapher.



Chapter 8 Section 6 **Question 9 Page 492**

a) $\vec{n}_1 = [1, 1, 1]; \ \vec{n}_2 = [2, 1, 3]; \ \vec{n}_3 = [2, -1, -2]$

For perpendicularity of two planes, the dot product of their normals must be zero.
In this case, $\vec{n}_1 \cdot \vec{n}_2 = 6, \ \vec{n}_1 \cdot \vec{n}_3 = -1,$ and $\vec{n}_2 \cdot \vec{n}_3 = -3$.
There are no orthogonal planes.

b) $\vec{n}_1 = [2, 1, 4]; \ \vec{n}_2 = [2, 3, 1]; \ \vec{n}_3 = [2, -1, 2]$

For perpendicularity of two planes, the dot product of their normals must be zero.
In this case, $\vec{n}_1 \cdot \vec{n}_2 = 11, \ \vec{n}_1 \cdot \vec{n}_3 = 11,$ and $\vec{n}_2 \cdot \vec{n}_3 = 3$.
There are no orthogonal planes.

c) $\vec{n}_1 = [5, -2, -7]; \ \vec{n}_2 = [1, -1, 1]; \ \vec{n}_3 = [3, 4, 1]$

For perpendicularity of two planes, the dot product of their normals must be zero.
In this case, $\vec{n}_1 \cdot \vec{n}_2 = 0, \ \vec{n}_1 \cdot \vec{n}_3 = 0,$ and $\vec{n}_2 \cdot \vec{n}_3 = 0$.
Each pair of two planes is orthogonal.

d) $\vec{n}_1 = [2, -5, -1]; \ \vec{n}_2 = [1, 2, 2]; \ \vec{n}_3 = [2, 8, 3]$

For perpendicularity of two planes, the dot product of their normals must be zero.
In this case, $\vec{n}_1 \cdot \vec{n}_2 = -10, \ \vec{n}_1 \cdot \vec{n}_3 = -39,$ and $\vec{n}_2 \cdot \vec{n}_3 = 24$.
There are no orthogonal planes.

Chapter 8 Section 6 **Question 10 Page 492**

a) $\begin{aligned} \vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 &= [2, 1, 1] \cdot [4, 3, -3] \times [4, 2, 2] \\ &= [2, 1, 1] \cdot [12, -20, -4] \\ &= 0 \end{aligned}$

b) $\begin{aligned} \vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 &= [1, 3, -1] \cdot [3, 8, -4] \times [1, 2, -2] \\ &= [1, 3, -1] \cdot [-8, 2, -2] \\ &= 0 \end{aligned}$

c) $\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 = [1, 9, 3] \cdot [1, 15, 3] \times [4, -13, 12]$

$= [1, 9, 3] \cdot [219, 0, -73]$

$= 0$

d) $\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 = [1, -6, 1] \cdot [1, -1, 0] \times [2, -12, 2]$

$= [1, -6, 1] \cdot [-2, -2, -10]$

$= 0$

Since the triple scalar product is always zero, the normals for planes intersecting in a line will always be coplanar.

Chapter 8 Section 6 **Question 11 Page 492**

a) First examine the normals. None are scalar multiples of each other.
Check if the normals are coplanar.

$\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 = [3, 2, -1] \cdot [2, 1, -2] \times [2, -3, 4]$

$= [3, 2, -1] \cdot [-2, -12, -8]$

$= -22$

The normals are not coplanar; there is an intersection point.

Eliminate one of the variables from two pair of equations. Eliminate z.

$6x + 4y - 2z = -4$ $2①$

$2x + y - 2z = 7$ $②$

$4x + 3y = -11$ $④$ $2①-②$

$12x + 8y - 4z = -8$ $4①$

$2x - 3y + 4z = -3$ $③$

$14x + 5y = -11$ $⑤$ $4①+③$

Solve equations ④ and ⑤ for x and y.

$20x + 15y = -55$ $5④$

$42x + 15y = -33$ $3⑤$

$-22x = -22$ $5④-3⑤$

$x = 1$

Substitute $x = 1$ in ⑤.

$14(1) + 5y = -11$ ⑤

$y = -5$

Substitute $x = 1$ and $y = -5$ in ①.

$3(1) + 2(-5) - z = -2$

$z = -5$

The three planes intersect at the point (1, –5, –5).

b) $\vec{n}_1 = [3, -4, 2]$; $\vec{n}_2 = [6, -8, 4]$; $\vec{n}_3 = [15, -20, 10]$

$10\vec{n}_1 = 5\vec{n}_2$

$= 2\vec{n}_3$

So, the normals are parallel.

Examine the equations to determine if the planes are coincident or parallel and distinct.

$30x - 40y + 20z = 10$ 10①

$30x - 40y + 20z = 50$ 5②

$30x - 40y + 20z = -6$ 2③

The equations are not scalar multiples of each other.
The planes are parallel but distinct.

c) $\vec{n}_1 = [2, 1, 6]$; $\vec{n}_2 = [5, 1, -3]$; $\vec{n}_3 = [3, 2, 15]$

By inspection, none of the normals are parallel.

Check if the normals are coplanar.

$\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 = [2, 1, 6] \cdot [5, 1, -3] \times [3, 2, 15]$

$= [2, 1, 6] \cdot [21, -84, 7]$

$= 0$

Since the triple scalar product is zero, the normals are coplanar and the planes intersect either in a line or not at all.

Solve the system algebraically to determine if there is a solution.

$2x + y + 6z = 5$ ①

$5x + y - 3z = 1$ ②

$3x + 2y + 15z = 9$ ③

$-3x + 9z = 4$ ④ ①–②

$7x - 21z = -7$ ⑤ 2②–③

$-21x + 63z = 28$ ⑥ 7④

$21x - 63z = -21$ ⑦ 3⑤

$0 = 7$ ⑥+⑦

The system has no solution. The planes intersect in pairs only.

d) $\vec{n}_1 = [1, -1, 1]$; $\vec{n}_2 = [1, 1, 3]$; $\vec{n}_3 = [2, -5, -1]$

By inspection, none of the normals are parallel.

Check if the normals are coplanar.

$$\begin{aligned}\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 &= [1, -1, 1] \cdot [1, 1, 3] \times [2, -5, -1] \\ &= [1, -1, 1] \cdot [14, 7, -7] \\ &= 0\end{aligned}$$

Since the triple scalar product is zero, the normals are coplanar and the planes intersect either in a line or not at all.

Solve the system algebraically to determine if there is a solution.

$$\begin{aligned}
x - y + z &= 20 && ① \\
x + y + 3z &= -4 && ② \\
2x - 5y - z &= -6 && ③ \\
-2y - 2z &= 24 && ④ \quad ①-② \\
7y + 7z &= -2 && ⑤ \quad 2②-③ \\
-14x - 14z &= 168 && ⑥ \quad 7④ \\
14x + 14z &= -4 && ⑦ \quad 2⑤ \\
0 &= 164 && ⑥+⑦
\end{aligned}$$

The system has no solution. The planes intersect in pairs only.

e) $\vec{n}_1 = [4, 8, 4]$; $\vec{n}_2 = [5, 10, 5]$; $\vec{n}_3 = [3, -1, -4]$

Two of the normals are parallel since $5\vec{n}_1 = 4\vec{n}_2$

However, $5① \neq 4②$. The equations are not scalar multiples of each other.

The first planes are parallel but distinct and the third plane intersects both of them.
There are no solutions to the system.

f) First examine the normals. None are scalar multiples of each other.

Check if the normals are coplanar.

$$\begin{aligned}\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 &= [2, 2, 1] \cdot [5, 4, -4] \times [3, 5, -2] \\ &= [2, 2, 1] \cdot [12, -2, 13] \\ &= 33\end{aligned}$$

The normals are not coplanar; there is an intersection point.

Eliminate one of the variables from two pair of equations. Eliminate z.

$$\begin{array}{rll} 8x+8y+4z=40 & 4① & \\ \underline{5x+4y-4z=13} & ② & \\ 13x+12y=53 & ④ & 4①+② \end{array}$$

$$\begin{array}{rll} 4x+4y+2z=20 & 2① & \\ \underline{3x+5y-2z=6} & ③ & \\ 7x+9y=26 & ⑤ & 2①+③ \end{array}$$

Solve equations ④ and ⑤ for x and y.

$$\begin{array}{rll} 39x+36y=159 & 3④ & \\ \underline{28x+36y=104} & 4⑤ & \\ 11x=55 & 3④-4⑤ & \end{array}$$

$$x=5$$

Substitute $x = 5$ in ⑤.

$$7(5)+9y=26 \quad ⑤$$

$$y=-1$$

Substitute $x = 5$ and $y = -1$ in ①.

$$2(5)+2(-1)+z=10$$

$$z=2$$

The three planes intersect at the point (5, –1, 2).

g) First examine the normals. None are scalar multiples of each other.

Check if the normals are coplanar.

$$\begin{aligned} \vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 &= [2, 5, 0] \cdot [0, 4, 1] \times [7, 0, -4] \\ &= [2, 5, 0] \cdot [-16, 7, -28] \\ &= 3 \end{aligned}$$

The normals are not coplanar; there is an intersection point.

Eliminate one of the variables from two pair of equations. Eliminate z.

$$\begin{array}{rll} 16y+4z=4 & 4② & \\ \underline{7x-4z=1} & ③ & \\ 7x+16y=5 & ④ & 4②+③ \end{array}$$

Solve equations ① and ④ for x and y.

$14x + 35y = 7$ 7①

$14x + 32y = 10$ 2④

$3y = -3$ 7①–2④

$y = -1$

Substitute $y = -1$ in ① .

$2x + 5(-1) = 1$ ①

$x = 3$

Substitute $y = -1$ in ②.

$4(-1) + z = 1$

$z = 5$

The three planes intersect at the point (3, –1, 5).

h) These planes have an intersection point at the origin since the constant term for each equation is zero. Check to see if this point is unique or if the planes intersect in a line.

Check if the normals are coplanar.

$$\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 = [3, -2, -1] \cdot [1, 0, -1] \times [3, 2, -5]$$
$$= [3, -2, -1] \cdot [2, 2, 2]$$
$$= 0$$

The normals are coplanar and the planes intersect in a line.

Eliminate one of the variables from two pair of equations. Eliminate y.

$3x - 2y - z = 0$ ①

$3x + 2y - 5z = 0$ ③

$6x - 6z = 0$ ①+③

$x - z = 0$ ④

Equations ② and ④ are identical.

Let $z = t$.

$x - t = 0$ ④

$x = t$

Substitute $x = t$ and $z = t$ in ①

$3t - 2y - t = 0$

$y = t$

The planes intersect in the line $[x, y, z] = [0, 0, 0] + t[1, 1, 1]$, $t \in \mathbb{R}$.

i) First examine the normals. $3\vec{n}_2 = 4\vec{n}_3$. These planes are parallel.
However, $3② = 4③$ and therefore these planes are coincident.

Since the first plane is not parallel, the three planes will intersect in a line.
Since x does not appear in any equation, let $x = t$. Solve equations ① and ② for y and z.

$$\begin{aligned} 4y + 8z &= 44 && 4① \\ 4y - 4z &= -16 && ② \\ \hline 12z &= 60 && 4①-② \\ z &= 5 \end{aligned}$$

Substitute $z = 5$ in ①.

$$\begin{aligned} y + 2(5) &= 11 && ① \\ y &= 1 \end{aligned}$$

The planes intersect in the line $[x, y, z] = [0, 1, 5] + t[1, 0, 0],\ t \in \mathbb{R}$.

Chapter 8 Section 6 **Question 12 Page 492**

a) The planes intersect at the point (5, 10, –3).

b) The three planes intersect in a line. Let $z = t$. The equation of the line is
$[x, y, z] = [-4, 8, 0] + t[-1, 3, 1],\ t \in \mathbb{R}$.

c) There is no point on all three planes. The three planes intersect in pairs.

Chapter 8 Section 6 **Question 13 Page 493**

a) The planes can intersect at a point or a line.

b) The planes can intersect at one line (if the two parallel planes are coincident) or in pairs (if the two parallel planes are distinct). There are no intersection points for the system in the latter case.

c) The planes will either not intersect (if at least two of the planes are distinct) or have an infinite number of solutions (if they are all coincident).

Chapter 8 Section 6 **Question 14 Page 493**

a) Check the dot product of pairs of normals.

$$\vec{n}_1 \cdot \vec{n}_2 = [5, 1, 4] \cdot [1, 3, -2] = 0$$
$$\vec{n}_1 \cdot \vec{n}_3 = [5, 1, 4] \cdot [1, -1, -1] = 0$$
$$\vec{n}_2 \cdot \vec{n}_3 = [1, 3, -2] \cdot [1, -1, -1] = 0$$

The planes are mutually perpendicular and must intersect in a single point.

Using algebra, the solution is (1, –3, 4).

b) Check the dot product of pairs of normals.

$\vec{n}_1 \cdot \vec{n}_2 = [2, -6, 1] \cdot [7, 3, 4] = 0$

$\vec{n}_1 \cdot \vec{n}_3 = [2, -6, 1] \cdot [27, 1, -48] = 0$

$\vec{n}_2 \cdot \vec{n}_3 = [7, 3, 4] \cdot [27, 1, -48] = 0$

The planes are mutually perpendicular and must intersect in a single point.

Using algebra, the solution is (3, 4, 21).

Chapter 8 Section 6 **Question 15 Page 493**

Solutions for Achievement Checks are shown in the Teacher Resource.

Chapter 8 Section 6 **Question 16 Page 493**

a) All three equations have no y-coordinate. Any solution will have $y = t$ and x and z values determined by the three equations.

Solve ① and ② for x and z.

$2x - 6z = -6$ $2①$

$2x - z = 4$ ②

$-5z = -10$ $2① - ②$

$z = 2$

Substitute z = 2 in ② .

$2x - 2 = 4$ ②

$x = 3$

For consistency, $x = 3$ and $z = 2$ must be a solution to ③.

$\text{L.S.} = 3x + 5z$

$= 3(3) + 5(2)$

$= 19$

For a consistent, dependent system, change the constant term in ③ to 19.

b) Find the intersection of the first two planes. This will be a line since their normals are not scalar multiples.

Solve using algebra.

Let $z = t$.

$y + 5t = 20$ ②

$y = 20 - 5t$

Substitute $y = 20 - 5t$ in ① .

$2x + 8(20 - 5t) = -6$ ①

$x = -83 + 20t$

The line of intersection is $[x, y, z] = [-83, 20, 0] + t[20, -5, 1], \quad t \in \mathbb{R}$.

If the system is to be consistent, the parametric equations for the line must satisfy the third equation.

L.S. $= 3(-83 + 20t) + 12(20 - 5t) + 6t$ R.S. $= -9$

$= 6t - 9$

Clearly the z-term in the third equation must be $0z$ instead of $6z$ for the system to be consistent and dependent.

Chapter 8 Section 6 **Question 17 Page 493**

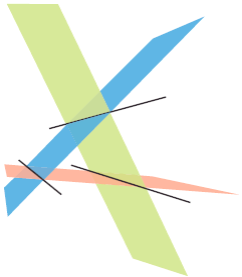
Answers are not unique for any parts of this question.
Sketches are included to aid visualization. They are not drawn to correct scale or orientation.

a) Choose normals that are identical but D terms that are not.
A possible set of planes is $x + y + z = 1$, $x + y + z = 4$, and $x + y + z = -11$.

b) A simple set of three planes is $x = 3$, $y = 1$, and $z = -9$.

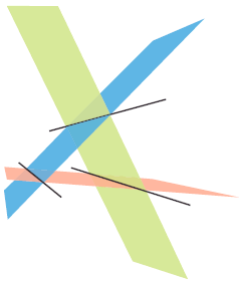
c) Choose three coplanar normals; $\vec{n}_3 = \vec{n}_1 + \vec{n}_2$ will work. Then choose $D_3 = D_1 + D_2$ for consistency of the system. A possible set of planes is $x + y + z = 1$, $x + 2y + 3z = 5$, and $2x + 3y + 4z = 6$.

d) Choose three coplanar normals; $\vec{n}_3 = \vec{n}_1 + \vec{n}_2$ will work. Then choose $D_3 \neq D_1 + D_2$ for inconsistency of the system. A possible set of planes is $x + y + z = 1$, $x + 2y + 3z = 5$, and $2x + 3y + 4z = 13$.

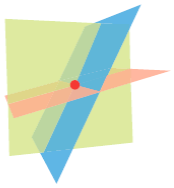


e) The normal vectors for the planes must be perpendicular to the direction vector of the line (dot product is zero).
Choose D terms for the planes so that the point (1, 3, –4) satisfies the equations.
A possible set of planes is $x - 4y = -11$, $9y - z = 31$, and $9x - 4z = 25$.

f) Choose three equations with no y-term. Choose planes with non-parallel normals to assure intersections.
A possible set of planes is $3x + z = 4$, $x + z = 2$, and $9x - 4z = 0$.



g) The normal vectors for the planes must be perpendicular to each other (dot product is zero). Choose D terms for the planes so that the point (–2, 4, –4) satisfies the equations.
A possible set of planes: $x = -2$, $y = 4$, and $z = -4$.



h) Choose planes with normals perpendicular to the z-axis, i.e., with no $-z$-term. Also points such as (0, 0, 0) must satisfy the equations.
A possible set of planes is $x = 0$, $y = 0$, and $3x - 2y = 0$.

Chapter 8 Section 6 **Question 18 Page 493**

The first three planes are parallel to the y-axis, forming a triangular prism.
The last two planes indicate that the prism is a right triangular prism and that the height is 14 units.

Find the intersection points of the planes with the base of the prism ($y = -4$).

$$\begin{aligned} x + z &= -3 \quad ① \\ \underline{10x - 3z = 22} \quad &② \\ 13z &= -52 \quad 10①-② \\ z &= -4 \\ x &= 1 \end{aligned}$$

$$\begin{aligned} x + z &= -3 \quad ① \\ \underline{4x - 9z = -38} \quad &③ \\ 13z &= 26 \quad 4①-② \\ z &= 2 \\ x &= -5 \end{aligned}$$

$$\begin{aligned} 10x - 3z &= 22 \quad ② \\ \underline{4x - 9z = -38} \quad &③ \\ 26x &= 104 \quad 3②-③ \\ x &= 4 \\ z &= 6 \end{aligned}$$

The points on the base are (1, –4, –4), (–5, –4, 2) and (4, –4, 6).

The lengths of the sides are:

$$\begin{aligned} \sqrt{((-5)-1)^2 + 0^2 + (2-(-4))^2} &= \sqrt{72} \\ &\doteq 8.49 \end{aligned}$$

$$\begin{aligned} \sqrt{(4-1)^2 + 0^2 + (6-(-4))^2} &= \sqrt{109} \\ &\doteq 10.44 \end{aligned}$$

$$\begin{aligned} \sqrt{(4-(-5))^2 + 0^2 + (6-2)^2} &= \sqrt{97} \\ &\doteq 9.85 \end{aligned}$$

Find the area of the triangle using a convenient method (trigonometry, analytic geometry formula given on page 359 of text, Heron's formula).

Use Heron's formula: $A = \frac{1}{2}\sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$.

$$\begin{aligned} s &= \frac{8.49 + 10.44 + 9.85}{2} \\ &\doteq 14.39 \end{aligned}$$

$$\begin{aligned} A &= \sqrt{14.39(14.39-8.49)(14.39-10.44)(14.39-9.85)} \\ &\doteq 39.02 \end{aligned}$$

Volume = (39)(14)
= 546
The volume is 546 units^3.

Chapter 8 Section 6 **Question 19 Page 493**

First eliminate one variable from three of the equations. Eliminate w.

$$2w + x + y + 2z = -9 \quad ①$$
$$w - x + y + 2z = 1 \quad ②$$
$$2w + x + 2y - 3z = 18 \quad ③$$
$$3w + 2x + 3y + z = 0 \quad ④$$

$$3x - y - 2z = -11 \quad ⑤ \qquad ①-2②$$
$$-y + 5z = -27 \quad ⑥ \qquad ①-③$$
$$-x - 3y + 4z = -27 \quad ⑦ \qquad 3①-2④$$

Then eliminate x from two of the equations.

$$-y + 5z = -27 \quad ⑧ \qquad ①-③$$
$$-10y + 10z = -92 \quad ⑨ \qquad ⑤+3⑦$$

Then eliminate y from one equation.

$$40z = -178 \quad ⑩ \qquad 10⑧-⑨$$
$$z = -4.45$$

Substitute $z = -4.45$ in ⑧.

$$-y + 5(-4.45) = -27$$
$$y = 4.75$$

Substitute $y = 4.75$ and $z = -4.45$ in ⑤

$$3x - 4.75 - 2(-4.45) = -11$$
$$x = -5.05$$

Lastly substitute $x = -5.05$, $y = 4.75$, and $z = -4.45$ in ①

$$2w + (-5.05) + 4.75 + 2(-4.45) = -9$$
$$w = 0.1$$

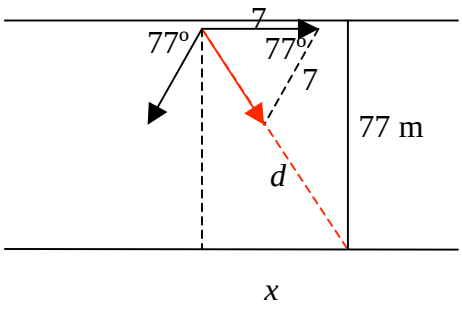
The point of intersection is (0.1, –5.05, 4.75, –4.45).

Chapter 8 Section 6 **Question 20 Page 493**

Many solutions are possible. One example is:

$$2x + y - 2z + 2t = 27$$
$$x - y + z + t = 4$$
$$x + y - z - t = 2$$
$$3x + y = 10$$

Chapter 8 Section 6 **Question 21 Page 493**



Calculate the resultant speed.

$$\left|\vec{R}\right|^2 = 7^2 + 7^2 - 2(7)(7)\cos 77^\circ$$

$$\left|\vec{R}\right| \doteq 8.72$$

Since the triangle is isosceles, the other tow angles in the resultant triangle are 51.5°.

The distance, d, the canoeist will travel is found using trigonometry.

$$\sin 51.5^\circ = \frac{77}{d}$$

$$d \doteq 98.4$$

The distance the canoeist is down the opposite bank, x, can be found by using the Pythagorean theorem.

$$x^2 + 77^2 = 98.4^2$$

$$x \doteq 61.3$$

She is approximately 61.3 m downstream to her original starting point.

Chapter 8 Section 6 **Question 22 Page 493**

$$V = \left|\vec{a} \cdot \vec{b} \times \vec{c}\right|$$

$$147 = \left|[1, 2, -3] \cdot [1, k, -3] \times [2, k, 1]\right|$$

$$147 = \left|[1, 2, -3] \cdot [4k, -7, -k]\right|$$

$$147 = \left|4k - 14 + 3k\right|$$

$$147 = \left|7k - 14\right|$$

This leads to two possibilities:

$147 = 7k - 14$ or $147 = -(7k - 14)$

$k = 23$ $147 = -7k + 14$

$k = -19$

Chapter 8 Extension **Solve Systems of Equations Using Matrices**

Chapter 8 Extension **Question 1 Page 500**

a) $\left[\begin{array}{ccc|c} 3 & 2 & 5 & 8 \\ 1 & -8 & 4 & 1 \\ -2 & 10 & 6 & 7 \end{array}\right]$

b) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 14 \\ 0 & 3 & -1 & 10 \\ 5 & 2 & 0 & 1 \end{array}\right]$

c) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -9 \end{array}\right]$

d) $\left[\begin{array}{ccc|c} 4 & 8 & 24 & 28 \\ 3 & 4 & 3 & -8 \\ 1 & -5 & -2 & -10 \end{array}\right]$

Chapter 8 Extension **Question 2 Page 500**

a) $\begin{aligned} x + 3y - 2z &= 5 \\ 8x - y + 4z &= -9 \\ 2x - 3y + 7z &= -6 \end{aligned}$

b) $\begin{aligned} -2x + z &= 10 \\ 6z &= 3 \\ 3x - y &= -10 \end{aligned}$

c) $\begin{aligned} 3x - 2y - z &= 6 \\ y - 2z &= 7 \\ 5z &= 15 \end{aligned}$

d) $\begin{aligned} 4x - y - 7z &= 9 \\ 5y &= 12 \\ 8z &= 2 \end{aligned}$

Chapter 8 Extension **Question 3 Page 500–501**

A variety of solutions are shown for this question.

a) $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 3 & 13 \\ 0 & 0 & 1 & 5 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$

From the third row,
$z = 5$.

From the second row,
$2y + 3z = 13$
$2y + 3(5) = 13$
$y = -1$

From the first row,
$x + 2y + z = 4$
$x + 2(-1) + 5 = 4$
$x = 1$

The solution is $(1, -1, 5)$.

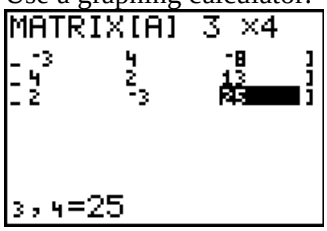
b) $\left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 0 & 3 & 2 & 24 \\ 0 & 1 & -3 & -20 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \rightarrow \left[\begin{array}{ccc|c} 2 & 0 & -3 & -24 \\ 0 & \boxed{3} & 2 & 24 \\ 0 & 0 & 11 & 84 \end{array}\right] \begin{array}{c} R_1 - R_2 \\ \\ R_2 - 3R_3 \end{array} \rightarrow \left[\begin{array}{ccc|c} 22 & 0 & 0 & -12 \\ 0 & 33 & 0 & 96 \\ 0 & 0 & \boxed{11} & 84 \end{array}\right] \begin{array}{c} 11R_1 + 3R_3 \\ 11R_2 - 2R_3 \\ R_3 \end{array}$

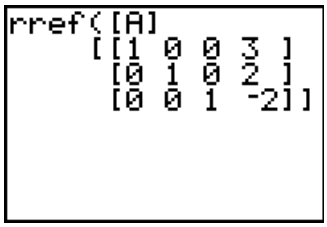
The solution is $\left(-\frac{6}{11}, \frac{32}{11}, \frac{84}{11}\right)$.

c) $\left[\begin{array}{ccc|c} 4 & 0 & 3 & -6 \\ 0 & 1 & -4 & 12 \\ 0 & 0 & 6 & -12 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \rightarrow \left[\begin{array}{ccc|c} 8 & 0 & 0 & 0 \\ 0 & 6 & 0 & 24 \\ 0 & 0 & \boxed{6} & -12 \end{array}\right] \begin{array}{c} 2R_1 - R_3 \\ 6R_2 + 4R_3 \\ R_3 \end{array}$

The solution is $(0, 4, -2)$.

d) Use a graphing calculator.





The solution is $(3, 2, -2)$.

Chapter 8 Extension **Question 4 Page 501**

a) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -2 & 3 & 15 \\ 1 & 3 & -1 & -4 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$

$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 4 \\ 0 & 3 & -2 & -11 \\ 0 & -2 & 2 & 8 \end{array}\right] \begin{array}{l} \\ R_1 - R_2 \\ R_1 - R_3 \end{array}$

$\left[\begin{array}{ccc|c} 3 & 0 & 5 & 23 \\ 0 & \boxed{3} & -2 & -11 \\ 0 & 0 & 2 & 2 \end{array}\right] \begin{array}{l} 3R_1 - R_2 \\ \\ 2R_2 + 3R_3 \end{array}$

$\left[\begin{array}{ccc|c} 6 & 0 & 0 & 36 \\ 0 & 3 & 0 & -9 \\ 0 & 0 & \boxed{2} & 2 \end{array}\right] \begin{array}{l} 2R_1 - 5R_3 \\ R_2 + R_3 \\ R_3 \end{array}$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1 \div 6 \\ R_2 \div 3 \\ R_3 \div 2 \end{array}$

The solution is (6, –3, 1).

b) $\left[\begin{array}{ccc|c} 1 & 1 & 3 & 10 \\ 2 & -1 & 2 & 10 \\ 3 & 2 & 1 & 20 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$

$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 3 & 10 \\ 0 & 3 & 4 & 10 \\ 0 & 1 & 8 & 10 \end{array}\right] \begin{array}{l} \\ 2R_1 - R_2 \\ 3R_1 - R_3 \end{array}$

$\left[\begin{array}{ccc|c} 3 & 0 & 5 & 20 \\ 0 & \boxed{3} & 4 & 10 \\ 0 & 0 & -20 & -20 \end{array}\right] \begin{array}{l} 3R_1 - R_2 \\ \\ R_2 - 3R_3 \end{array}$

$\left[\begin{array}{ccc|c} 3 & 0 & 5 & 20 \\ 0 & 3 & 4 & 10 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3 \div (-20) \end{array}$

$$\left[\begin{array}{ccc|c} 3 & 0 & 0 & 15 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right] \begin{array}{c} R_1 - 5R_3 \\ R_1 - 4R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} R_1 \div 3 \\ R_2 \div 3 \\ R_3 \end{array}$$

The solution is (5, 2, 1).

c) $\left[\begin{array}{ccc|c} 3 & 4 & -2 & 13 \\ 2 & -3 & 1 & 4 \\ 4 & 1 & -5 & 22 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$

$$\left[\begin{array}{ccc|c} \boxed{3} & 4 & -2 & 13 \\ 0 & 17 & -7 & 14 \\ 0 & 13 & 7 & -14 \end{array}\right] \begin{array}{c} \\ 2R_1 - 3R_2 \\ 4R_1 - 3R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 51 & 0 & -6 & 165 \\ 0 & \boxed{17} & -7 & 14 \\ 0 & 0 & -210 & 420 \end{array}\right] \begin{array}{c} 17R_1 - 4R_2 \\ R_2 \\ 13R_2 - 17R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 17 & 0 & -2 & 55 \\ 0 & 17 & -7 & 14 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} R_1 \div 3 \\ R_2 \\ R_3 \div -210 \end{array}$$

$$\left[\begin{array}{ccc|c} 17 & 0 & 0 & 51 \\ 0 & 17 & 0 & 0 \\ 0 & 0 & \boxed{1} & -2 \end{array}\right] \begin{array}{c} R_1 + 2R_3 \\ R_2 + 7R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} R_1 \div 17 \\ R_2 \div 17 \\ R_3 \end{array}$$

The solution is (3, 0, –2).

Chapter 8 Extension **Question 5 Page 501**

a) $\left[\begin{array}{ccc|c} 4 & 12 & -8 & 1 \\ 3 & 5 & 6 & -1 \\ 6 & 18 & -12 & -15 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$

$\left[\begin{array}{ccc|c} \boxed{4} & 12 & -8 & 1 \\ 0 & 16 & -48 & 7 \\ 0 & 0 & 0 & 66 \end{array}\right] \begin{array}{c} R_1 \\ 3R_1 - 4R_2 \\ 6R_1 - 4R_3 \end{array}$

The third row indicates that there is no solution to the system.
The normals for the first and third planes are scalar multiples of each other; these planes are distinct.

b) $\left[\begin{array}{ccc|c} 1 & 8 & 5 & 1 \\ 3 & -2 & 1 & 6 \\ 5 & 1 & 4 & -10 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$

$\left[\begin{array}{ccc|c} \boxed{1} & 8 & 5 & 1 \\ 0 & 26 & 14 & -3 \\ 0 & 39 & 21 & 15 \end{array}\right] \begin{array}{c} R_1 \\ 3R_1 - R_2 \\ 5R_1 - R_3 \end{array}$

$\left[\begin{array}{ccc|c} 26 & 0 & 18 & 50 \\ 0 & \boxed{26} & 14 & -3 \\ 0 & 0 & 0 & -237 \end{array}\right] \begin{array}{c} 26R_1 - 8R_2 \\ R_2 \\ 39R_2 - 8R_3 \end{array}$

The third row indicates that there is no solution to the system.
The normals for the planes are not scalar multiples of each other; these planes only intersect in pairs as in a triangular prism.

c) $\left[\begin{array}{ccc|c} 1 & -6 & 2 & 3 \\ 1 & -3 & 1 & 1 \\ 2 & -12 & 4 & 6 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$

$\left[\begin{array}{ccc|c} \boxed{1} & -6 & 2 & 3 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 \\ R_1 - R_2 \\ 2R_1 - R_3 \end{array}$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & \boxed{-3} & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 - 2R_2 \\ R_2 \\ R_3 \end{array}$

The third row is 0 = 0. This indicates the solution is consistent and dependent.

There are ‘leading ones’ in columns 1 and 2. Let $z = t$.
From row 1, $x = -1$.
From row 2,
$-3y + t = 2$

$$y = -\frac{2}{3} + \frac{1}{3}t$$

The three planes intersect in a line that can be defined as $[x, y, z] = \left[-1, -\frac{2}{3}, 0\right] + t\left[0, \frac{1}{3}, 1\right],\ t \in \mathbb{R}$.

d) $\left[\begin{array}{ccc|c} 4 & -5 & 2 & 7 \\ 3 & -2 & 1 & 6 \\ 1 & 0 & -1 & 4 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$

$\left[\begin{array}{ccc|c} \boxed{4} & -5 & 2 & 7 \\ 0 & -7 & 2 & -3 \\ 0 & -5 & 6 & -9 \end{array}\right] \begin{array}{c} R_1 \\ 3R_1 - 4R_2 \\ R_1 - 4R_3 \end{array}$

$\left[\begin{array}{ccc|c} 28 & 0 & 4 & 64 \\ 0 & \boxed{-7} & 2 & -3 \\ 0 & 0 & -32 & 48 \end{array}\right] \begin{array}{c} 7R_1 - 5R_2 \\ R_2 \\ 5R_2 - 7R_3 \end{array}$

$\left[\begin{array}{ccc|c} 224 & 0 & 0 & 560 \\ 0 & -112 & 0 & 0 \\ 0 & 0 & \boxed{-32} & 48 \end{array}\right] \begin{array}{c} 8R_1 + R_3 \\ 16R_2 + R_3 \\ R_3 \end{array}$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2.5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1.5 \end{array}\right] \begin{array}{c} R_1 \div 224 \\ R_2 \div -112 \\ R_3 \div -32 \end{array}$

The planes intersect at the point (2.5, 0, –1.5).

e) $\left[\begin{array}{ccc|c} 2 & 1 & -3 & 10 \\ 10 & 5 & -15 & -3 \\ 6 & 3 & -9 & 4 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$

$\left[\begin{array}{ccc|c} \boxed{2} & 1 & -3 & 10 \\ 0 & 0 & 0 & 53 \\ & & & \end{array}\right] \begin{array}{c} R_1 \\ 5R_1 - R_2 \\ \end{array}$

Stop the row reduction process at this point. Row 2 shows there is no solution.

Examination of the normals to the planes indicates that all three planes are parallel to each other. (Normals are all scalar multiples of each other and the planes are distinct.)

f) $\left[\begin{array}{ccc|c} 3 & 2 & 2 & 15 \\ 1 & 5 & -6 & 46 \\ 4 & 2 & 1 & 9 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$

$\left[\begin{array}{ccc|c} \boxed{3} & 2 & 2 & 15 \\ 0 & -13 & 20 & -123 \\ 0 & 2 & 5 & 33 \end{array}\right] \begin{array}{c} R_1 \\ R_1 - 3R_2 \\ 4R_1 - 3R_3 \end{array}$

$\left[\begin{array}{ccc|c} 39 & 0 & 66 & -51 \\ 0 & \boxed{-13} & 20 & -123 \\ 0 & 0 & 105 & 183 \end{array}\right] \begin{array}{c} 13R_1 + 2R_2 \\ R_2 \\ 2R_2 + 13R_3 \end{array}$

$\left[\begin{array}{ccc|c} 4095 & 0 & 0 & -17433 \\ 0 & -1365 & 0 & -16575 \\ 0 & 0 & \boxed{105} & 183 \end{array}\right] \begin{array}{c} 105R_1 - 66R_3 \\ 105R_2 - 20R_3 \\ R_3 \end{array}$

$\left[\begin{array}{ccc|c} 105 & 0 & 0 & -447 \\ 0 & 7 & 0 & 85 \\ 0 & 0 & 35 & 61 \end{array}\right] \begin{array}{c} R_1 \div 39 \\ R_2 \div 195 \\ R_3 \div 3 \end{array}$

The planes intersect at the point $\left(-\frac{447}{105}, \frac{85}{7}, \frac{61}{35}\right)$.

Chapter 8 Extension **Question 6 Page 501**

a) The three planes intersect at the point (–7, 4, –10).

b) The three planes do not intersect.

c) The planes intersect in a line that can be defined by $[x, y, z] = [8, 1, 0] + t[-2, -7, 1],\ t \in \mathbb{R}$

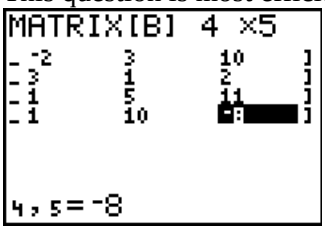
d) The three planes do not intersect.

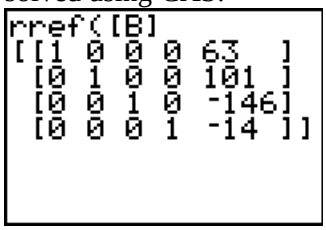
Chapter 8 Extension **Question 7 Page 501**

Answers may vary.

Chapter 8 Extension **Question 8 Page 501**

This question is most efficiently solved using CAS.





The solution point is (63, 101, –146, –14).

Use matrices and row reduction,

$$\left[\begin{array}{cccc|c} 1 & -3 & -2 & 3 & 10 \\ 4 & 2 & 3 & 1 & 2 \\ 2 & 1 & 1 & 5 & 11 \\ -2 & 4 & 1 & 10 & -8 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -3 & -2 & 3 & 10 \\ 0 & -14 & -11 & 11 & 38 \\ 0 & -7 & -5 & 1 & 9 \\ 0 & -2 & -3 & 16 & 12 \end{array}\right] \begin{array}{c} R_1 \\ 4R_1 - R_2 \\ 2R_1 - R_3 \\ 2R_1 + R_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 14 & 0 & 5 & 9 & 26 \\ 0 & -14 & -11 & 11 & 38 \\ 0 & 0 & -1 & 9 & 20 \\ 0 & 0 & 10 & -101 & -46 \end{array}\right] \begin{array}{c} 14R_1 - 3R_2 \\ R_2 \\ R_2 - 2R_3 \\ R_2 - 7R_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 14 & 0 & 0 & 54 & 126 \\ 0 & -14 & 0 & -88 & -182 \\ 0 & 0 & -1 & 9 & 20 \\ 0 & 0 & 0 & -11 & 154 \end{array}\right] \begin{array}{c} R_1 + 5R_3 \\ R_2 - 11R_3 \\ R_3 \\ 10R_3 + R_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 154 & 0 & 0 & 0 & 9702 \\ 0 & -14 & 0 & 0 & -1414 \\ 0 & 0 & -11 & 0 & -1166 \\ 0 & 0 & 0 & -11 & 154 \end{array}\right] \begin{array}{c} 11R_1 + 54R_4 \\ R_2 - 8R_4 \\ 11R_3 + 9R_4 \\ R_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 63 \\ 0 & 1 & 0 & 0 & 101 \\ 0 & 0 & 1 & 0 & -146 \\ 0 & 0 & 0 & 1 & -14 \end{array}\right] \begin{array}{l} R_1 \div 154 \\ R_2 \div -14 \\ R_3 \div -11 \\ R_4 \div -11 \end{array}$$

The solution point is (63, 101, –146, –14) or $w = 63,\ x = 101,\ y = -146,$ and $z = -14$.

Chapter 8 Extension **Question 9 Page 501**

Take the 7 L jug and fill it up with water. Using this jug, fill the 3 L jug. Take the 4 L that now remain in the 7 L jug and pour into a pail. Refill the 7 L jug and once again pour this into the pail. The 7 L being added to the 4 L that was there before will make 11 L of water in the pail.

Chapter 8 Extension **Question 10 Page 501**

Fill the 3 oz glass using the 8 oz glass. Take the 3 oz glass and pour it into the 5 oz glass. Refill the 3 oz glass using the 8 oz glass that now has 5 oz remaining. At this point the 8 oz glass will now only have 2 oz in it. Pour the 3 oz glass into the 5 oz glass that currently has 3 oz in it, until it is full. At this point, the 3 oz glass has 1 oz in it, so pour this one oz out. Take the 5 oz glass that is now full and fill up the 3 oz glass. This will leave 2 oz in the 5 oz glass that can now be poured into the 8 oz glass that has 2 oz in it to make a total of 4 oz.

Chapter 8 Review

Chapter 8 Review Question 1 Page 502

$t \in \mathbb{R}$ for all equations.

a) Vector: $[x, y] = [-3, 2] + t[1, 2]$
Parametric equations: $x = -3 + t$
$y = 2 + 2t$

b) Vector: $[x, y, z] = [-9, 0, 4] + t[6, 5, 1]$
Parametric equations: $x = -9 + 6t$
$y = 5t$
$z = 4 + t$

c) Vector: $[x, y, z] = [0, 0, 7] + t[1, 0, 0]$
Parametric equations: $x = 0$
$y = t$
$z = 7 + t$

d) Lines perpendicular to the *xy*-plane are parallel to the *z*-axis and the vector $\vec{k} = [0, 0, 1]$.
Vector: $[x, y, z] = [3, 0, -4] + t[0, 0, 1]$
Parametric equations: $x = 3$
$y = 0$
$z = -4 + t$

Chapter 8 Review Question 2 Page 502

a) Find two points on the line. If $x = 3$, $y = 3$; if $x = 1$, $y = -2$.
The vector equation is $[x, y] = [3, 3] + t[2, 5],\ t \in \mathbb{R}$.

b) Find two points on the line. If $x = 3$, $y = 1$; if $x = 10$, $y = 0$.
The vector equation is $[x, y] = [3, 1] + t[-7, 1],\ t \in \mathbb{R}$.

c) Find two points on the line. $x = 8$, $y = 2$; $x = 8$, $y = 3$.
The vector equation is $[x, y] = [8, 2] + t[0, 1],\ t \in \mathbb{R}$.

d) Find two points on the line. If $x = 4$, $y = 1$; if $x = 0$, $y = 0$.
The vector equation is $[x, y] = [4, 1] + t[4, 1],\ t \in \mathbb{R}$.

Chapter 8 Review **Question 3 Page 502**

a) The normal and the direction vector are perpendicular. Therefore, $\vec{n} = [7, -2]$.
The equation is of the form $7x - 2y + C = 0$. Substituting (1, 4) into the equation gives $C = 1$.
The scalar equation is $7x - 2y + 1 = 0$.

b) The normal and the direction vector are perpendicular. Therefore, $\vec{n} = [7, 5]$.
The equation is of the form $7x + 5y + C = 0$. Substituting (10, –3) into the equation gives $C = -55$.
The scalar equation is $7x + 5y - 55 = 0$.

Chapter 8 Review **Question 4 Page 502**

a) The parametric equations are:
$x = 1 + 3t$
$y = -1 + 4t$
$z = 5 + 7t,\ t \in \mathbb{R}$

b) Substitute the coordinates into the parametric equations.

$13 = 1 + 3t$ $\quad$ $15 = -1 + 4t$ $\quad$ $23 = 5 + 7t$

$t = 4$ $\quad$ $t = 4$ $\quad$ $t = \frac{18}{7}$

Since the t-values are not identical, the point does not lie on the line.

Chapter 8 Review **Question 5 Page 502**

The direction vectors for the sides are:

$\overrightarrow{AB} = [4, 2]$ $\quad$ $\overrightarrow{BC} = [4, -4]$

$= [2, 1]$ $\quad$ $= [1, -1]$

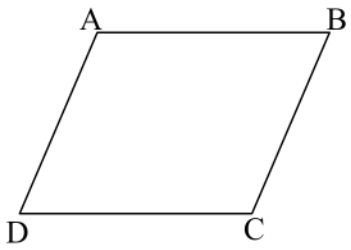
The vector equations are:

AB: $[x, y] = [-1, 4] + t[2, 1],\ t \in \mathbb{R}$

BC: $[x, y] = [3, 6] + t[1, -1],\ t \in \mathbb{R}$

CD: $[x, y] = [7, 2] + t[2, 1],\ t \in \mathbb{R}$

DA: $[x, y] = [-1, 4] + t[1, -1],\ t \in \mathbb{R}$



Chapter 8 Review **Question 6 Page 502**

For the x-intercept of the first line, let $y = 0$ and $z = 0$.

$0 = 8 + 4t$ $0 = 14 + 7t$

$t = -2$ $t = -2$

When $t = -2$:

$x = -21 - 12(-2)$

$= 3$

The x-intercept of the first line is 3.

For the y-intercept of the second line, let $x = 0$ and $z = 0$.

$0 = 6 + 2s$ $0 = 12 + 4s$

$s = -3$ $s = -3$

When $s = -3$:

$y = -8 - 5(-3)$

$= 7$

The y-intercept of the second line is 7.

Since (3, 0, 0) and (0, 7, 0) are points on the required line, a direction vector is [3, –7, 0].
Parametric equations for the line are:

$x = 3 + 3t$

$y = -7t$

$z = 0, \ t \in \mathbb{R}$

Chapter 8 Review **Question 7 Page 502**

a) When $s = 0$ and $t = 0$, (3, 4, –1) is a point on the plane.
When $s = 1$ and $t = 0$, (4, 5, –5) is a point on the plane.
When $s = 1$ and $t = 0$, (2, 3, 3) is a point on the plane.

b) When $x = 0$ and $y = 0$, (0, 0, 12) is a point on the plane.
When $y = 0$ and $z = 0$, (–12, 0, 0) is a point on the plane.
When $x = 0$ and $z = 0$, (0, –6, 0) is a point on the plane.

c) When $k = 1$ and $p = 1$, (7, –6, 3) is a point on the plane.
When $k = 0$ and $p = 0$, (0, –5, 2) is a point on the plane.
When $k = -1$ and $p = -1$, (–7, –4, 1) is a point on the plane.

Chapter 8 Review **Question 8 Page 502**

A direction vector for the plane is $[5, 1, 3]-[2, -9, 10]=[3, 10, -7]$.

A vector equation for the plane is $[x, y, z]=[2, -9, 10]+s[3, -8, 7]+t[3, 10, -7],\ s, t \in \mathbb{R}$.

Parametric equations for the plane are:

$x = 2 + 3s + 3t$

$y = -9 - 8s + 10t$

$z = 10 + 7s - 7t,\ s, t \in \mathbb{R}$

Chapter 8 Review **Question 9 Page 502**

a) If P(–3, 4, –5) lies on the plane, there exists a single set of s- and t-values that satisfy the equation.

$-3 = 1 + 2s + t$ ①

$4 = -5 + s + 7t$ ②

$-5 = 6 + 3s + t$ ③

Solve ① and ② for s and t.

$-4 = 2s + t$ ①

$18 = 2s + 14t$ 2②

$-22 = -13s$ ①–2②

$s = \frac{22}{13}$

Substitute $s = \frac{22}{13}$ in ①.

$-4 = 2\left(\frac{22}{13}\right) + t$ ①

$t = -\frac{96}{13}$

Now check if these values satisfy equation ③.

L.S. = 5

$\text{R.S.} = 6 + 3\left(\frac{22}{13}\right) + \left(-\frac{96}{13}\right)$

$= -\frac{48}{13}$

L.S. ≠ R.S.

Therefore, P(–3, 4, –5) does not lie on the plane.

b) Substitute the coordinates in the scalar equation.

$\text{L.S.} = 4(-3) + 4 - 2(-5) - 2$

$= 0$

R.S. = 0

L.S. = R.S.

Therefore, the point P(–3, 4, –5) lies on the plane.

Chapter 8 Review **Question 10 Page 502**

Calculate direction vectors.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
$= [-1, -1, 10] - [2, 1, 5]$
$= [-3, -2, 5]$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$
$= [8, 5, -5] - [2, 1, 5]$
$= [6, 4, -10]$

$\overrightarrow{AC} = 2\overrightarrow{AB}$

Since the direction vectors are scalar multiples of each other, the three points are collinear and do not define a unique plane.

Chapter 8 Review **Question 11 Page 502**

a) The x-, y-, and z- intercepts are 16, –4, and 8. (Points: (16, 0, 0), (0, –4, 0), and (0, 0, 8))
Possible direction vectors are [–16, –4, 0] or [4, 1, 0] and [0, 4, 8] or [0, 1, 2].

b) (16, 0, 0), (0, –4, 0), and (0, 0, 8), so x-intercept = 16, y = intercept = –4, and z-intercept = 8.

c) Vector equation is $[x, y, z] = [16, 0, 0] + s[4, 1, 0] + t[0, 1, 2],\ s, t \in \mathbb{R}$.
Parametric equations are:
$x = 16 + 4s$
$y = s + t$
$z = 2t,\ s, t \in \mathbb{R}$

Chapter 8 Review **Question 12 Page 502**

The scalar equation is of the form $x + 2y - 9z + D = 0$,

Substitute the coordinates of P to determine D.
$3 + 2(-4) - 9(0) + D = 0$
$D = 5$

The equation of the plane is $x + 2y - 9z + 5 = 0$.

Chapter 8 Review **Question 13 Page 502**

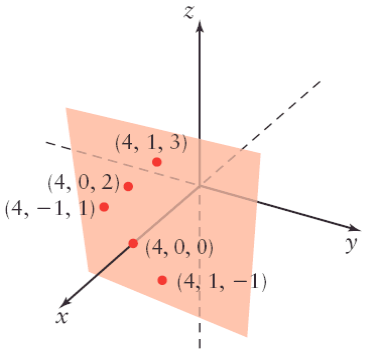
Since $\vec{j}$ and $\vec{k}$ are direction vectors of the plane:
$\vec{n} = \vec{j} \times \vec{k}$.
$= \vec{i}$
$= [1, 0, 0]$

The equation is of the form $x + D = 0$.
Since (5, 4, –7) is on the plane, $D = -5$.

The equation of the plane is $x - 5 = 0$ or $x = 5$.

Chapter 8 Review **Question 14 Page 502**

a) By observation, equation is $x = 4$.



b) [3, –7, 1] and [0, 1, 0] are direction vectors for the plane.

$$\vec{n} = [3, -7, 1] \times [0, 1, 0]$$
$$= [-1, 0, 3]$$

The scalar equation is of the form $-x + 3z + D = 0$.
The point (1, 2, 4) is on the plane.

$$-1 + 3(4) + D = 0$$
$$D = -11$$

The equation is $x - 3z + 11 = 0$.

Chapter 8 Review **Question 15 Page 502**

a) The direction vectors are [5, 2] and [7, 3] which are not parallel.
Substitute the parametric equations in the scalar equation.

$$2(-9 + 7t) - 5(-4 + 3t) = 6$$
$$-18 + 14t + 20 - 15t = 6$$
$$t = -4$$

Substitute $t = -4$ in the parametric equations.
The one solution is (–37, –16).

b) Equate the corresponding coordinates and solve for s and t.

$9 + s = 3t$ ①

$\underline{4 + s = 9 - 4t}$ ②

$5 = -9 + 7t$ ①–②

$t = 2$

Substitute $t = 2$ in ① .

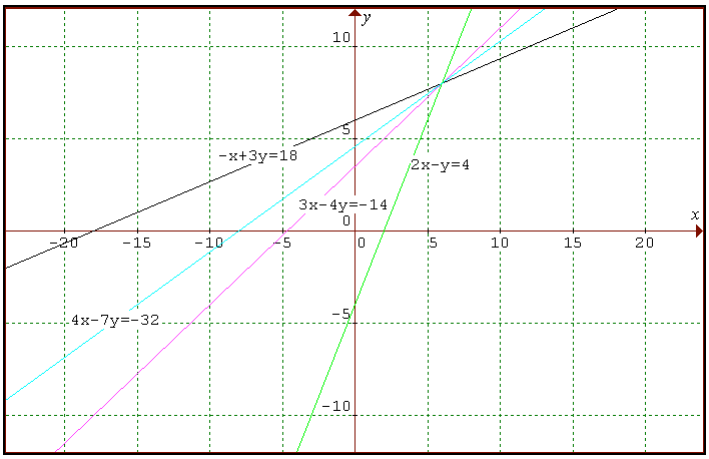
$9 + s = 6$ ①

$s = -3$

Substitute either $s = -3$ in given equation with parameter s or $t = 2$ in given equation with parameter t.
Then one solution is (6, 1).

Chapter 8 Review **Question 16 Page 503**

Scalar multiples and sums or differences of these equations will pass through the intersection point.
Therefore, add the equations to get $2x - y = 4$. Subtract the equations to get $4x - 7y = -32$.
These two equations will pass through the intersection point as shown in the graph.



Chapter 8 Review **Question 17 Page 503**

a) The two direction vectors are equal. The lines are parallel and either coincident or distinct.

Check if (1, 5, –2) is on the second line.

$1 = -3 + t$ $\quad 5 = -23 + 7t$ $\quad -2 = 10 - 3t$

$t = 4$ $\quad t = 4$ $\quad t = 4$

Since the t-values are equal, the point lies on the second line and the two lines are coincident.
There are an infinite number of points of intersection.

b) The direction vectors are not scalar multiples of each other. The lines must intersect.

To find the intersection point, equate like coordinates.

$15 + 4s = 13 - 5t$ ①

$2 + s = -5 + 2t$ ②

$-1 - s = -4 + 3t$ ③

Solve ① and ② for s and t.

$4s + 5t = -2$ ①

$4s - 8t = -28$ 4②

$13t = 26$ ①–4②

$t = 2$

Substitute $t = 2$ in .

$s - 2(2) = -7$ ②

$s = -3$

Check if these values satisfy ③.

L.S. $= -1 - (-3)$ R.S. $= -4 + 3(2)$

$= 2$ $= 2$

L.S. = R.S.

Substitute $s = -3$ or $t = 2$ to find the point of intersection.

The two lines intersect at the point (3, –1, 2).

Chapter 8 Review **Question 18 Page 503**

For the distance between skew lines, $d = \dfrac{\left|\overrightarrow{P_1P_2} \cdot \vec{n}\right|}{\left|\vec{n}\right|}$ where $P_1 \in \ell_1$, $P_2 \in \ell_2$, and $\vec{n} = \vec{m}_1 \times \vec{m}_2$.

Let $P_1(1, 0, -1)$, $P_2(8, 1, 3)$, $\vec{m}_1 = [2, 3, -4]$, and $\vec{m}_2 = [4, -5, 1]$.

$\overrightarrow{P_1P_2} = [7, 1, 4]$

$\vec{n} = \vec{m}_1 \times \vec{m}_2$

$= [-17, -18, -22]$

$d = \dfrac{\left|[7, 1, 4] \cdot [-17, -18, -22]\right|}{\left|[-17, -18, -22]\right|}$

$d = \dfrac{225}{\sqrt{1097}}$

$d \doteq 6.8$

Chapter 8 Review **Question 19 Page 503**

a) Substitute the parametric equations into the scalar equation of the plane and solve for t.

$$5(-17+4t)-2(7+t)+4(-6-3t)=23$$
$$-85+20t-14-2t-24-12t=23$$
$$6t=146$$
$$t=\frac{73}{3}$$

Substitute this value of t in the parametric equations.

$$x=-17+4\left(\frac{73}{3}\right) \qquad y=7+\frac{73}{3} \qquad z=-6-3\left(\frac{73}{3}\right)$$
$$=\frac{241}{3} \qquad =\frac{94}{3} \qquad =-79$$

The point of intersection of the line and the plane is $\left(\frac{241}{3}, \frac{94}{3}, -79\right)$.

b) Substitute the parametric equations into the scalar equation of the plane and solve for t.

$$(-1+3t)+4(-9+3t)+3(16-5t)=11$$
$$-1+3t-36+12t+48-15t=11$$
$$0t=0$$

This equation is true for every value of t.
The two lines intersect at every point on the line; the line is on the plane.

Chapter 8 Review **Question 20 Page 503**

The distance between a point P and a plane is given by $d=\frac{\left|\vec{n}\cdot\overrightarrow{PQ}\right|}{\left|\vec{n}\right|}$ where Q is any point on the plane with normal $\vec{n}$.

$$\overrightarrow{PQ}=[0,-2,0]-[3,-2,0]$$
$$=[-3,0,0]$$

$$d=\frac{\left|[4,-1,8]\cdot[-3,0,0]\right|}{\sqrt{4^2+(-1)^2+8^2}}$$
$$=\frac{12}{9}$$
$$=\frac{4}{3}$$
$$\doteq 1.33$$

The distance is $\frac{4}{3}$ or approximately 1.3 units.

Chapter 8 Review **Question 21 Page 503**

Use elimination.

$$\begin{aligned} 12x + 4y + 4z &= 10 && 4① \\ 5x + 4y - 2z &= 31 && ② \\ \hline 7x \quad &+ 6z = -21 && ③ \qquad 4①-② \end{aligned}$$

Let $z = t$.

③ becomes $x = \dfrac{-21-6t}{7}$

② becomes:

$$5\left(\frac{-21-6t}{7}\right) + 4y - 2t = -21$$

$$y = \frac{22t - 21}{14}$$

The parametric equations for the line are:

$$x = -3 - \frac{6}{7}t$$

$$y = -\frac{3}{2} + \frac{11}{7}t$$

$$z = t,\ t \in \mathbb{R}$$

A vector equation for the line is $[x,\ y,\ z] = [-3,\ -1.5,\ 0] + t[-6,\ 11,\ 7],\ t \in \mathbb{R}$.

Chapter 8 Review **Question 22 Page 503**

a) First examine the normals. None are scalar multiples of each other.

Check if the normals are coplanar.

$$\begin{aligned} \vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 &= [2,\ 5,\ 2] \cdot [1,\ 2,\ -3] \times [2,\ 1,\ 5] \\ &= [2,\ 5,\ 2] \cdot [13,\ -11,\ -3] \\ &= -35 \end{aligned}$$

The normals are not coplanar; there is an intersection point.

Eliminate one of the variables from two pairs of equations. Eliminate x.

$$\begin{aligned} 2x + 5y + 2z &= 3 && ① \\ 2x + 4y - 6z &= -22 && 2② \\ \hline y + 8z &= 25 && ④ \qquad ①-2② \end{aligned}$$

$$\begin{aligned} 2x + 5y + 2z &= 3 && ① \\ 2x + y + 5z &= 8 && ③ \\ \hline 4y - 3z &= -5 && ⑤ \qquad ①-③ \end{aligned}$$

Solve equations ④ and ⑤ for y and z.

$4y + 32z = 100$ 4④

$4y - 3z = -5$ ⑤

$35z = 105$ 4④–⑤

$z = 3$

Substitute $z = 3$ in ④.

$y + 8(3) = 25$ ④

$y = 1$

Substitute $y = 1$ and $z = 3$ in ②.

$x + 2(1) - 3(3) = -11$

$x = -4$

The three planes intersect at the point (–4, 1, 3).

b) First examine the normals. None are scalar multiples of each other.

Check if the normals are coplanar.

$$\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 = [1, 3, 2] \cdot [3, -5, 1] \times [6, 4, 7]$$
$$= [1, 3, 2] \cdot [-39, -15, 42]$$
$$= 0$$

Since the triple scalar product is zero, the normals are coplanar and the planes intersect either in a line or not at all.

Solve the system algebraically to determine if there is a solution.

$x + 3y + 2z = 10$ ①

$3x - 5y + z = 1$ ②

$6x + 4y + 7z = -5$ ③

$14y + 5z = 29$ ④ 3①–②

$14y + 5z = 65$ ⑤ 6①–③

$0 = -36$ ④–⑤

The system has no solution. The planes intersect in pairs only.

c) First examine the normals. None are scalar multiples of each other.

Check if the normals are coplanar.

$$\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 = [1, 3, -1] \cdot [3, 1, 1] \times [5, 7, 1]$$
$$= [1, 3, -1] \cdot [-6, 2, 16]$$
$$= -16$$

The normals are not coplanar; there is an intersection point.

Eliminate one of the variables from two pairs of equations. Eliminate z.

$$\begin{aligned} x+3y-z&=-2 && ① \\ 3x+y+z&=14 && ② \\ \hline 4x+4y&=12 && ④ \quad ①+② \end{aligned}$$

$$\begin{aligned} x+3y-z&=-2 && ① \\ 5x+7y+z&=10 && ③ \\ \hline 6x+10y&=8 && ⑤ \quad ①+③ \end{aligned}$$

Solve equations ④ and ⑤ for x and y.

$$\begin{aligned} 12x+12y&=36 && 3④ \\ 12x+20y&=16 && 2⑤ \\ -8y&=20 && 3④-2⑤ \\ \hline y&=-2.5 \end{aligned}$$

Substitute $y = -2.5$ in ④.

$$\begin{aligned} 4x+4(-2.5)&=12 && ④ \\ x&=5.5 \end{aligned}$$

Substitute $x = 5.5$ and $y = -2.5$ in ②.

$$\begin{aligned} 3(5.5)+(-2.5)+z&=14 \\ z&=0 \end{aligned}$$

The three planes intersect at the point (5.5, –2.5, 0).

Chapter 8 Review **Question 23 Page 503**

a) First examine the normals. None are scalar multiples of each other. The planes are not parallel.

Check if the normals are coplanar.

$$\begin{aligned} \vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 &= [2, 5, 3] \cdot [1, -3, 6] \times [3, 2, 9] \\ &= [2, 5, 3] \cdot [-39, 9, 11] \\ &= 0 \end{aligned}$$

Since the triple scalar product is zero, the normals are coplanar and the planes intersect either in a line or not at all.

The usual way to check this is to solve the system algebraically. However, in this case, observe that $\vec{n}_1 + \vec{n}_2 = \vec{n}_3$

If the planes intersect in a line, then ① + ② = ③, but $0 + 19 \neq -7$.

Therefore, the three planes do not have a common intersection.

b) First examine the normals. None are scalar multiples of each other.

Check if the normals are coplanar.

$$\begin{aligned}\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 &= [8,\ 20,\ 16] \cdot [3, 15, 12] \times [2, -5,\ 4] \\ &= [8,\ 20,\ 16] \cdot [120, 12, 15] \\ &= 1440\end{aligned}$$

The normals are not coplanar. These planes will intersect at a point.

Chapter 8 Practice Test

Chapter 8 Practice Test **Question 1 Page 504**

D is the best answer.

$[A, B] = [3, 5]$ is the normal to the line.
Check which vector has dot product of zero with this normal. $[3, 5] \cdot [-5, 3] = 0$

Chapter 8 Practice Test **Question 2 Page 504**

B is the best answer.

All equations have parallel direction vectors. None can be eliminated on this criterion.
Check if $(1, -8)$ is a point on each line.

A: $t = \frac{1}{4}$ gives $x = 1$, but $y \neq -8$.

B: $t = -\frac{3}{2}$ gives $x = 1$ and $y = -8$.

C: $t = \frac{11}{12}$ gives $x = 1$, but $y \neq -8$.

D: $t = 3$ gives $x = 1$, but $y \neq -8$.

Chapter 8 Practice Test **Question 3 Page 504**

B is the best answer.

Substitute the point in each equation.
A: $3(10) + 6(-3) - 2(5) - 2 = 0$
B: $1(10) + (-3) - 1(5) - 12 = -10$
C: $2(10) - 2(-3) - 3(5) - 11 = 0$
D: $4(10) + 5(-3) + (5) - 30 = 0$

Chapter 8 Practice Test **Question 4 Page 504**

D is the best answer.

Lines in three-space have vector, parametric, and symmetric equations.
The slope-intercept equation only exists for lines in two-space.

Chapter 8 Practice Test **Question 5 Page 504**

D is the best answer.

Scalar equations represent lines in two-space and planes in three-space.

Chapter 8 Practice Test **Question 6 Page 504**

A is the best answer.

Check which vector is a scalar multiple of $\vec{n} = [1, 2, -3]$.

A: $[2, -3, 4] \neq k[1, 2, -3]$

B: $[-1, -2, 3] = -[1, 2, -3]$

C: $[2, 4, -6] = 2[1, 2, -3]$

D: $[3, 6, -9] = 3[1, 2, -3]$

Chapter 8 Practice Test **Question 7 Page 504**

D is the best answer.

Substitute each point in the equation.

A: $3(-8) - 4(-9) + (0) - 12 = 0$

B: $3(4) - 4(1) + (4) - 12 = 0$

C: $3(16) - 4(10) + (4) - 12 = 0$

D: $3(18) - 4(12) + (2) - 12 = -4$

Chapter 8 Practice Test **Question 8 Page 504**

B is the best answer.

The normals $[10, -7]$ and $[4, 5]$ are not scalar multiples of each other. Therefore, the lines must not be parallel and they intersect in a unique point.

Chapter 8 Practice Test **Question 9 Page 504**

C is the best answer.

The lines have the same direction vector. They are either parallel or coincident.

Check if (4, –2, 3) is a point on the second line.

$4 = 1 + 5t$ $\quad -2 = -3 - 3t$ $\quad 3 = 1 + 2t$

$t = \frac{3}{5}$ $\quad t = -\frac{1}{3}$ $\quad t = 1$

Since the t-values are different, the point is not on the line and the lines are parallel and distinct.

A: means there is an intersection point.
B: means the lines are identical.
C: means the lines do not intersect
D: means the lines do not intersect but they are also not parallel

Chapter 8 Practice Test **Question 10 Page 504**

A is the best answer.

Planes can intersect in a line, be parallel ands distinct, or be coincident.
Two planes cannot intersect in just one point.

Chapter 8 Practice Test **Question 11 Page 504**

a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OC}$

$= [2, -9, 0] - [1, 5, -4]$

$= [1, -14, 4]$

A possible vector equation is $[x, y, z] = [1, 5, -4] + t[1, -14, 4],\ t \in \mathbb{R}$.

Possible parametric equations are:

$x = 1 + t$

$y = 5 - 14t$

$z = -4 + 4t,\ t \in \mathbb{R}$

b) Let $t = 2$ and $t = -1$. The resulting points are: (3, –23, 4) and (0, 19, –8).

Chapter 8 Practice Test **Question 12 Page 504**

The normal to the first line is [4, 8] or [1, 2]. This vector will be a direction vector for the required line.

To find the x-intercept, let $y = 0$.

$0 = 7 + 3t$

$t = -\frac{7}{3}$

When $t = -\frac{7}{3}$:

$x = 2 + \left(-\frac{7}{3}\right)(-10)$

$= \frac{76}{3}$

The x-intercept is $\frac{76}{3}$.

The parametric equations of the required line are:

$x = \frac{76}{3} + t$

$y = 2t,\ t \in \mathbb{R}$

Chapter 8 Practice Test **Question 13 Page 504**

For a perpendicular direction, find the cross product of the direction vectors of the two given lines.
$\vec{m} = [4, 6, -3] \times [3, 2, 4] = [30, -25, -10]$. Use $[6, -5, -2]$.
The parametric equations are
$x = -6 + 6t$
$y = 4 - 5t$
$z = 3 - 2t, \quad t \in \mathbb{R}$

Chapter 8 Practice Test **Question 14 Page 504**

A plane parallel to the *xz*-plane has equation of the form $y = k$.
Since the plane must contain the point (3, –1, 5), the required equation is $y = -1$ or $y + 1 = 0$.
(Note that the line is actually parallel to the *xz*-plane since every point has $y = -1$.)

Chapter 8 Practice Test **Question 15 Page 505**

a) Substitute the parametric equations in the scalar equation of the first line.
$6(4 + t) + 2(-7 - 3t) = 5$
$24 + 6t - 14 - 6t = 5$
$10 = -5$

Since there are no solutions for t, the lines do not intersect

b) Use elimination.
$2x + 3y = 21$ ①
$4x - y = 7$ ②
$14x = 42$ ①+3②
$x = 3$

Substitute $x = 3$ in ② .
$4(3) - y = 7$ ②
$y = 5$

The intersection point is (3, 5).

c) The direction vectors are not parallel since the direction vectors are not scalar multiples of each other.

Equate the expressions for like coordinates
$-2 + 3s = 7 + t$ $\quad 4 - 3s = 10 + 4t$ $\quad -1 + s = 4 + 3t$
$3s - t = 9$ ① $\quad 3s + 4t = -6$ ② $\quad s - 3t = 5$ ③

Solve equations ① and ② for s and t.

$3s - t = 9$ ①

$3s + 4t = -6$ ②

$-5t = 15$ ①–②

$t = -3$

Substitute $t = -3$ in ②

$3s + 4(-3) = -6$

$3s = 6$

$s = 2$

Check that s and t satisfy ③.

L.S. $= 2 - 3(-3)$ R.S. $= 5$

$= 11$

L.S. $\neq$ R.S.

Therefore, the lines do not intersect.

d) The direction vectors are not parallel since the direction vectors are not scalar multiples of each other.

Equate the expressions for like coordinates

$3 + 5s = 1 + 10t$ $4 + 2s = -4 + 4t$ $-6 - 2s = 4 - 4t$

$5s - 10t = -2$ $2s - 4t = -8$ $-2s + 4t = 10$

$s - 2t = -1$ ① $s - 2t = -4$ ② $s - 2t = -5$ ③

Since the same expression has three different values, the lines do not intersect.

Chapter 8 Practice Test **Question 16 Page 505**

a) The direction vectors are not parallel since the direction vectors are not scalar multiples of each other. Now show that the lines do not intersect.

Equate the expressions for like coordinates

$2 + 5s = 1 - 2t$ $6 + s = 5 + 4t$ $1 + 3s = -3 + t$

$5s + 2t = -1$ ① $s - 4t = -1$ ② $3s - t = -4$ ③

Solve equations ① and ② for s and t.

$10s + 4t = -2$ 2①

$s - 4t = -1$ ②

$11s = -3$ 2①+②

$s = -\frac{3}{11}$

Substitute in $s = -\frac{3}{11}$ in ②.

$$-\frac{3}{11} - 4t = -1$$

$$t = \frac{2}{11}$$

Check that s and t satisfy③.

$$\text{L.S.} = 3\left(-\frac{3}{11}\right) - \left(\frac{2}{11}\right) \qquad \text{R.S.} = -4$$

$$= -1$$

$$\text{L.S.} \neq \text{R.S.}$$

Therefore the lines do not intersect.
Since the two lines do not intersect and are not parallel, they are skew lines.

b) The distance between skew lines is $d = \frac{\left|\overrightarrow{P_1P_2} \cdot \vec{n}\right|}{\left|\vec{n}\right|}$ where $P_1 \in \ell_1$, $P_2 \in \ell_2$, and $\vec{n} = \vec{m}_1 \times \vec{m}_2$

Let $P_1(2, 6, 1)$, $P_2(1, 5, -3)$, $\vec{m}_1 = [5, 1, 3]$, and $\vec{m}_2 = [-2, 4, 1]$.

$$\overrightarrow{P_1P_2} = [-1, -1, -4]$$

$$\vec{n} = \vec{m}_1 \times \vec{m}_2$$
$$= [-11, -11, 22]$$

$$d = \frac{\left|[-1, -1, -4] \cdot [-11, -11, 22]\right|}{\left|[-11, -11, 22]\right|}$$

$$d = \frac{66}{\sqrt{726}}$$

$$d \doteq 2.45$$

The distance between the lines is approximately 2.45 units.

Chapter 8 Practice Test **Question 17 Page 505**

Find the scalar equation of the plane.

$\overrightarrow{AB} = [-9, -4, -2]$

$\overrightarrow{BC} = [13, -3, -13]$

Find the normal vector.

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{BC} &= [-4(-13) - (-3)(-2), -2(13) - (-13)(-9), -9(-3) - 13(-4)] \\ &= [46, -143, 79]\end{aligned}$$

The equation is of the form $46x - 143y + 79z + D = 0$.

Use the coordinates of A to find D.

$$\begin{aligned}46(2) - 143(5) + 79(6) + D &= 0 \\ D &= 149\end{aligned}$$

The scalar equation of the plane is $46x - 143y + 79z + 149 = 0$.

Now check if the line is on the plane.

$$\begin{aligned}46(-3 + 22k) - 143(-6 + k) + 79(-11 - 11k) + 149 &= 0 \\ -138 + 1012k + 858 - 143k - 869 - 869k + 149 &= 0 \\ 0k &= 0\end{aligned}$$

Since this equation is true for all values of t, the line does lie in the plane.

Chapter 8 Practice Test **Question 18 Page 505**

a) The normal vectors are not scalar multiples of each other and so the planes are not parallel.
The two planes intersect at a line.

b) The normal vectors are not scalar multiples of each other and so the planes are not parallel.

Check if the normals are coplanar.

$$\begin{aligned}\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 &= [1, 1, -1] \cdot [2, 3, 4] \times [2, -2, 1] \\ &= [1, 1, -1] \cdot [11, 6, -10] \\ &= 27\end{aligned}$$

The normals are not coplanar. These planes will intersect at a point.

c) Check the normals. $2\vec{n}_1 = \vec{n}_3$, but 2 times equation ① does not equal equation ③.
Therefore two planes are parallel and distinct and the system has no solutions.
The third plane is not parallel to the other two planes and intersects both of them.

d) The normal vectors are not scalar multiples of each other and so the planes are not parallel.
The planes intersect at a line.

Chapter 8 Practice Test **Question 19 Page 505**

a) $\overrightarrow{BA} = [3, 19, -3]$ and $\overrightarrow{CA} = [2, 14, 5]$

A vector equation is $[x, y, z] = [1, 13, 2] + s[3, 19, -3] + t[2, 14, 5]$, $s, t \in \mathbb{R}$.

Parametric equations are:

$x = 1 + 3s + 2t$

$y = 13 + 19s + 14t$

$z = 2 - 3s + 5t$, $s, t \in \mathbb{R}$

b) First, find the normal to the plane.

$\vec{n} = \overrightarrow{BA} \times \overrightarrow{CA}$

$= [137, -21, 4]$

The scalar equation is of the form $137x - 21y + 4z + D = 0$

Use one of the points to find D. Use C.

$137(-1) - 21(-1) + 4(-3) + D = 0$

$D = 128$

The scalar equation is $137x - 21y + 4z + 128 = 0$.

c) When $s = 1$ and $t = 1$, a point is (6, 46, 4).
When $s = 0$ and $t = 1$, a point is (3, 27, 7).

Chapter 8 Practice Test **Question 20 Page 505**

a) Substitute the coordinates for the origin in the parametric equations.

$0 = 2 + s + 2t$ $0 = 1 + s$ $0 = 3 + s + 2t$

$s + 2t = -2$ ① $s = -1$ ② $s + 2t = -3$ ③

From ① and ②, $s = -1$ and $t = -\frac{1}{2}$.

Check if these values satisfy ③.

L.S. $= (-1) + 2\left(-\frac{1}{2}\right)$ R.S. $= -3$

$= -2$

L.S. $\neq$ R.S.

Therefore the origin is not a point on the line.

b) The distance between a point P and a plane is given by $d = \dfrac{\left|\vec{n} \cdot \overrightarrow{PQ}\right|}{\left|\vec{n}\right|}$ where Q is any point on the plane with normal $\vec{n}$.

$$\begin{aligned}\vec{n} &= [1, 1, 1] \times [2, 0, 2] \\ &= [2, 0, -2]\end{aligned}$$

Choose Q(2, 1, 3).

$$\begin{aligned}\overrightarrow{PQ} &= [2, 1, 3] - [0, 0, 0] \\ &= [2, 1, 3]\end{aligned}$$

$$\begin{aligned}d &= \frac{\left|[2, 0, -2] \cdot [2, 1, 3]\right|}{\sqrt{2^2 + 0^2 + (-2)^2}} \\ &= \frac{2}{\sqrt{8}} \\ &\doteq 0.71\end{aligned}$$

The distance is approximately 0.71 units.

Chapter 8 Practice Test **Question 21 Page 505**

Choose three coplanar normal vectors.

$\vec{n}_1 = [2, -3, 5]$. Choose any non-parallel vector for $\vec{n}$, Choose [1, 1, 1].

For a third normal, choose a coplanar vector. Choose $\vec{n}_1 + \vec{n}_2 = [3, -2, 6]$.

Possible equations for the other planes are $x + y + z = 2$ and $3x - 2y + 6z = 2$.

To be sure that the planes do not intersect in a common line, check that ① + ② ≠ ③ (i.e. $D_1 + D_2 \neq D_3$).

Chapter 8 Practice Test **Question 22 Page 505**

a) Substitute the parametric equations in the scalar equation.

$$2(3r) + 3(2r) + 4(6 - 3r) = 24$$
$$0r = 0$$

$$2(14 + s) + 3(-12 - 2s) + 4(8 + s) = 24$$
$$0s = 0$$

$$2(9 + t) + 3(-10 + 6t) + 4(9 - 5t) = 24$$
$$0t = 0$$

Since each of the equations is true for all values of *r*, *s*, and *t* respectively, all three lines are contained in the plane.

b) To find the first vertex, equate like coordinates for lines ℓ_1 and ℓ_2.

$3r = 14 + s$ ①

$2r = -12 - 2s$ ②

$6 - 3r = 8 + s$ ③

Solve ① and ② for s and t.

$3r - s = 14$ ①

$r + s = -6$ 0.5②

$4r = 8$ ①+0.5②

$r = 2$

Substitute $r = 2$ in ②.

$2 + s = -6$ 0.5②

$s = -8$

Check if these values satisfy ③.

L.S. $= 6 - 3(2)$ R.S. $= 8 + (-8)$

$= 0$ $= 0$

L.S. = R.S.

Substitute $r = 2$ or $s = -8$ to find the point of intersection.
The two lines intersect at the point (6, 4, 0).

To find the second vertex, equate like coordinates for lines ℓ_2 and ℓ_3.

$14 + s = 9 + t$ ①

$-12 - 2s = -10 + 6t$ ②

$8 + s = 9 - 5t$ ③

Solve ① and ② for s and t.

$s - t = -5$ ①

$s + 3t = -1$ 0.5②

$-4t = -4$ ①–0.5②

$t = 1$

Substitute $t = 1$ in ①.

$s - 1 = -5$ ①

$s = -4$

Check if these values satisfy ③.

L.S. $= 8 + (-4)$ R.S. $= 9 - 5(1)$

$= 4$ $= 4$

L.S. = R.S.

Substitute $t = 1$ or $s = -4$ to find the point of intersection.
The two lines intersect at the point (10, –4, 4).

To find the third vertex, equate like coordinates for lines ℓ_1 and ℓ_3.

$3r = 9 + t$ ①

$2r = -10 + 6t$ ②

$6 - 3r = 9 - 5t$ ③

Solve ① and ③ for s and t.

$3r - t = 9$ ①

$3r - 5t = -3$ ③

$4t = 12$ ①–③

$t = 3$

Substitute $t = 3$ in ①.

$3r - 3 = 9$ ①

$r = 4$

Check if these values satisfy ②.

L.S. $= 2(4)$ R.S. $= -10 + 6(3)$

$= 8$ $= 8$

L.S. = R.S.

Substitute $t = 3$ or $r = 4$ to find the point of intersection.
The two lines intersect at the point (12, 8, –6).

The vertices are A(12, 8, –6), B(10, –4, 4), and C(6, 4, 0).

c) $|AB| = \sqrt{(10-12)^2 + (-4-8)^2 + (4-(-6))^2}$

$= \sqrt{248}$

$|AC| = \sqrt{(6-12)^2 + (4-8)^2 + (0-(-6))^2}$

$= \sqrt{88}$

$|BC| = \sqrt{(10-6)^2 + (-4-4)^2 + (4-0)^2}$

$= \sqrt{96}$

Perimeter $= \sqrt{248} + \sqrt{88} + \sqrt{96}$

$\doteq 34.9$

The perimeter is approximately 34.9 units.

d) $\text{Area} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$

$= \frac{1}{2}\left|\left[-2, -12, 10\right] \times \left[-4, 8, -4\right]\right|$

$= \frac{1}{2}\left|\left[-32, -48, -64\right]\right|$

$= \frac{1}{2}\sqrt{(-32)^2 + (-48)^2 + (-64)^2}$

$= \frac{1}{2}\sqrt{7424}$

$\doteq 43.1$

The area of the triangle is approximately 43.1 units^2.

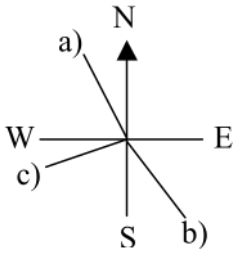
Chapters 6 to 8 Review

Chapter 6 to 8 Review **Question 1 Page 506**

a) 340º

b) 144º

c) 260º

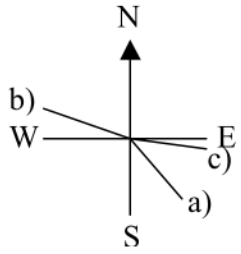


Chapter 6 to 8 Review **Question 2 Page 506**

a) S50ºE

b) N70ºW

c) S86ºE



Chapter 6 to 8 Review **Question 3 Page 506**

a) $\overrightarrow{DB}$ or $\overrightarrow{EA}$

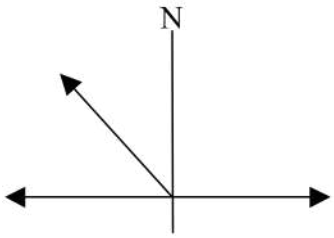
b) No. The magnitudes of the vectors are marked equal in the diagram. The vectors look parallel in the diagram but the angles in the isosceles triangles involved may be different in magnitude and so we cannot be sure the vectors are parallel. (e.g., $\angle FAE$ could equal 45º while $\angle BDC$ could equal 46º.)

c) The two isosceles triangles involved need to be congruent. This requires either $\angle FAE = \angle BDC$ or $|FE| = |BC|$. Both conditions would lead to AF and CD having equal lengths and being parallel.

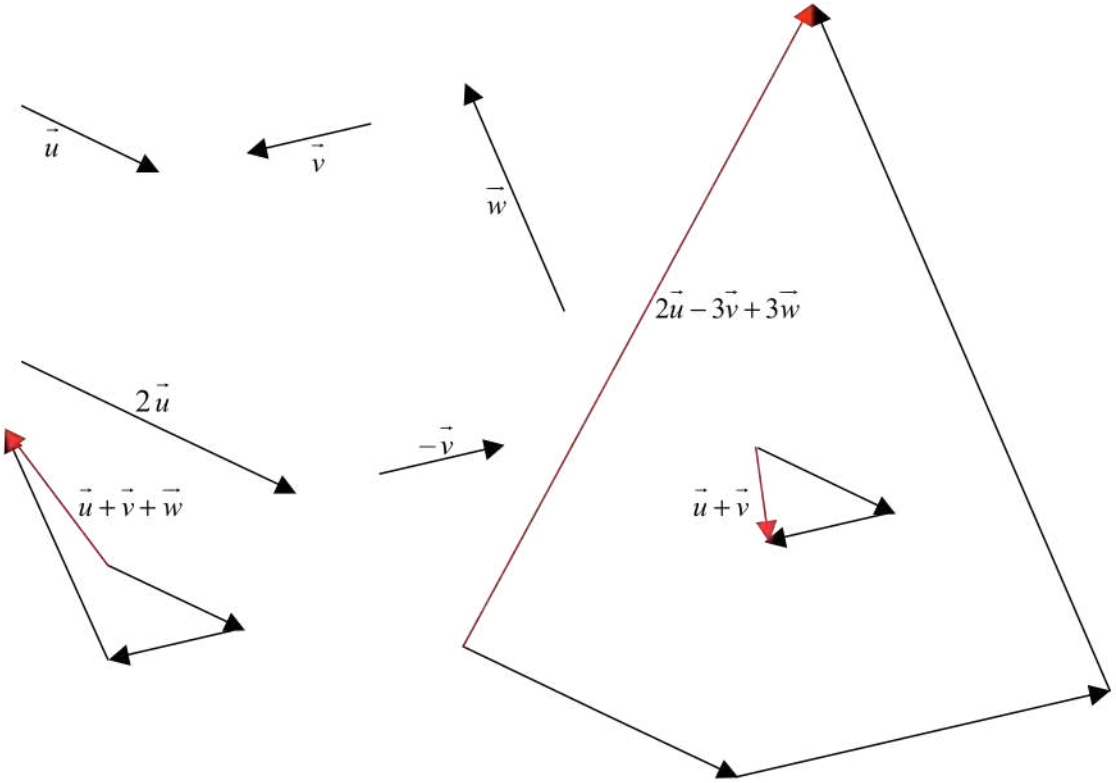
d) $\begin{aligned}\overrightarrow{AB} &= \overrightarrow{EB} - \overrightarrow{EA} \\ &= \overrightarrow{AE} - \overrightarrow{BE} \\ &= \overrightarrow{DB} - \overrightarrow{DA} \\ &= \overrightarrow{AD} - \overrightarrow{BD}\end{aligned}$

Chapter 6 to 8 Review **Question 4 Page 506**

The first two vectors are opposites of each other and have a vector sum of $\vec{0}$. The resultant force is 50 N in the northwest direction.

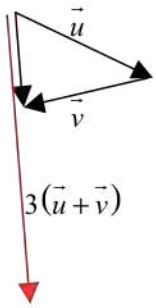


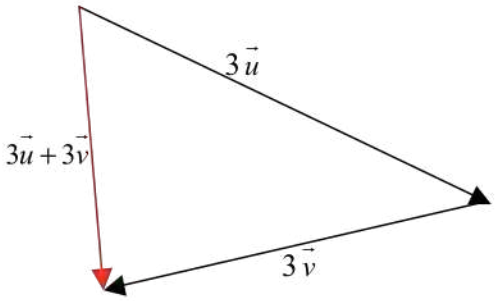
Chapter 6 to 8 Review **Question 5 Page 506**



Chapter 6 to 8 Review **Question 6 Page 506**

Draw a scale diagram using the vectors from question 5.





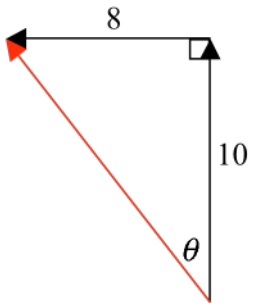
Chapter 6 to 8 Review **Question 7 Page 506**

a) Normal acceleration due to gravity on Earth is 9.8 m/s^2. On a roller coaster at an amusement park the acceleration felt by a rider is twice as great at the bottom of a short dip.

b) A ball hits a racquet with a force of 100 N. The force of the racquet on the ball is ten times greater and in the opposite direction.

Chapter 6 to 8 Review **Question 8 Page 506**

a) Draw a diagram. Use the Pythagorean theorem and trigonometry.

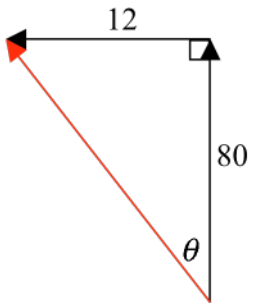


$$\left|\vec{R}\right| = \sqrt{10^2 + 8^2}$$
$$\doteq 12.8$$

$$\tan\ \theta = \frac{8}{10}$$
$$\theta = \tan^{-1}\left(\frac{8}{10}\right)$$
$$\theta \doteq 38.7^\circ$$

The resultant is approximately 12.8 N in a direction N38.7°W.

b) Draw a diagram. Use the Pythagorean theorem and trigonometry.

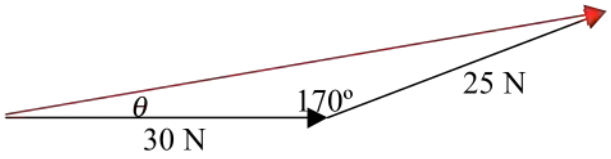


$$\left|\vec{R}\right| = \sqrt{80^2 + 12^2}$$
$$\doteq 80.9$$

$$\tan\theta = \frac{8}{10}$$
$$\theta = \tan^{-1}\left(\frac{12}{80}\right)$$
$$\theta \doteq 8.5^{\circ}$$

The resultant is approximately 80.9 m/s, 8.5° up from the horizontal.

c) Draw a diagram of the situation.



Use the cosine law to find the magnitude of the resultant displacement.

The angle between the displacements in the diagram is 170°.

$$\left|\vec{R}\right|^2 = 30^2 + 25^2 - 2(30)(25)\cos 170^0$$
$$\left|\vec{R}\right|^2 \doteq 3002.2$$
$$\left|\vec{R}\right| \doteq 54.8$$

To find the direction of the resultant, use the sine law. Let θ represent the angle of $\vec{R}$ to the horizontal direction.

$$\frac{\sin\ \theta}{25} = \frac{\sin\ 170^\circ}{54.8}$$

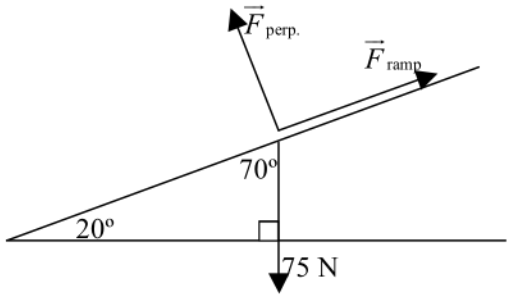
$$\sin\ \theta = \frac{25\sin\ 170^\circ}{54.8}$$

$$\theta = \sin^{-1}\left(\frac{25\sin\ 170^\circ}{54.8}\right)$$

$$\theta \doteq 4.5^\circ$$

The resultant force has a magnitude of approximately 54.8 N, 4.5° from the horizontal

Chapter 6 to 8 Review **Question 9 Page 506**



$$\left|\vec{F}_{\text{ramp}}\right| = 75\cos\ 70^\circ \qquad \left|\vec{F}_{\text{perp.}}\right| = 75\sin\ 70^\circ$$

$$\doteq 25.7 \qquad \doteq 70.5$$

The component parallel to the ramp is 25.7 N.
The component perpendicular to the ramp is 70.5 N.

Chapter 6 to 8 Review **Question 10 Page 506**

a) Unit vectors parallel to $\vec{u}$ are of the form $\pm\frac{1}{|\vec{u}|}\vec{u}$.

$$|\vec{u}| = \sqrt{2^2 + 3^2}.$$

$$= \sqrt{13}$$

The required unit vectors are $\left[\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right]$ and $\left[-\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}\right]$.

b) $\sqrt{13}$

c) Let Q(x, y) be the point.

$\overrightarrow{PQ} = [x + 7,\ y - 5]$

$x + 7 = 2$ $\qquad$ $y - 5 = 3$

$x = -5$ $\qquad$ $y = 8$

The point is Q(–5, 8).

Chapter 6 to 8 Review $\qquad$ **Question 11 Page 506**

Answers may vary. For example:

For $k = 2$ and $m = 3$ and $\vec{u} = [5, 6]$.

$\text{L.S.} = (k + m)\vec{u}$

$= (2 + 3)[5, 6]$

$= 5[5, 6]$

$= [25, 30]$

$\text{R.S.} = k\vec{u} + m\vec{u}$

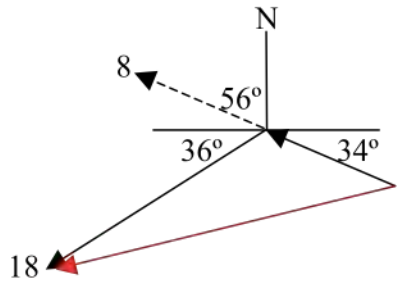
$= 2[5, 6] + 3[5, 6]$

$= [10, 12] + [15, 18]$

$= [25, 30]$

L.S. = R.S.

Chapter 6 to 8 Review $\qquad$ **Question 12 Page 506**



The boat's vector is $[18\cos 216^\circ, 18\sin 216^\circ] = [-14.5623, -10.5801]$.

The current's vector is $[8\cos 146^\circ, 8\sin 146^\circ] = [-6.6323, 4.4735]$.

The resultant vector is $[-21.1946, -6.1066]$.

The magnitude of the resultant is $\sqrt{(-21.1946)^2 + (-6.1066)^2} \doteq 22.05$.

The angle of the resultant with the *east* direction is:

$\theta = \tan^{-1}\left(\dfrac{-6.1066}{-21.1946}\right)$

$\doteq 16.1^\circ + 180^\circ$

$= 196.1^\circ$

(The resultant is pointing SW or at a bearing of 253.9°.)

Chapter 6 to 8 Review **Question 13 Page 506**

$$\vec{a} \cdot \vec{b} = \left|\vec{a}\right|\left|\vec{b}\right| \cos\ \theta$$
$$= (90)(108) \cos\ 40^{\circ}$$
$$\doteq 7446.0$$

The force is approximately 7446.0 N.

Chapter 6 to 8 Review **Question 14 Page 506**

a) $\vec{m} \cdot \vec{n} = [-2, -8] \cdot [9, 0]$
$= -2(9) + (-8)(0)$
$= -18$

b) $\vec{p} \cdot \vec{q} = [4, 5, -1] \cdot [6, -2, 7]$
$= 4(6) + 5(-2) + (-1)(7)$
$= 7$

Chapter 6 to 8 Review **Question 15 Page 506**

Choose any vector the makes a dot product of zero with $\vec{u}$.
$\vec{v} = [5, 6]$

Chapter 6 to 8 Review **Question 16 Page 506**

$$\cos\ \theta = \frac{\vec{u} \cdot \vec{v}}{\left|\vec{u}\right|\left|\vec{v}\right|}$$
$$\cos\ \theta = \frac{[5, 7, -1] \cdot [8, 7, 8]}{\sqrt{5^2 + 7^2 + (-1)^2}\sqrt{8^2 + 7^2 + 8^2}}$$
$$\cos\ \theta = \frac{81}{\sqrt{75}\sqrt{177}}$$
$$\cos\ \theta \doteq 0.7030$$
$$\theta = \cos^{-1}(0.7030)$$
$$\theta = 45.3^{\circ}$$

The angle between the vectors is approximately 45°.

Chapter 6 to 8 Review **Question 17 Page 507**

$$\begin{aligned} W &= \vec{F} \cdot \vec{s} \\ &= \left|\vec{F}\right|\left|\vec{s}\right| \cos\ \theta \\ &= (80)(20) \cos\ 10^{\circ} \\ &\doteq 1575.7 \end{aligned}$$

Tyler does 1575.7 J of mechanical work.

Chapter 6 to 8 Review **Question 18 Page 507**

$$\vec{v} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$\vec{v} = [9, 1, 5] - [3, 7, -2]$$
$$\vec{v} = [6, -6, 7]$$

$$\left|\vec{v}\right| = \sqrt{6^2 + (-6)^2 + 7^2}$$
$$\left|\vec{v}\right| = 11$$

Chapter 6 to 8 Review **Question 19 Page 507**

a) $\vec{b} \cdot \vec{a} \times \vec{c} = [7, 3, 4] \cdot [2, -4, 5] \times [-3, 7, 1]$

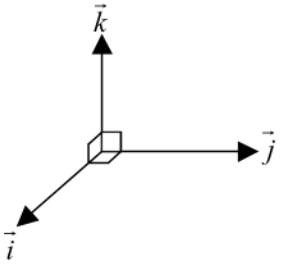
$$\begin{aligned} &= [7, 3, 4] \cdot [-4(1) - 7(5), 5(-3) - 1(2), 2(7) - (-3)(-4)] \\ &= [7, 3, 4] \cdot [-39, -17, 2] \\ &= 7(-39) + 3(-17) + 4(2) \\ &= -316 \end{aligned}$$

b) $\vec{a} \times \vec{b} - \vec{b} \times \vec{c} = [2, -4, 5] \times [7, 3, 4] - [7, 3, 4] \times [-3, 7, 1]$

$$\begin{aligned} &= [-31, 27, 34] - [-25, -19, 58] \\ &= [-6, 46, -24] \end{aligned}$$

Chapter 6 to 8 Review **Question 20 Page 507**

No, $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.



$\vec{i} \times \vec{j} = \vec{k}$

$\vec{j} \times \vec{i} = -\vec{k}$

Chapter 6 to 8 Review **Question 21 Page 507**

$$\begin{aligned}\vec{c} \times \vec{d} &= \left[6, 3, -2\right] \times \left[4, 5, -7\right] \\ &= \left[3(-7) - 5(-2), -2(4) - (-7)(6), 6(5) - 4(3)\right] \\ &= \left[-11, 34, 18\right]\end{aligned}$$

$$\begin{aligned}\vec{d} \times \vec{c} &= -\vec{c} \times \vec{d} \\ &= \left[11, -34, -18\right]\end{aligned}$$

Two possible orthogonal vectors are (–11, 34, 18) and (11, –34, –18).

Chapter 6 to 8 Review **Question 22 Page 507**

$$\begin{aligned}A &= \left|\vec{u} \times \vec{v}\right| \\ &= \left|\vec{u}\right|\left|\vec{v}\right|\left|\sin\ \theta\right|\end{aligned}$$

$$\sin\ \theta = \frac{95}{(12)(10)}$$

$$\theta = \sin^{-1}\left(\frac{95}{(12)(10)}\right)$$

$$\theta \doteq 52.3^{\circ}$$

Adjacent interior angles in a parallelogram add to 180°. The interior angles are 52.3° and 127.7°.

Chapter 6 to 8 Review **Question 23 Page 507**

Assume the force acts at right angles to the shaft.

$$\begin{aligned}\left|\vec{\tau}\right| &= \left|\vec{r}\right|\left|\vec{F}\right|\sin\theta \\ &= (0.18)(40)\sin 90^\circ \\ &\doteq 7.2\end{aligned}$$

The magnitude of the torque is 7.2 N·m.

Chapter 6 to 8 Review **Question 24 Page 507**

a) $\begin{aligned}\overrightarrow{PQ} &= [-4, 7] - [3, 5] \\ &= [-7, 2]\end{aligned}$

A vector equation is $[x, y] = [3, 5] + t[-7, 2],\ t \in \mathbb{R}$.

Parametric equations are:

$x = 3 - 7t$

$y = 5 + 2t,\ t \in \mathbb{R}$

b) $\begin{aligned}\overrightarrow{AB} &= [-2, -3, 6] - [6, -1, 5] \\ &= [-8, -2, 1]\end{aligned}$

A vector equation is $[x, y, z] = [6, -1, 5] + t[-8, -2, 1],\ t \in \mathbb{R}$.

Parametric equations are:

$x = 6 - 8t$

$y = -1 - 2t$

$z = 5 + t,\ t \in \mathbb{R}$

Chapter 6 to 8 Review **Question 25 Page 507**

The direction of the line is $\vec{m} = [-7, 3]$.

A normal is any perpendicular vector (i.e., a vector having a dot product of zero with $\vec{m}$). Use $\vec{n} = [3, 7]$.

The scalar equation is of the form $3x + 7y + C = 0$.

The point (6, –2) is on the line and leads to C = –4.

The equation of the line is $3x + 7y - 4 = 0$.

Chapter 6 to 8 Review **Question 26 Page 507**

The scalar equation is of the form $5x + y + C = 0$.

Substitute (–4, 6).
$5(-4) + 6 + C = 0$
$C = 14$

The scalar equation is $5x + y + 14 = 0$.

A direction vector is any vector perpendicular to the normal. Use $[-1, 5]$.
A vector equation is $[x, y] = [-4, 6] + t[-1, 5],\ t \in \mathbb{R}$.

Chapter 6 to 8 Review **Question 27 Page 507**

a) Lines parallel to the x- and y-axes have scalar equations of the form $z = k$.
A vector equation is $[x, y, z] = [2, 5, -1] + s[1, 0, 0] + t[0, 1, 0],\ s, t \in \mathbb{R}$
A scalar equation is $z = -1$.

b) A vector equation is $[x, y, z] = [6, 2, 3] + s[1, 1, 6] + t[1, -5, 5],\ s, t \in \mathbb{R}$.
For a scalar equation, $\vec{n} = \overrightarrow{BA} \times \overrightarrow{CA}$
$= [1, 1, 6] \times [1, -5, 5]$
$= [35, 1, -6]$

The scalar equation is of the form $35x + y - 6z + D = 0$.

Substitute (6, 2, 3).
$35(6) + 2 - 6(3) + D = 0$
$D = -194$

A scalar equation is $35x + y - 6z - 194 = 0$.

c) Parallel planes have equal normals.
The scalar equation is of the form $2x + 6y + 4z + D = 0$.

Substitute (3, 2, 1).
$2(3) + 6(2) + 4(1) + D = 0$
$D = -22$

A scalar equation is $2x + 6y + 4z - 22 = 0$ or $x + 3y + 2z - 11 = 0$.

Chapter 6 to 8 Review **Question 28 Page 507**

If three direction vectors were mutually perpendicular, they would define a three-dimensional space, not a plane. Two perpendicular vectors can be used to form a plane, and any other vectors in the plane would be a linear combination of these two vectors and hence coplanar with them.

Chapter 6 to 8 Review **Question 29 Page 507**

a) $\vec{n} = \vec{m}_1 \times \vec{m}_2$

$= [-1, 3, 4] \times [6, 1, -2]$

$= [-10, 22, -19]$

The scalar equation is of the form $10x - 22y + 19z + D = 0$.

Substitute (2, 3, 5).

$10(2) - 22(3) + 19(5) + D = 0$

$D = -49$

A scalar equation is $10x - 22y + 19z - 49 = 0$.

b) The scalar equation is of the form $x + 2y - 5z + D = 0$.

Substitute (6, –2, 3).

$6 + 2(-2) - 5(3) + D = 0$

$D = 13$

A scalar equation is $x + 2y - 5z + 13 = 0$.

Chapter 6 to 8 Review **Question 30 Page 507**

The angle between planes is defined as the angle between their normal vectors.

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{\left|\vec{n}_1\right|\left|\vec{n}_2\right|}$$

$$= \frac{[2, 2, 7] \cdot [3, -4, 4]}{\sqrt{2^2 + 2^2 + 7^2}\sqrt{3^2 + (-4)^2 + 4^2}}$$

$$= \frac{26}{\sqrt{57}\sqrt{41}}$$

$$\theta = \cos^{-1}\left(\frac{26}{\sqrt{57}\sqrt{41}}\right)$$

$$\doteq 57.5^\circ$$

The angle is about 57.5°.

Chapter 6 to 8 Review **Question 31 Page 507**

The direction vectors are not scalar multiples of each other. The lines must intersect.
To find the intersection point, equate like coordinates.

$2-5s=-11+2t$ ①

$1+3s=7-3t$ ②

$-4+s=2+3t$ ③

Solve ② and ③ for s and t.

$3s+3t=6$ ②

$s-3t=6$ ③

$4s=12$ ②+③

$s=3$

Substitute $s = 3$ in ③.

$3-3t=6$ ③

$t=-1$

Check if these values satisfy ①.

L.S. $=2-5(3)$ R.S. $=-11+2(1)$

$=-13$ $=-13$

L.S. = R.S.

Substitute $s=3$ or $t=-1$ to find the point of intersection.
The two lines intersect at the point (–13, 10, –1).

Chapter 6 to 8 Review **Question 32 Page 507**

For the distance between skew lines, $d=\dfrac{\left|\overrightarrow{P_1P_2}\cdot\vec{n}\right|}{\left|\vec{n}\right|}$ where $P_1\in\ell_1$, $P_2\in\ell_2$, and $\vec{n}=\vec{m}_1\times\vec{m}_2$.

Let $P_1(4, 2, -3)$, $P_2(-6, -2, 3)$, $\vec{m}_1=[-1,-2,2]$, and $\vec{m}_2=[-2,2,1]$.

$\overrightarrow{P_1P_2}=[-10,-4,6]$ and $\vec{n}=\vec{m}_1\times\vec{m}_2=[-6,-3,-6]$

$$d=\frac{\left|[-10,-4,6]\cdot[-6,-3,-6]\right|}{\left|[-6,-3,-6]\right|}$$

$$d=\frac{36}{\sqrt{81}}$$

$d=4$

The distance between the skew lines is 4 units.

Chapter 6 to 8 Review **Question 33 Page 507**

a) Substitute the parametric equations for the line in the equation for the plane and solve for s.

$$2(4+3s)+3(4+3s)-5(-1-2s)=3$$
$$8+6s+18+12s+5+10s=3$$
$$28s=-28$$
$$s=-1$$

Substitute $s = 1$ in the parametric equations.

$x = 4+3(-1)$ $\quad y = 6+4(-1)$ $\quad z = -1-2(-1)$

$= 1$ $\quad = 2$ $\quad = 1$

The point of intersection of the line and the plane is $(1, 2, 1)$.

b) Substitute the parametric equations into the scalar equation of the plane and solve for s.

$$(8+4s)+3(-2-2s)-2(-2-s)=6$$
$$8+4s-6-6s+4+2s=6$$
$$0s=0$$

This equation is true for every value of s.

The two lines intersect at every point on the line; the line is on the plane.

Chapter 6 to 8 Review **Question 34 Page 507**

The distance between a point P and a plane is given by $d=\dfrac{\left|\vec{n}\cdot\overrightarrow{PQ}\right|}{\left|\vec{n}\right|}$ where Q is any point on the plane with normal $\vec{n}$. Choose Q(1, 0, 0).

$$\overrightarrow{PQ}=[1, 0, 0]-[3, -2, 5]$$
$$=[-2, 2, -5]$$

$$d=\frac{\left|[2, 4, -1]\cdot[-2, 2, -5]\right|}{\sqrt{2^2+4^2+(-1)^2}}$$
$$=\frac{9}{\sqrt{21}}$$
$$\doteq 1.96$$

The distance is about 1.96 units.

Chapter 6 to 8 Review **Question 35 Page 507**

a) The normals are identical but the equations are different.
The planes are parallel and distinct.

b) The normals are scalar multiples of each other. $2\vec{n}_1 = \vec{n}_2$ and $2① = ②$.
The planes are parallel and coincident.

Chapter 6 to 8 Review **Question 36 Page 507**

a) First examine the normals. None are scalar multiples of each other.
Check if the normals are coplanar.

$$\begin{aligned}\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 &= [2, 3, -5] \cdot [5, -1, 2] \times [-1, 7, -12] \\ &= [2, 3, -5] \cdot [-2, 58, 34] \\ &= 0\end{aligned}$$

The normals are coplanar; there may be a line of intersection.

Use elimination to eliminate one of the variables, say y, from two pairs of equations.

$$\begin{array}{lll} 2x + 3y - 5z = 9 & ① & \\ \underline{15x - 3y + 6z = -9} & 3② & \\ 17x + z = 0 & ④ & ①+3② \end{array}$$

$$\begin{array}{lll} 35x - 7y + 14z = -21 & 7② & \\ \underline{-x + 7y - 12z = 21} & ③ & \\ 34x + 2z = 0 & ⑤ & 7②+③ \end{array}$$

Solve equations ④ and ⑤ for x and z.

$$\begin{array}{ll} 34x + 2z = 0 & 2④ \\ \underline{34x + 2z = 0} & ⑤ \\ 0 = 0 & 2④-⑤ \end{array}$$

Let $z = t$,

$$17x + t = 0 \quad ④$$

$$x = -\frac{1}{17}t$$

Substitute $x = -\frac{1}{17}t$ and $z = t$ in ②.

$$5\left(-\frac{1}{17}t\right) - y + 2t = -3 \quad ②$$

$$y = 3 + \frac{29}{17}t$$

The three planes intersect at a line that can be defined as

$$x = -\frac{1}{17}t$$
$$y = 3 + \frac{29}{17}t$$
$$z = t, \quad t \in \mathbb{R}$$

These equations can be simplified by multiplying the direction vector by 17. The simplified equations are ($t \in \mathbb{R}$):

$x = -t$ $\quad$ $y = 3 + 29t$ $\quad$ $z = 17t$

b) First examine the normals. None are scalar multiples of each other.
Check if the normals are coplanar.

$$\begin{aligned}\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 &= [2, 3, 1] \cdot [1, 4, -2] \times [7, 6, 5] \\ &= [2, 3, 1] \cdot [32, -19, -22] \\ &= -15\end{aligned}$$

The normals are not coplanar; there is an intersection point.

Use elimination to eliminate one of the variables, say x, from two pairs of equations.

$$\begin{aligned} 2x + 3y + z &= 5 && ① \\ \underline{2x + 8y - 4z = 20} && 2② \\ -5y + 5z &= -15 && ①-2② \\ -y + z &= -3 && ④ \end{aligned}$$

$$\begin{aligned} 7x + 28y - 14z &= 70 && 7② \\ \underline{7x + 6y + 5z = 7} && ③ \\ 22y - 19z &= 63 && ⑤ \quad 7②-③ \end{aligned}$$

Solve equations ④ and ⑤ for y and z.

$$\begin{aligned} -22y + 22z &= -66 && 22④ \\ \underline{22y - 19z = 63} && ⑤ \\ 3z &= -3 && 22④+⑤ \\ z &= -1 \end{aligned}$$

Substitute $z = -1$ in ④.

$$\begin{aligned} -y + (-1) &= -3 && ④ \\ y &= 2 \end{aligned}$$

Substitute $y = 2$ and $z = -1$ in ②.

$$\begin{aligned} x + 4(2) - 2(-1) &= 10 \\ x &= 0 \end{aligned}$$

The three planes intersect at the point (0, 2, –1).

c) First examine the normals. None are scalar multiples of each other.
Check if the normals are coplanar.

$$\begin{aligned}\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 &= [3, 1, -1] \cdot [4, -2, -3] \times [8, 6, -1] \\ &= [3, 1, -1] \cdot [20, -20, 40] \\ &= 0\end{aligned}$$

The normals are coplanar; there may be a line of intersection.

Use elimination to eliminate one of the variables, say y, from two pairs of equations.

$$\begin{array}{lll} 6x + 2y - 2z = 8 & 2① & \\ \underline{4x - 2y - 3z = 5} & ② & \\ 10x \quad -5z = 13 & ④ & 2①+② \end{array}$$

$$\begin{array}{lll} 18x + 6y - 6z = 24 & 6① & \\ \underline{8x + 6y \quad - z = 7} & ③ & \\ 10x \quad -5z = 17 & ⑤ & 6①-③ \end{array}$$

Solve equations ④ and ⑤ for x and z.

$$\begin{array}{ll} 10x - 5z = 13 & ④ \\ 10x - 5z = 17 & ⑤ \\ 0 = -4 & ④-⑤ \end{array}$$

This equation is never true.

The three planes do not have any common points. The planes intersect in pairs.